Otranto School

XXXII International Seminar of Nuclear and Subnuclear Physics
"Francesco Romano"

Gravitational Waves
part 2: detectors

Giovanni A. Prodi, Università di Trento and INFN-TIFPA, Trento
credits to Giacomo Ciani, Università di Padova

UNIVERSITÀ DI TRENTO

INFN Istituto Nazionale di Fisica Nucleare

TIFPA Trento Institute for Fundamental Physics and Applications

Beam detectors of GW

Measure with light the proper distance (invariant) between test masses in approximate free fall

- one-way beam detectors e.g. Pulsar timing signal $\rightarrow$ Earth
- two-way beam detectors
  - one beam e.g. radar
  - two beams e.g. interferometer

light travel time between test masses distant L along x axis
in a h_+ GW traveling orthogonal to x axis

for a continuous beam emission
at stabilized frequency f_{start}

- No dependence on the f_{start} value, which is not invariant
- dependence on h_+ at times of interaction with test masses

Interferometric detector of GWs

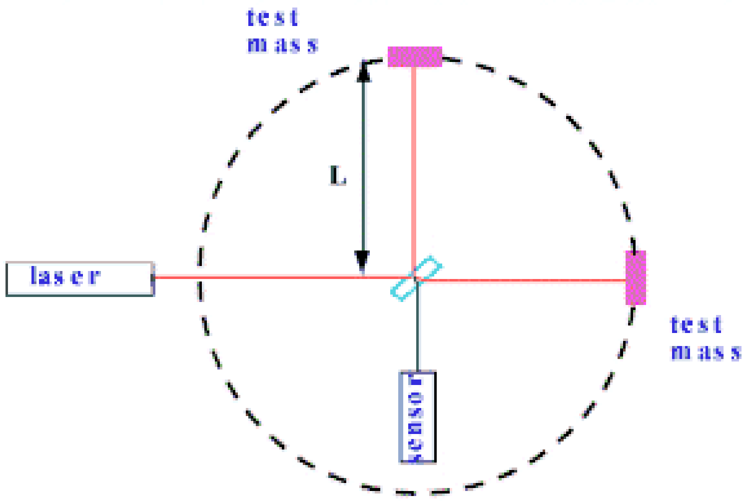
for one arm, length L along x axis,
in a h_+ GW traveling orthogonal to x axis

Interferometric detectors for GW

Measure differential arm changes ΔL by laser light beams:

optical phase difference $\Delta\phi$ at the antisymmetric port

$\Rightarrow$ light power variations at sensor



$$\frac{P_{out}}{P_{in}} = \sin^2(\Delta\phi) = \sin^2\left(\frac{2\pi\Delta L}{\lambda_{Laser}} + \Delta\phi_{GW}\right)$$

- 2 **balanced-length arms** allow common mode rejection of many technical noises.
- Operation close to **dark fringe** ($P_{out} \sim 0$) allows a «null measurement», i.e. more favorable $\Delta P_{out}/P_{out}$
- Want **high P_{in}** : signal scales with circulating power
- Target audio frequency gravitational waves:

optimal arm length: $L = \frac{\lambda_{GW}}{4} = 750km\left(\frac{100Hz}{f_{GW}}\right)$

$\Rightarrow$ **Km-scale arms** plus increase optical path by **optical resonant cavities**

detector layout

Fabry-Perot Optical Cavities

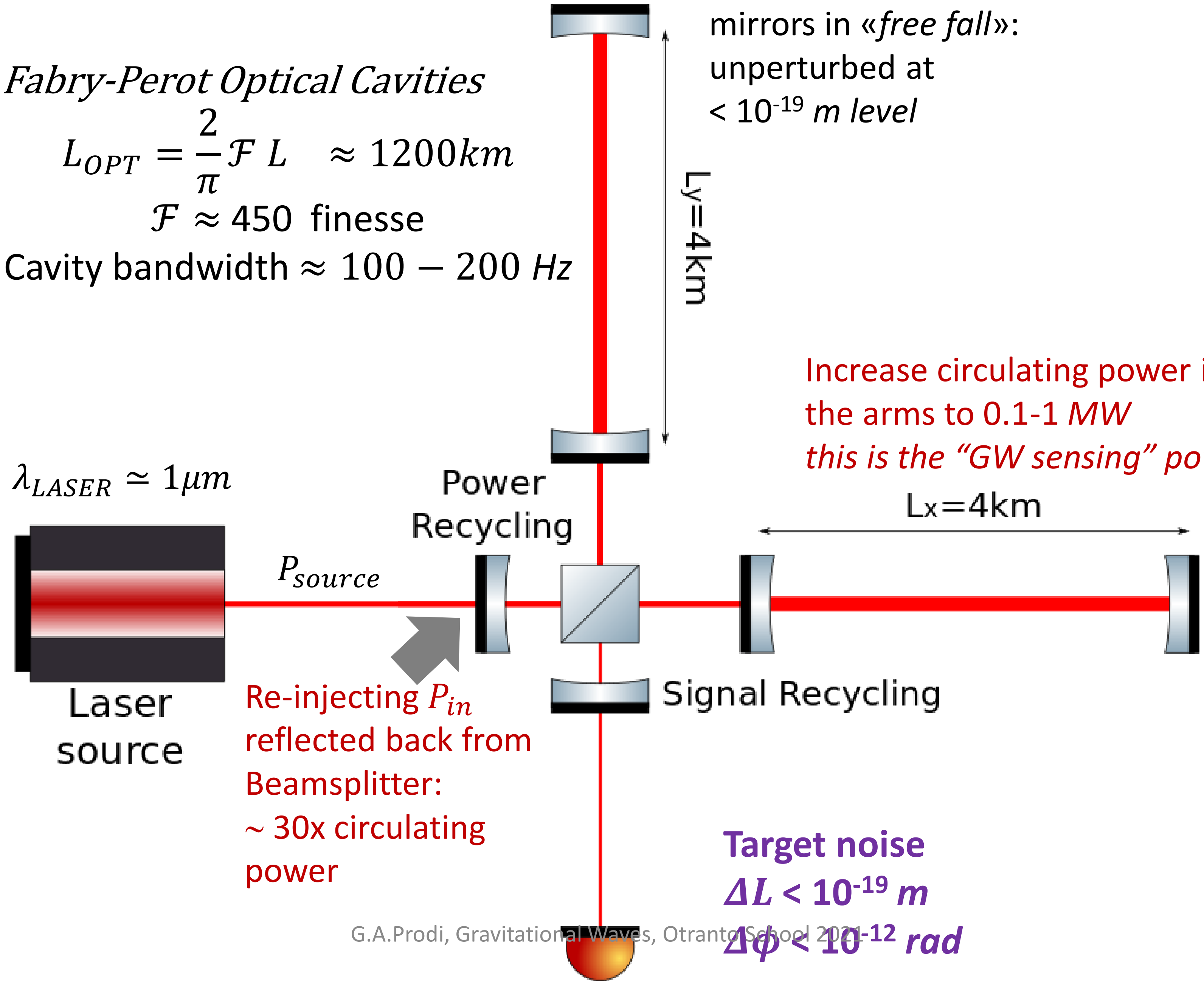
$$L_{OPT} = \frac{2}{\pi}\mathcal{F}\, L \quad \approx 1200km$$

$$\mathcal{F} \approx 450 \text{ finesse}$$

Cavity bandwidth $\approx 100 - 200\ Hz$

mirrors in «*free fall*»:
unperturbed at
< 10^{-19} *m level*

$\lambda_{LASER} \simeq 1\mu m$



Increase circulating power in the arms to 0.1-1 *MW*
this is the "GW sensing" power

Re-injecting P_{in} reflected back from Beamsplitter:
~ 30x circulating power

Target noise
$\Delta L < 10^{-19}\ m$
$\Delta\phi < 10^{-12}\ rad$

detector layout

Fabry-Perot Optical Cavities

$$L_{OPT} = \frac{2}{\pi}\mathcal{F}\,L \quad \approx 1200km$$

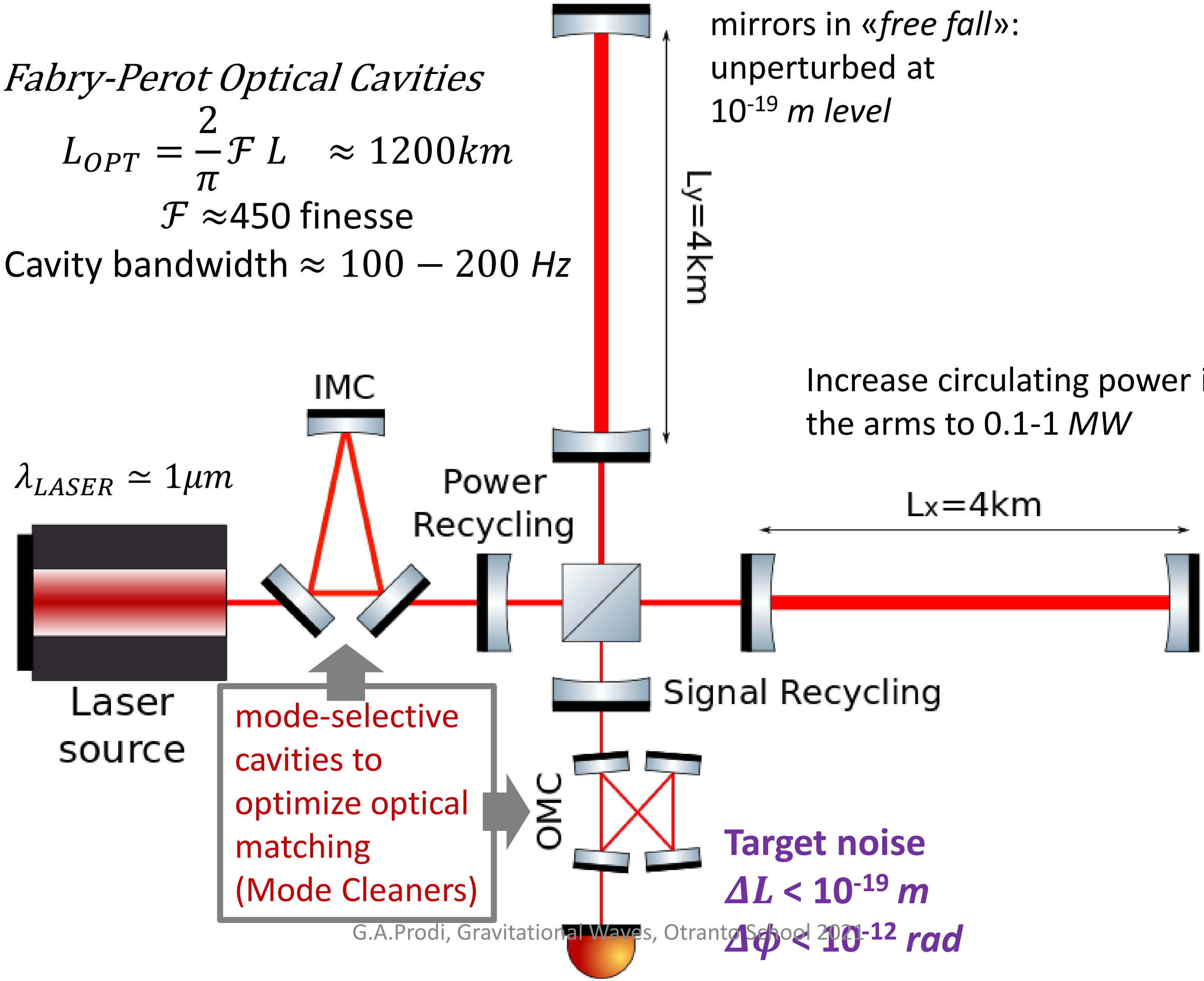
$\mathcal{F}$ ≈450 finesse

Cavity bandwidth $\approx 100 - 200\ Hz$

mirrors in «*free fall*»:
unperturbed at
10^{-19} *m level*

Increase circulating power in the arms to 0.1-1 *MW*

$\lambda_{LASER} \simeq 1\mu m$

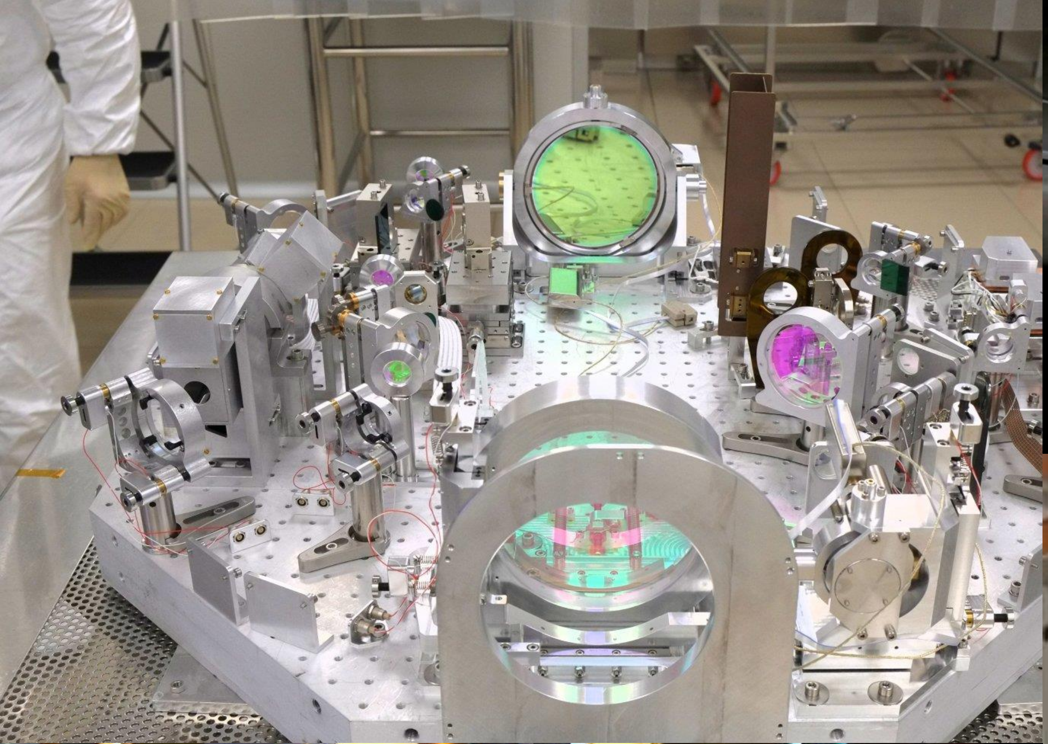


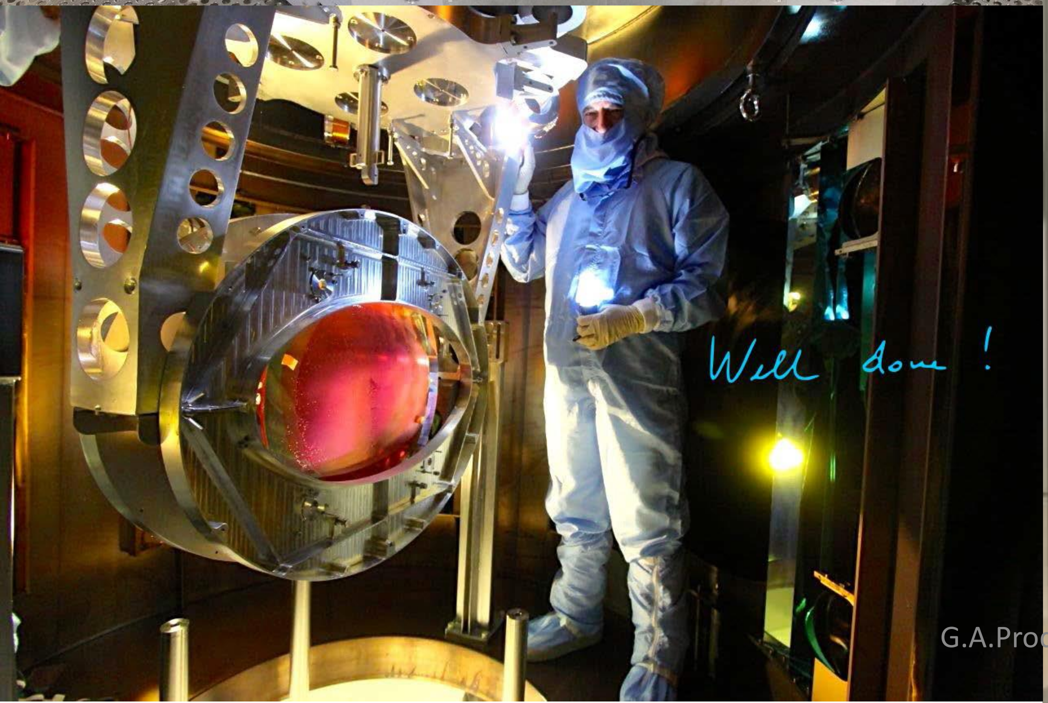
mode-selective cavities to optimize optical matching (Mode Cleaners)

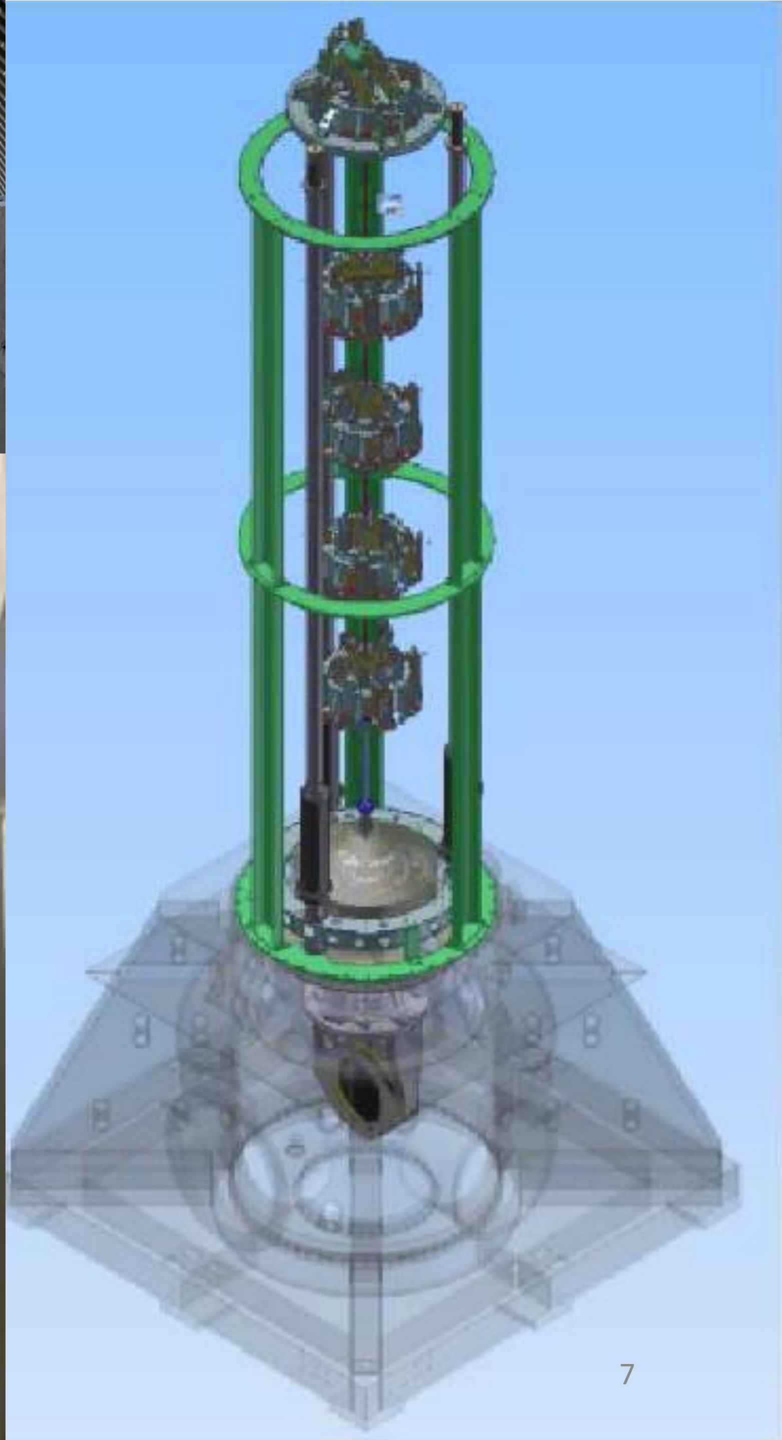
Target noise

ΔL* < 10^{-19} *m

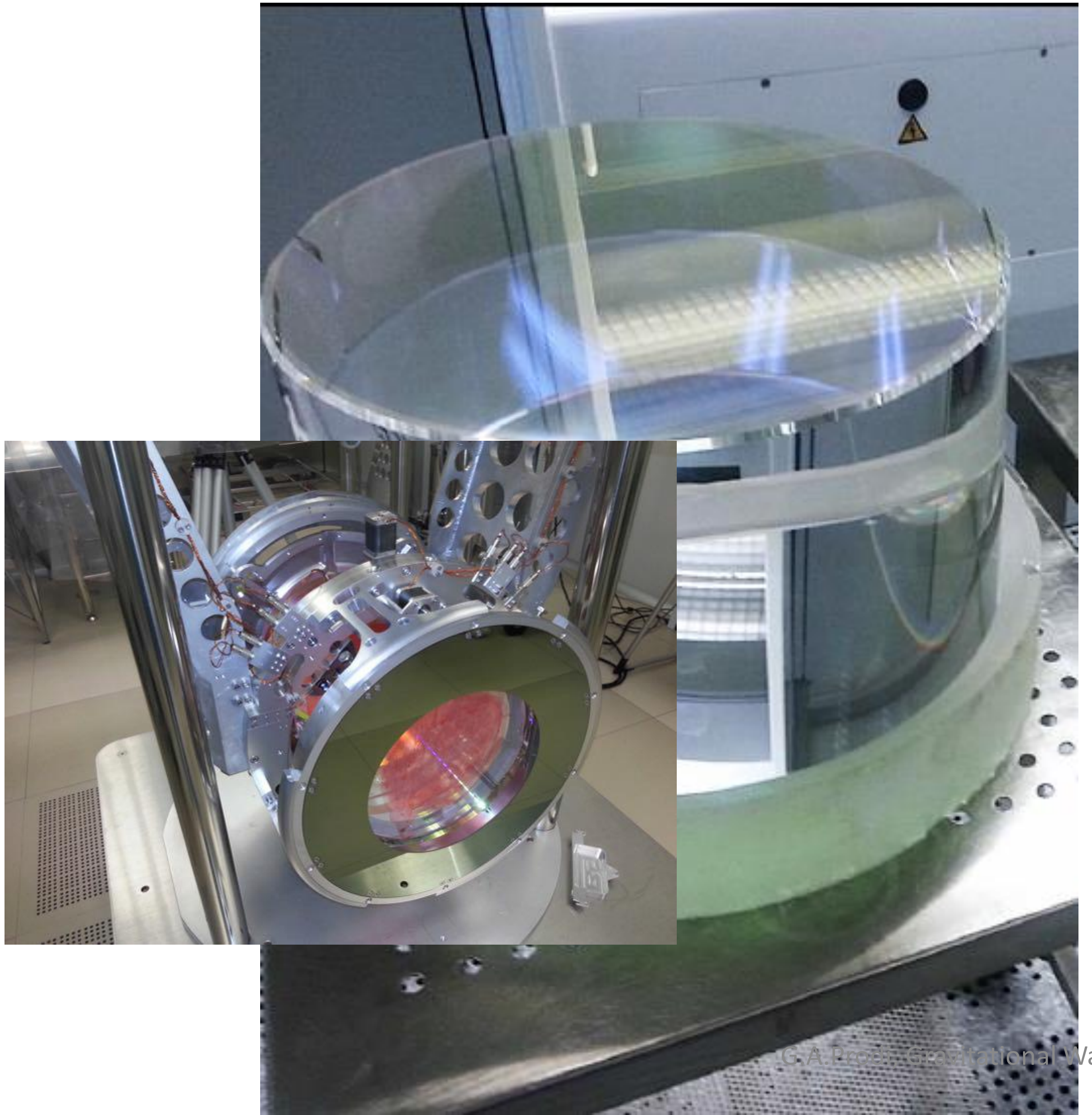
Δφ* < 10^{-12} *rad



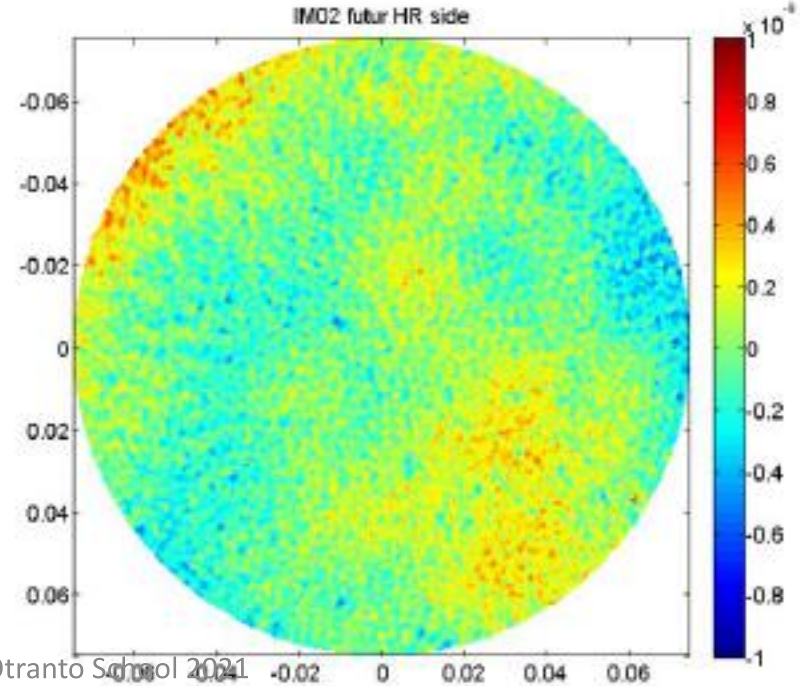

Well done !



Mirrors



- Low mechanical losses
- Low optical absorption
- Low scattering
- 42 kg, 35 cm diam., 20 cm thick
- Flatness < 0.5 nm rms
- Roughness < 0.1 nm rms
- Absorption < 0.5 ppm



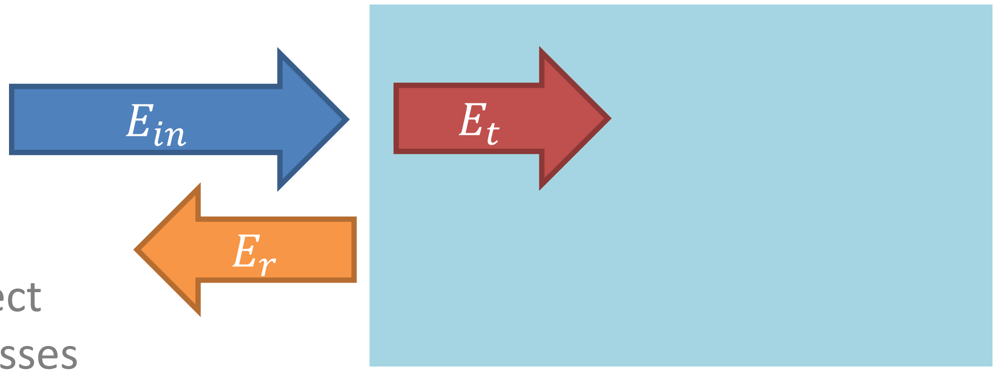
Dielectric mirrors

- Ideal mirror: sharp interface between two mediums
  - $E_r = rE_{in}, \quad E_t = tE_{in}$
  - Sharp interface: $r, t \in \mathbb{R}$
  - Energy conservation:

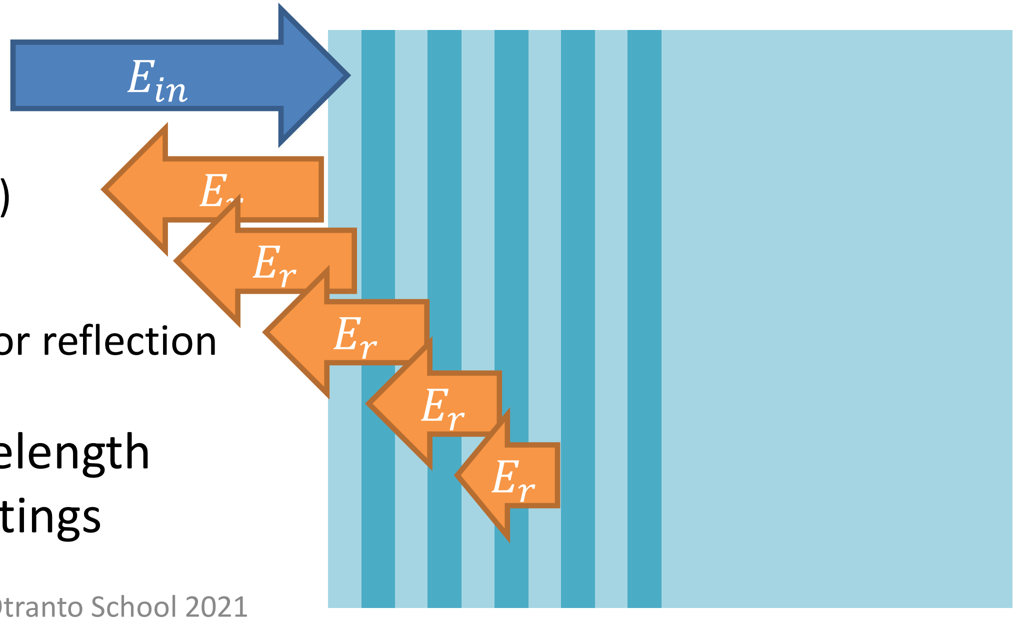
$$I_{in} = I_r + I_t \Rightarrow E_{in}^2 = E_r^2 + E_t^2 \Rightarrow r^2 + t^2 = 1$$

Only for a perfect mirror with no losses

  - $r_{n_{high} \to n_{low}} = -r_{n_{low} \to n_{high}}$

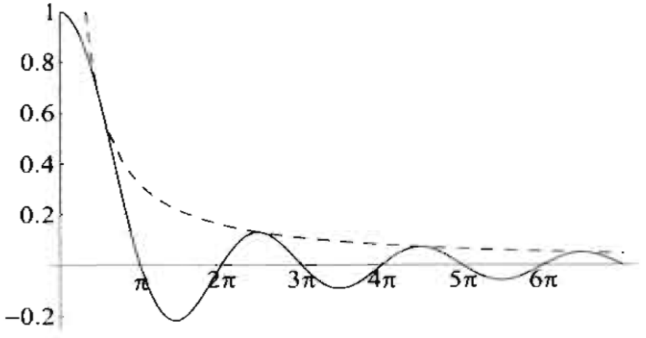


- Dielectric mirros: $\frac{\lambda}{4}$-stacks
  - Mechanism:
    - Alternating layers of high and low index of refraction
    - Each layer has an optical path of $\frac{\lambda}{4}$ (for desired wavelength)
    - Back and forth makes for $\frac{\lambda}{2}$ phase shift
    - $r_{n_{high} \to n_{low}} = -r_{n_{low} \to n_{high}}$ makes for total λ phase shift for reflection off different layers: constructive interference
  - Reflectivity can be accurately tuned in value and wavelength
  - Similar technique used to produce anti-reflection coatings



more on output phase difference

Sinc function



For the y arm, this sign is inverted

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{L_x}{c} h_+ \left(t_0 + \frac{L_x}{c}\right) \frac{\sin\left(\frac{\omega_{gw} L_x}{c}\right)}{\frac{\omega_{gw} L_x}{c}}$$

- For $\frac{L_x}{c} \ll T_{gw} = \frac{2\pi}{\omega_{gw}}$, the GW perturbation is "static" during the light's travel time
- For $\frac{L_x}{c} \gg T_{gw}$, h_+ keeps oscillating cancelling out its effect at every half period

- The phase difference of the two beams at recombination is thus $\Delta\phi = \omega_l (t^x - t^y)$. Usually, $L_x \approx L_y \approx L$ and, at first order in h:

$\Delta\phi_{gw}$, induced by gravitational waves

$$\Delta\phi \approx \omega_l \left(2 \frac{L_x - L_y}{c} + \frac{2L}{c} h_0 \cos(\omega_{gw} t + \alpha) \operatorname{sinc}\left(\frac{\omega_{gw} L}{c}\right)\right) = \Delta\phi_0 + \Delta\phi_{gw}$$

$\Delta\phi_0$, controlled by the experimentalist

note: E is evolving $e^{i\,t\,\omega_{laser}}$ GW creates **sidebands** at $\omega_l \pm \omega_{gw}$ (also mirror displacement noises do)

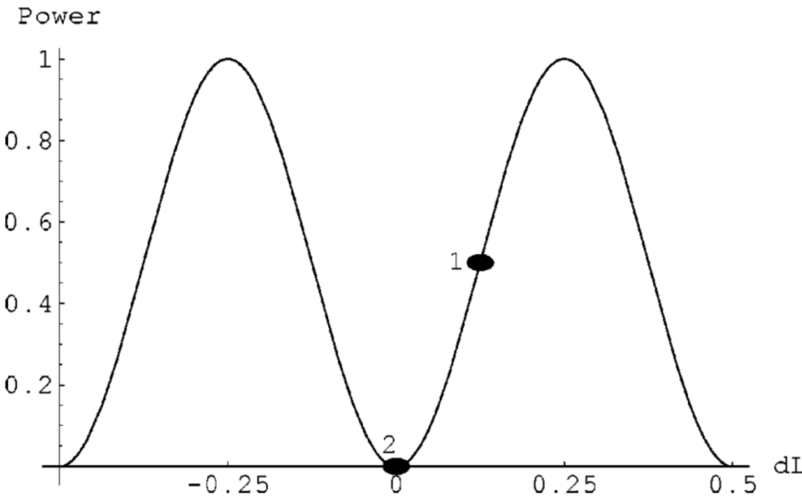
- $\Delta\phi_0$ contains two terms:
  - A (usually small) microscopic term to control the "working point" (=the output with no GW) of the interferometer and set a dark enough fringe
  - A macroscopic term (Schnupp asymmetry) to allow frequencies $\omega_l + \omega_{sb}$ to leak at the output

Black&Gutenkunst, Am J Phys 71 (2003) 365

null measurement

$$P_{out} = E_0^2 \sin^2(\Delta\phi_0 + \Delta\phi_{gw})$$

E_0^2: P_{in} $\Delta\phi_{gw}$: $\approx 2\omega_l Lh/c$



- operating point 1 : max signal gain BUT high dc output $P_{out} = P_{in}/2$
  - P_{in} *amplitude noise directly trasferred at output*
  - $\Delta P_{out}/P_{out} \approx 2\omega_{laser} Lh/c$ extremely low relative uncertainty

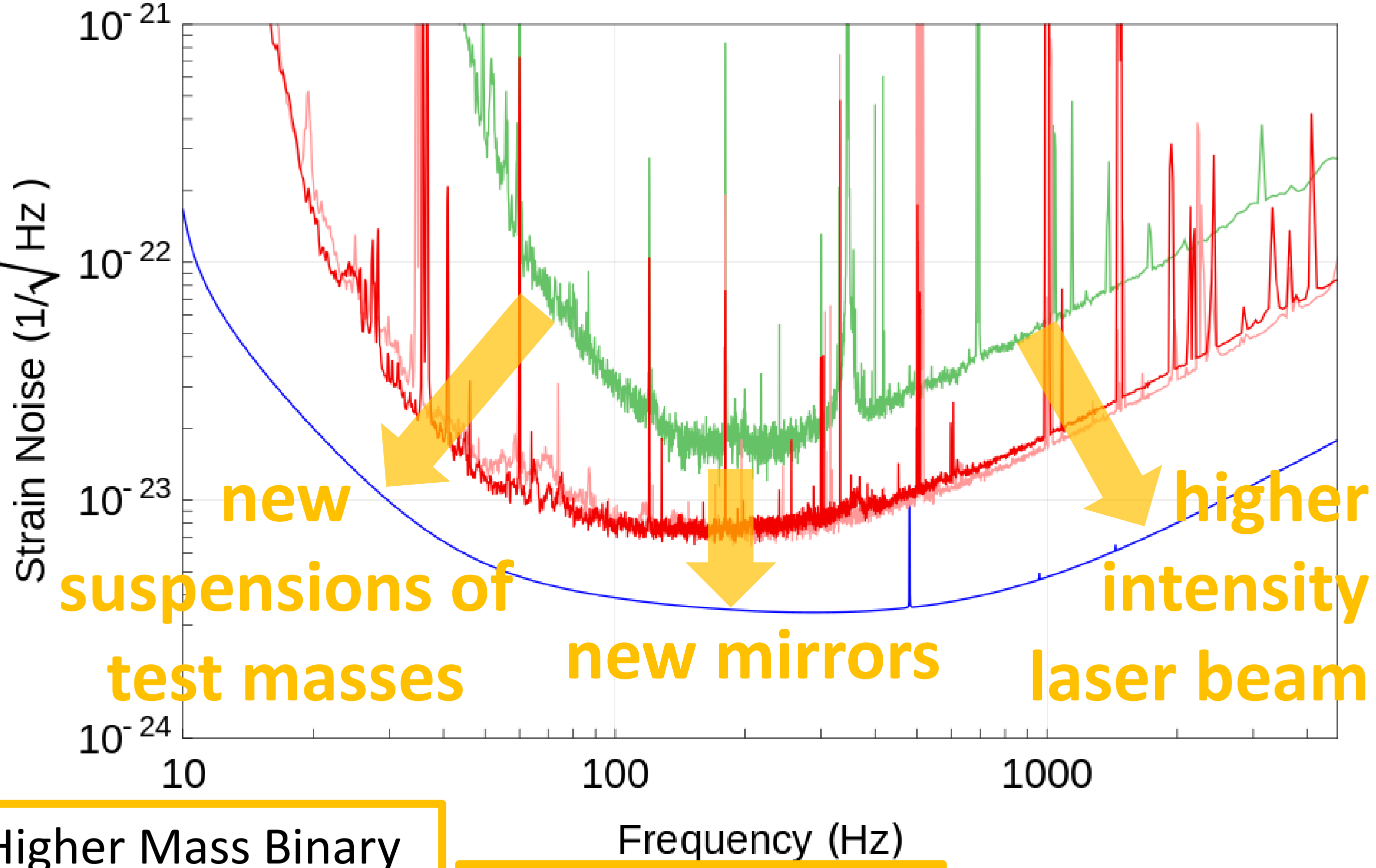
  NOT viable

- operating point 2 "dark fringe" : nulls dc output P_{out} BUT signal is second order effect
  - NOT viable with $L_x - L_y$=0
  - if you implement a lock-in homodyne detection: modulate P_{in} at ω_{SB}
  - but also GWs produce side-bands of the ω_{laser}
  - and realize a Schnupp asymmetry: $L_x - L_y \neq 0$, $L_x - L_y \ll L$, $L_x - L_y = n\pi\,\lambda_{laser}$

  to allow "**dark carrier**" but with some intensity at the sidebands ω_{SB} which modulates the GW signal

  $\approx 2\omega_{laser} Lh/c \sin\left(2\pi \frac{L_x - L_y}{\lambda_{SB}}\right) \approx 2\omega_{laser} Lh/c$ with feasible choices for the asymmetry

spectral sensitivity enhancement



--LIGO S6 run (2010)

--Advanced LIGO O1 run (2015)

--Advanced LIGO design goal

-- $10^{-20} m/\sqrt{Hz}$ displacement noise (single arm)

Higher Mass Binary BH Coalescences (Intermediate Mass)

Earlier detections of coalescences

BBH mergers
BH ring-down

NS transients

and yet UNKNOWN SOURCES

Stochastic background
NS periodic emission
SN

sample target sources

Volume of Universe surveyed scales as ≈ (noise amplitude)$^{-3}$

back-of-the-envelope sensitivity estimates

- target transient GW signal strain $\frac{\Delta L}{L} \approx 10^{-21}$ with signal bandwidth $\Delta f \approx$ 100Hz

$\Rightarrow$ required amplitude noise spectral density in GW strain $ASD \ll \frac{\frac{\Delta L}{L}}{\sqrt{\Delta f}} \approx 10^{-22} \frac{1}{\sqrt{Hz}}$

in terms of displacement ASD << $10^{-18} \frac{m}{\sqrt{Hz}}$

in terms of output phase ASD << $\frac{\Delta L}{\lambda} = \frac{10^{-18}\, m/\sqrt{Hz}}{10^{-6}\, m} \approx 10^{-12} \frac{1}{\sqrt{Hz}}$

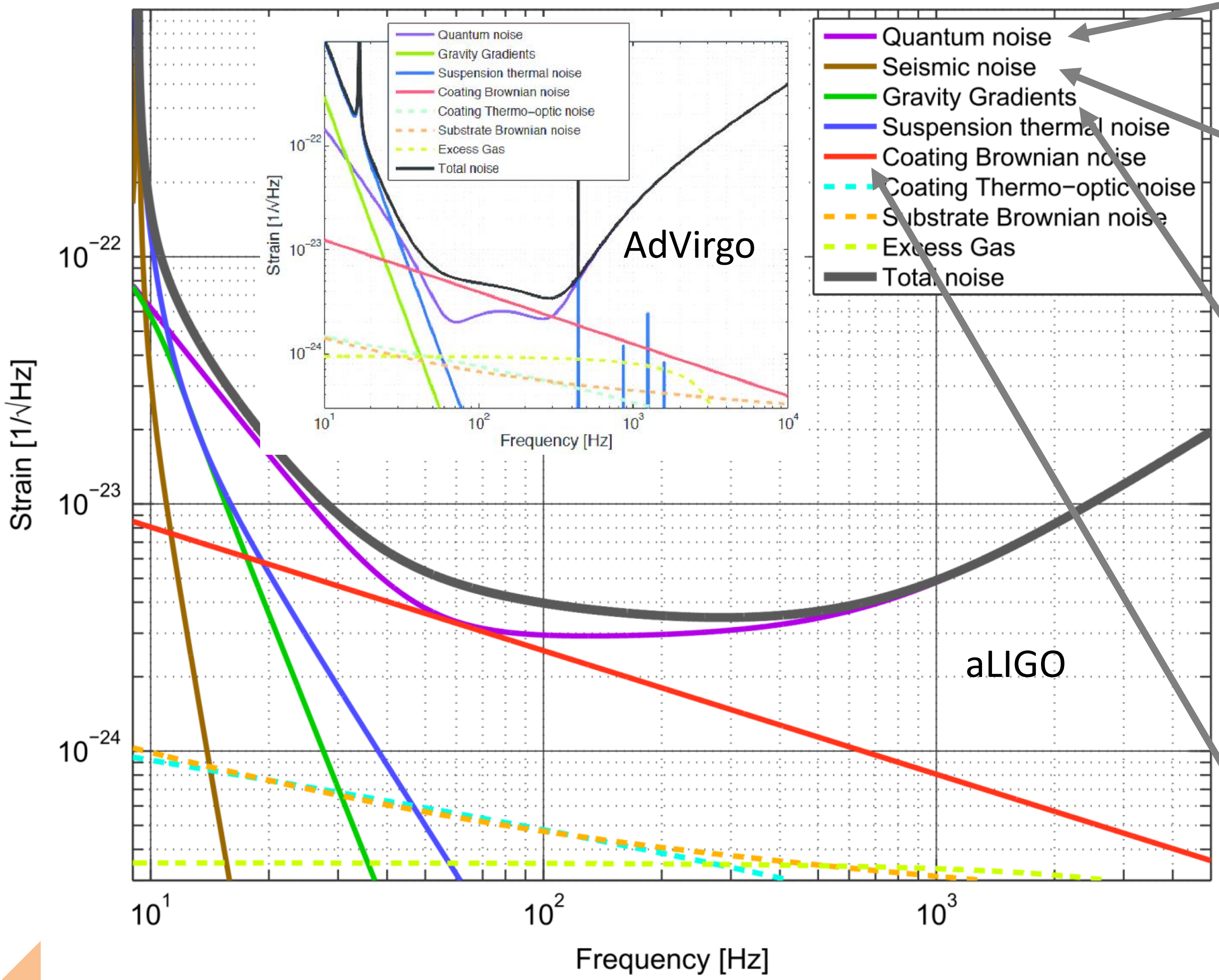
- photon counting noise (Poisson's statistics): beam splitter and e.g. input FP cavity mirrors randomize photon counts per unit time in each arm

$\Rightarrow$ fluctuation in the power and electric field in each arm:

$$10^{-12} \frac{1}{\sqrt{Hz}} > \frac{1}{\sqrt{N}} \Rightarrow N > 10^{24} \frac{1}{s}$$

- Power needed: $P > N\, \hbar\, \omega = N\, \hbar\, 2\, \pi \frac{c}{\lambda} \approx 10^{24} \frac{1}{s} \times 10^{-34} \mathrm{J} \cdot \mathrm{s} \times 2 \cdot 10^{15} \frac{1}{s} \approx 200\, kW$

noise budget of advanced detectors



Noise imposed by quantum mechanics. Dominates most of the design sensitivity. Not as fundamental as it looks like.

Ground vibration. In principle can be suppressed paying enough for suspension systems

Maybe the most "fundamental", since it acts at the input of the detector in the same way as GW (in fact, it is also a metric perturbation!)

Thermal noise, especially in mirror coating, is currently the limiting factor to the improvement of the detector's design sensitivity

Here the limit is the ability to keep the mirrors from moving

Here the limit is the ability to measure small displacements

Quantum noise

- Shot noise (counting error of photons)
  - Photon arrival events are uncorrelated: the autocorrelation function is $\delta(t)$, and the noise spectrum is flat:

$$\Delta P^2 = \frac{\Delta E^2}{T^2} = \frac{\Delta N^2 \hbar^2 \omega_{laser}^2}{T^2} = \frac{N \hbar^2 \omega_{\mathrm{l}}^2}{T^2} = \frac{P_0 \hbar \omega_l}{T} = \int_0^{\frac{1}{T}} d\omega\, S_P(\omega) = \frac{S_P(\omega)}{T} \Rightarrow S_P(\omega) = P_0 \hbar \omega_l$$

The output of the detector is proportional to P_0, so the relative noise is

Accounts for working point and photodetector efficiency. Is of order unity.

$$S_{\Delta\phi,shot}^{1/2}(\omega) = \frac{C}{P_0}\sqrt{P_0 \hbar \omega_l} = C\sqrt{\frac{\hbar \omega_l}{P_0}} \qquad \omega_l = \frac{2\pi c}{\lambda_l}$$

Referring the noise at the input we get: $S_{h,shot}^{1/2} = \frac{S_{\Delta\phi,shot}^{1/2}(\omega)}{T_{FP}} = \frac{C}{8\mathcal{F}L}\sqrt{\frac{2\pi\hbar c \lambda_l}{P_0}}\sqrt{1+\left(\frac{f_{gw}}{f_p}\right)^2}$

- Radiation pressure (uncorrelated photon impacts on mirrors)
  - Each photons that arrives at the mirror exchanges a momentum $\frac{2\hbar\omega_l}{c}$. Reasoning as above:

$$S_{F,rp}^{1/2}(\omega) = 2\sqrt{\frac{2P_0\hbar\omega_l}{c^2}} \Rightarrow S_{x,rp}^{1/2}(\omega) = \frac{2}{M\omega^2}\sqrt{\frac{2P_0\hbar\omega_l}{c^2}} \Rightarrow S_{h,rp}^{1/2} = \frac{16\sqrt{2}\mathcal{F}}{ML\omega^2}\sqrt{\frac{P_0\hbar}{2\pi c^2 \lambda_l}}\frac{1}{\sqrt{1+\left(\frac{f_{gw}}{f_p}\right)^2}}$$

Fluctuations at the beamsplitter are amplified in the cavities by a factor $\mathcal{F}$

Amplitude fluctuations are low-pass filtered by the arm cavities

Standard quantum limit

- Shot- and quantum radiation pressure-noise have opposite dependence on laser power (direct manifestation of the Heisenberg uncertainty principle):

$$S_{h,shot}^{1/2} \sim \frac{1}{\sqrt{\mathrm{P}_0}}, \qquad S_{h,rp}^{1/2} \sim \sqrt{P_0}$$

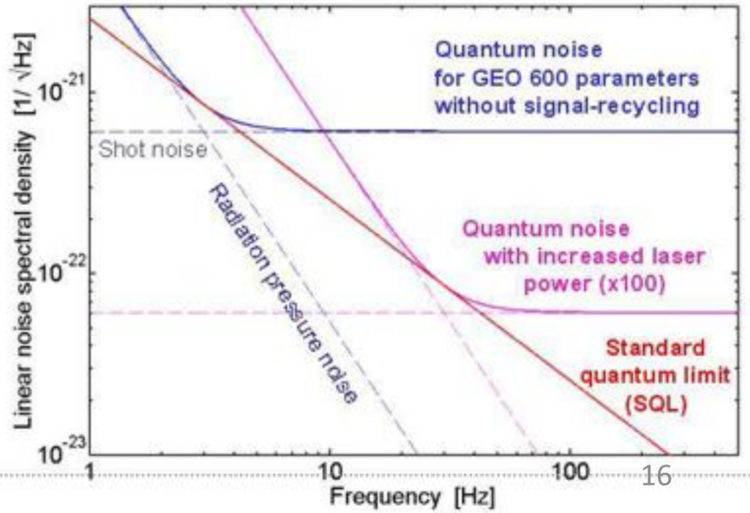
- The sum of the two is the total quantum noise

$$S_{h,qn}^{\frac{1}{2}}(f) = \frac{1}{L\pi f_0}\sqrt{\frac{\hbar}{M}\left(\left(1+\frac{f^2}{f_p^2}\right)+\frac{f_0^4}{f^4}\frac{1}{1+f^2/f_p^2}\right)}, \qquad f_0 = \frac{4\mathcal{F}}{\pi}\sqrt{\frac{P_0}{\pi\lambda_l c M}}$$

- For each value of f, this can be minimized by changing P_0 so that the two contributions are equal. The envelope of the minima is the **standard quantum limit**:

Note that it cannot be achieved on the entire band simultaneously!

$$S_{h,SQL}^{1/2}(f) = \frac{1}{2\pi f L}\sqrt{\frac{8\hbar}{M}}$$



NOTE: there exist *quantum-nondemolition techniques* to circumvent the SQL, since we are not really interested in measuring conjugate variables (i.e. position and velocity) of the mirrors at the same time with arbitrary accuracy.

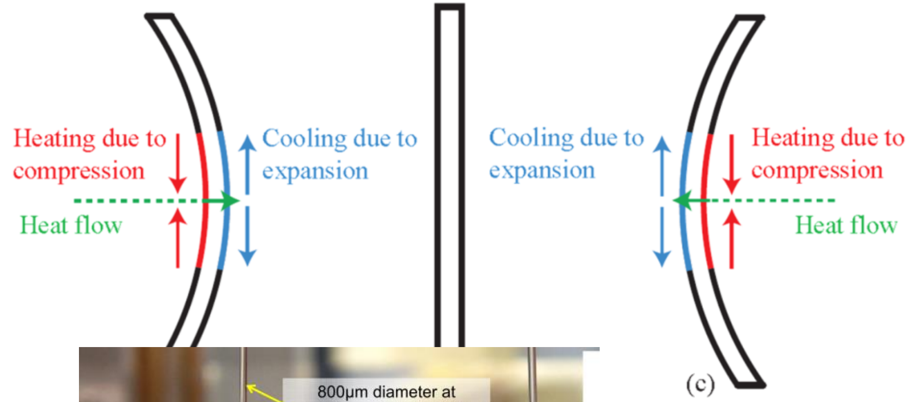
Thermal noise

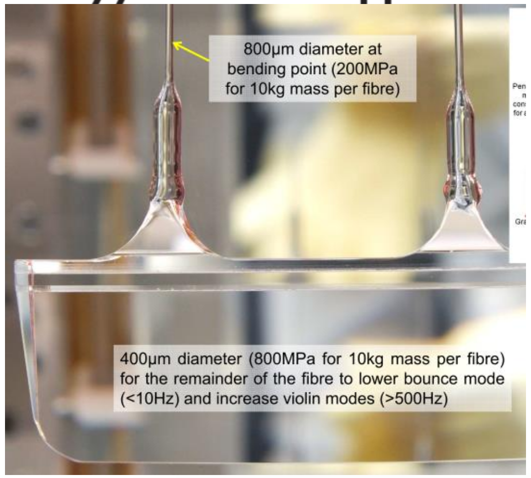
- In general contributed by any source of dissipation (not only mechanical), internal or external
- Ultimate limit is the internal dissipation:

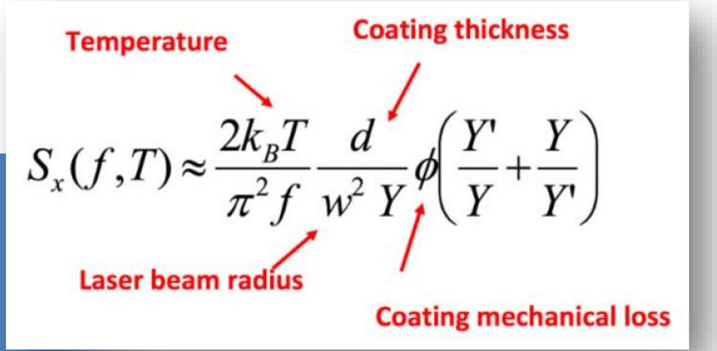
$$S_{F,th}(\omega) = 4k_bT\,\mathrm{Re}\big(Z(\omega)\big)$$

- Mirror suspensions:
  - At the level of the last mirror, it is typically higher than residual seismic
  - Typical loss sources: clamping and (relatively) high-loss materials
  - Solutions:
    - Use of low-loss materials (fused silica, $Q \sim 10^9$)
    - Monolithic mirrors and suspensions: SiO2 fibers welded to SiO2 mirrors
- Mirror coating Brownian thermal noise:
  - High index material in multilayer coating is relatively high-loss
  - Solutions under study:
    - New materials
    - Heat treatment to relax favor material relaxation
    - Innovative layer structures
    - Monolithic crystalline coatings

Thermo-elastic noise

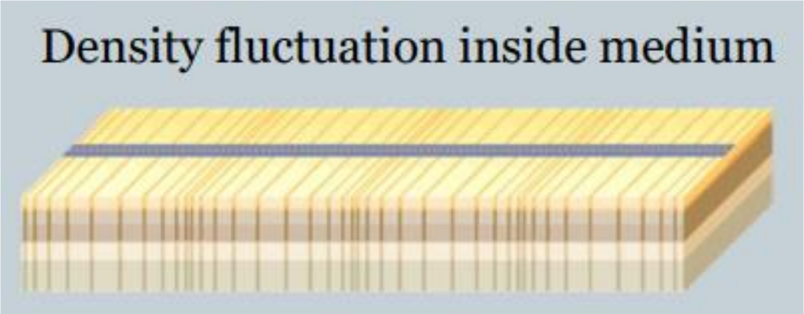


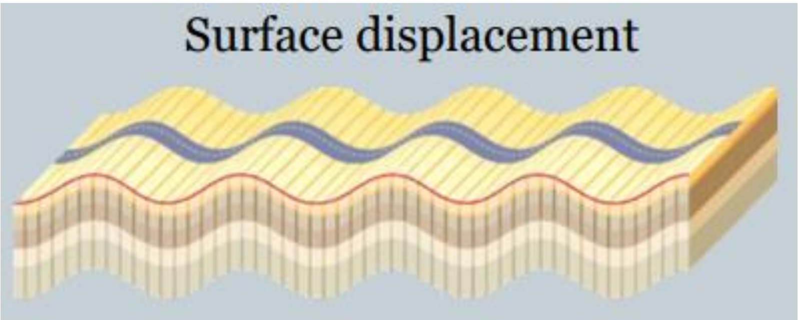




$$S_x(f,T) \approx \frac{2k_BT}{\pi^2 f}\frac{d}{w^2 Y}\phi\left(\frac{Y'}{Y}+\frac{Y}{Y'}\right)$$

Newtonian noise



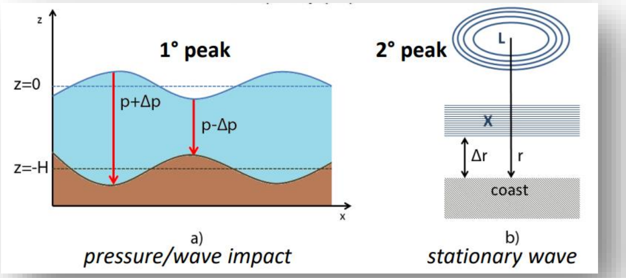


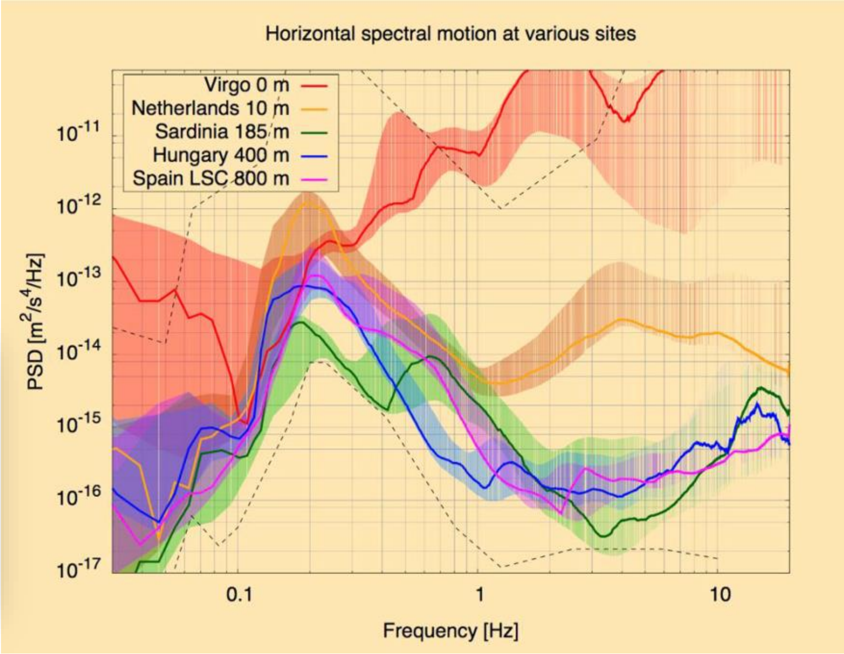
- Stochastic fluctuations of the local gravitational field
- Main sources:
  - Seismic vibrations (Lowe waves, compression waves)
  - Infrasound (variation in atmospheric pressure)
- Cannot be shielded
  - And even if it could, GW would be shielded too!
- Active strategy:
  - Deploy a network of sensor to measure ground displacement and atmospheric pressure variations
  - Model the effect of the measured disturbances on the mirrors and subtract from the GW signal
- Passive strategy:
  - Go underground, where surface waves don’t matter and atmospheric variations are reduced

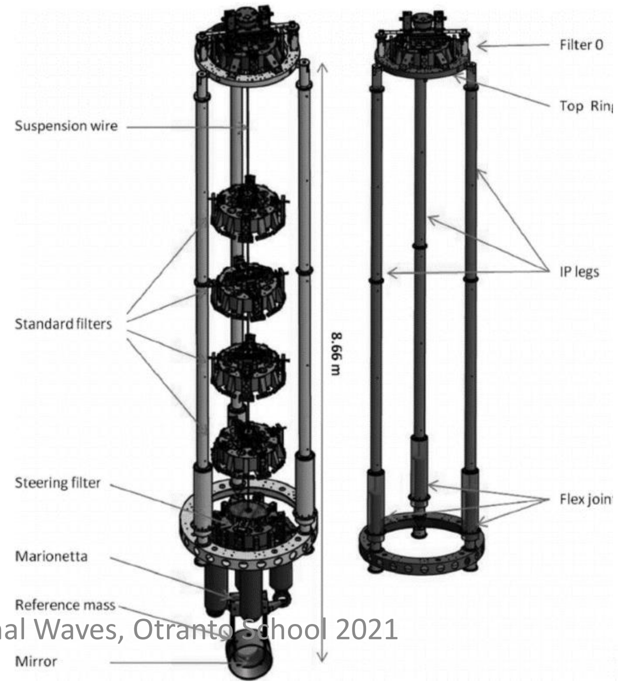
Seismic noise

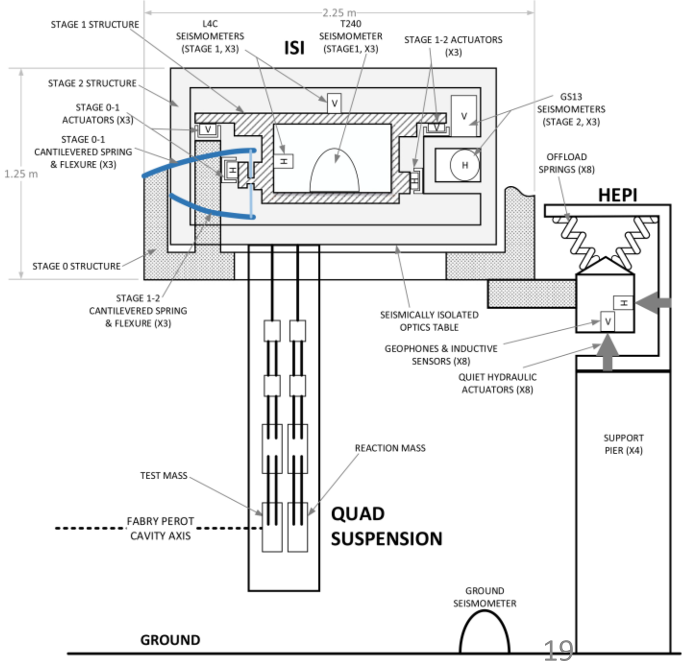
- Ground vibration is $O(10^{-6} m)$, more than 10 orders of magnitude higher than permissible for GW detection
  - Human activity (1-10 Hz)
  - "Oceanic peak" (0.1 Hz)
- Two main issues:
  - Out-of-band RMS: mirrors would be pushed out of alignment/resonance, and need to be actively controlled. Relatively low requirements, but high range.
  - In-band noise spectral density: mimics GW signal and must be suppressed. Stringent requirements.
    - Passive strategy works best, to avoid injecting readout/actuation noise
    - Cascade of pendula (and active pre-isolation stages)

$$H_{x_0 \to x} \sim \frac{1}{(1 - f^2/f_0^2)}, \quad f_0 = \sqrt{\frac{g}{l}} \sim 1\,Hz, \quad n \sim 5$$



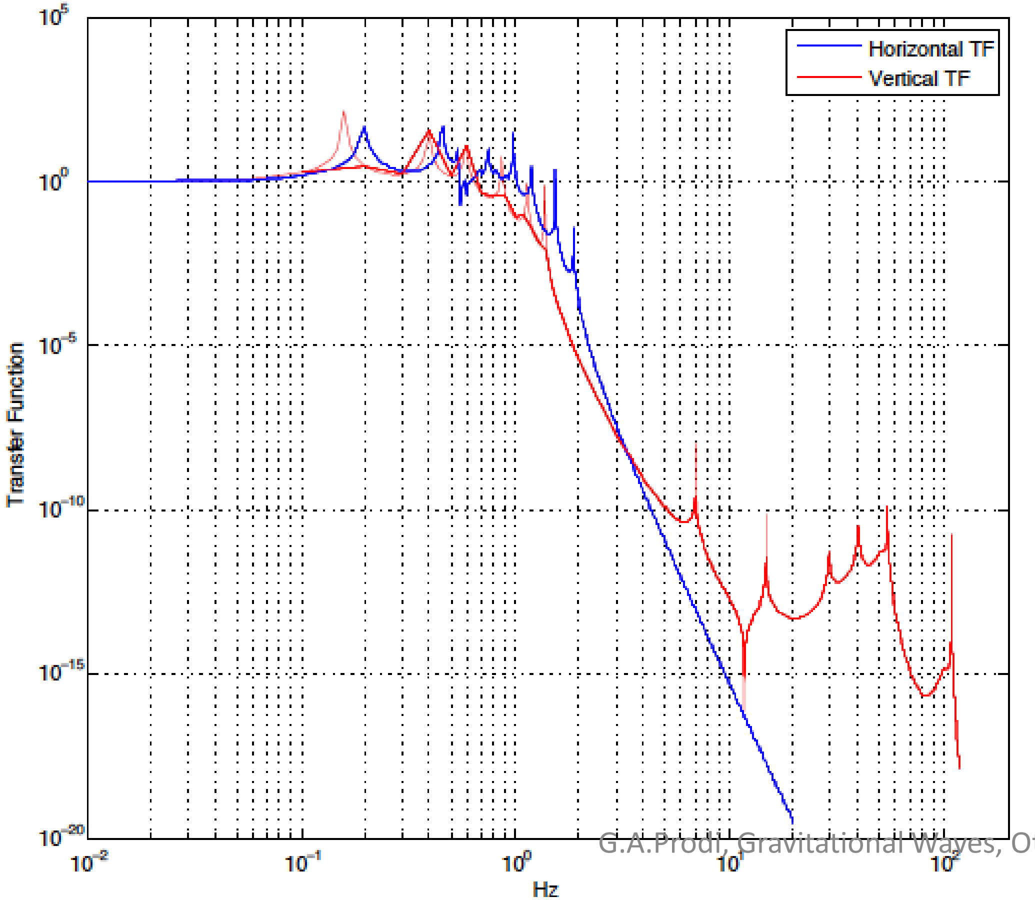


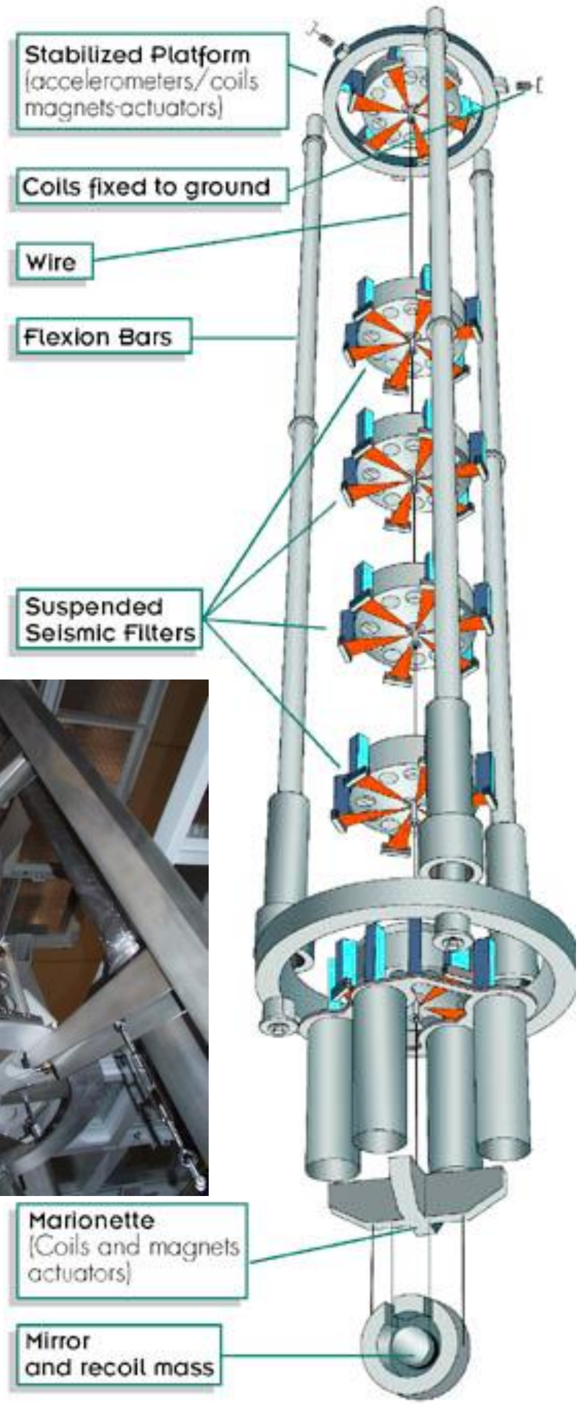




Seismic noise

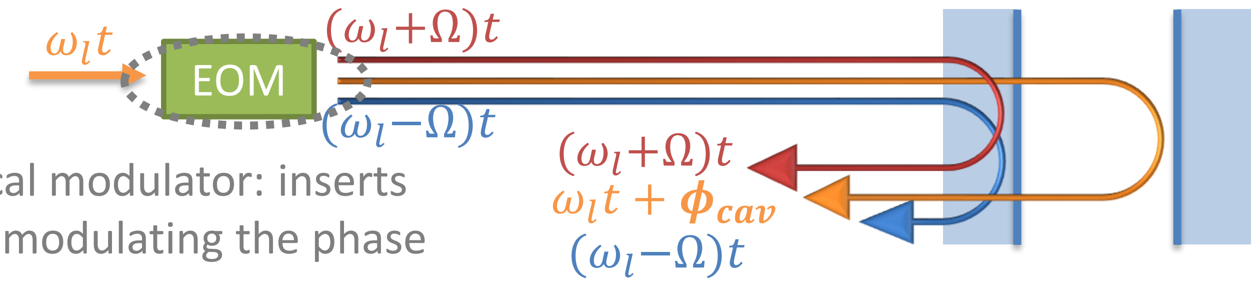
- *Typical RMS seismic noise in detection band (>10 Hz) is of the order of nm/√Hz*
  - *Need attenuation of about 10 orders of magnitude!*
- *Virgo superattenuator:*
  - *N* cascaded mechanical filters: $(f_0^2/f^2)^N$
  - 6 filters: total isolation 10^{12} @ 10 Hz



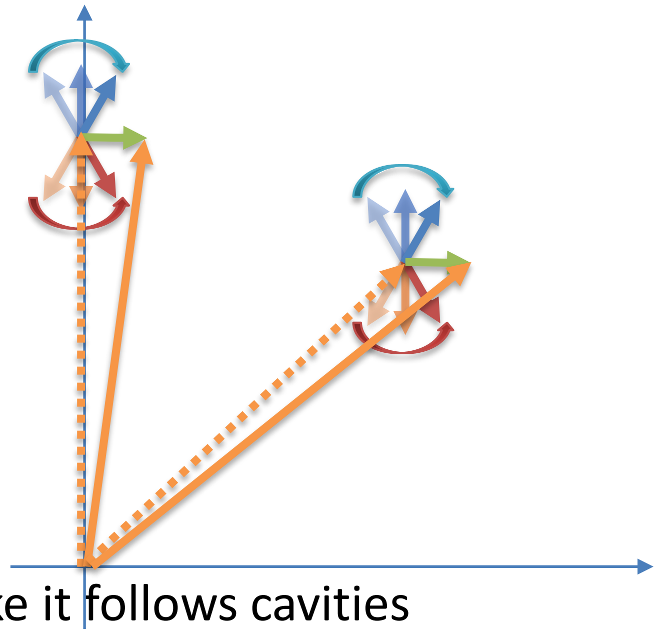


Locking and alignment

- All the sensitivity calculations we did assumed that the cavities were on (or very close to) resonance. What does it mean in terms of "stability"?
  - Arm cavities: $\mathcal{F} \sim 500,\ FSR = \frac{c}{2L} \sim 50\ kHz \Rightarrow FWHM \sim \frac{FSR}{\mathcal{F}} \sim 500\ Hz \Rightarrow \Delta L = L\frac{\Delta f}{f} = L\frac{FWHM}{c/\lambda} \sim 10^{-9}\ m$
- Why is that difficult? Ground motion, moon's tidal pull, intrinsic laser noise, etc..
- Need a technique to keep laser and cavity length "locked"
  - Pound-Drever-Hall technique: use sidebands as local oscillators to detect departure from resonance



Electro-optical modulator: inserts sidebands by modulating the phase

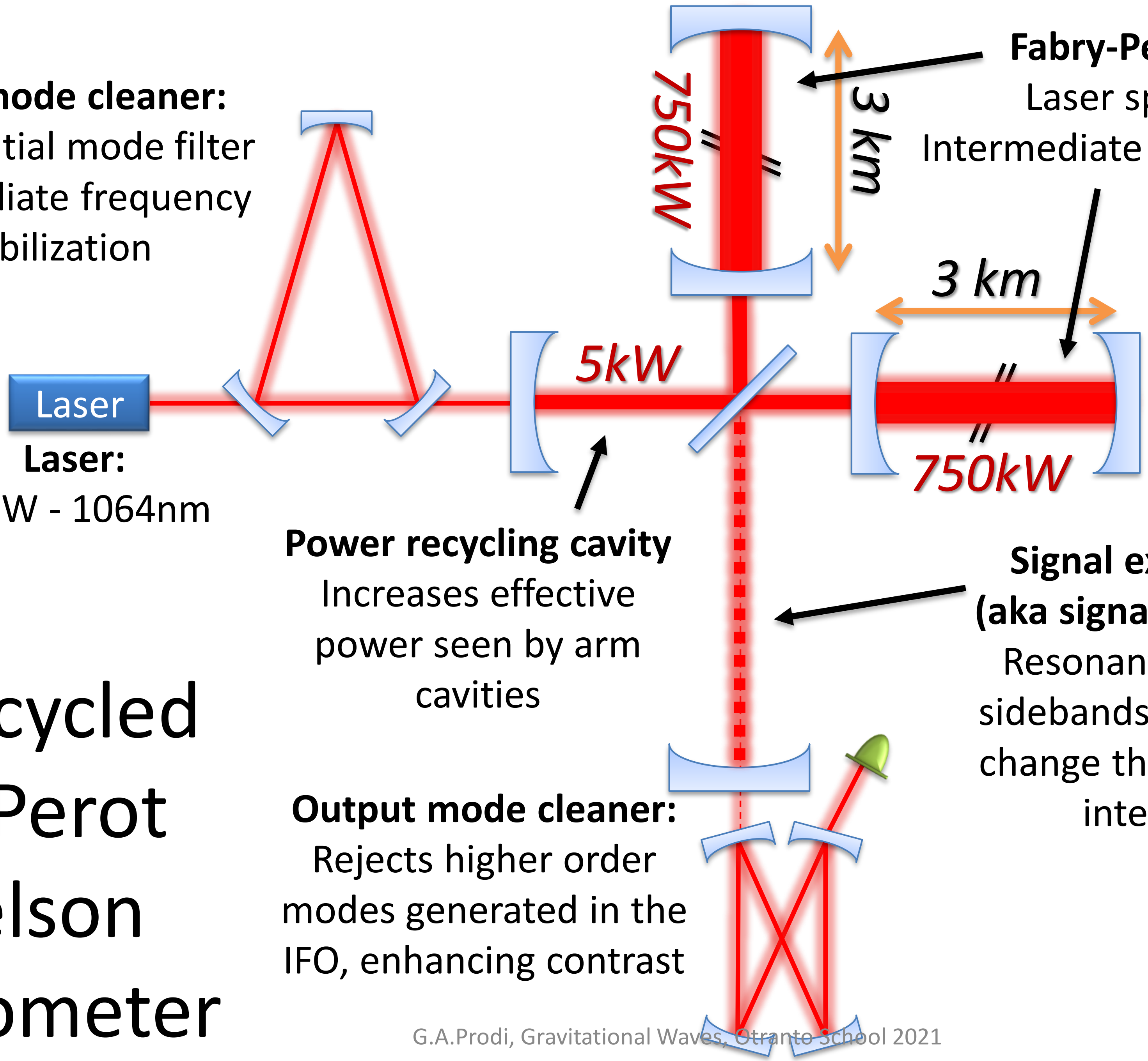


  - Arm cavities are good references at high frequency: actuate on laser to make it follows cavities
  - Independently-stabilized laser is good reference at low frequency: actuate on cavities to follow laser
- Also need sensors and actuators to keep everything aligned
  - E.g., keeping the laser beams centered at km distance requires angular control of order $\frac{1mm}{3km} \sim 0.1\ \mu rad$

Actually, to avoid extra noise the requirement is much more stringent: $\sim 1\ nrad\ RMS$!

- spare slides

Dual-recycled Fabry-Perot Michelson Interferometer



Input mode cleaner:
Laser spatial mode filter
Intermediate frequency stabilization

Laser:
100W - 1064nm

Fabry-Perot Arm cavities:
Laser spatial mode filter
Intermediate frequency stabilization

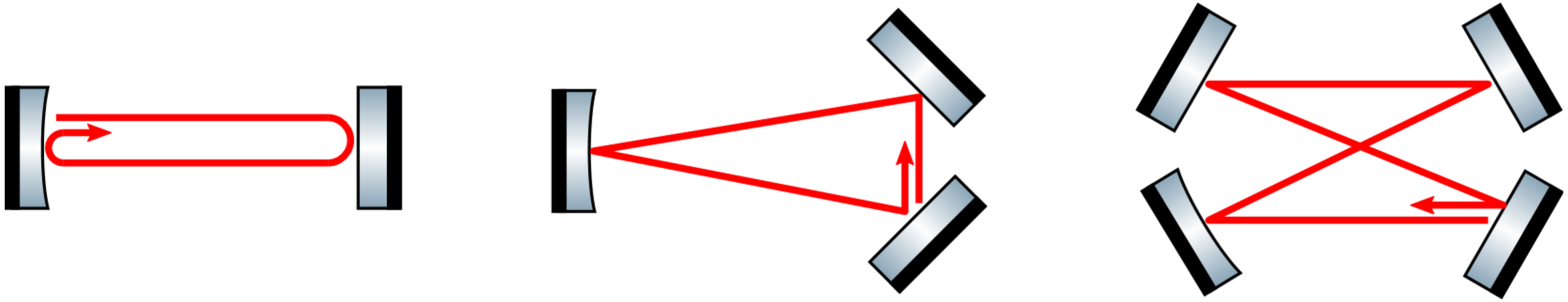
Power recycling cavity
Increases effective power seen by arm cavities

Signal extraction cavity (aka signal recycling cavity)
Resonantly enhance GW sidebands. Can be tuned to change the response of the interferometer.

Output mode cleaner:
Rejects higher order modes generated in the IFO, enhancing contrast

Optical cavities

- Optical cavity: an arrangement of mirrors that allows a closes path for the light



- Circulating field:

Dominates the frequency dependence

$$E_c = t_1 E_{in} + r_1 r_2 E_c e^{-ik2l} \Rightarrow E_c = E_{in} \frac{t_1}{1 - r_1 r_2 e^{-ik2L}}$$

Round-trip gain

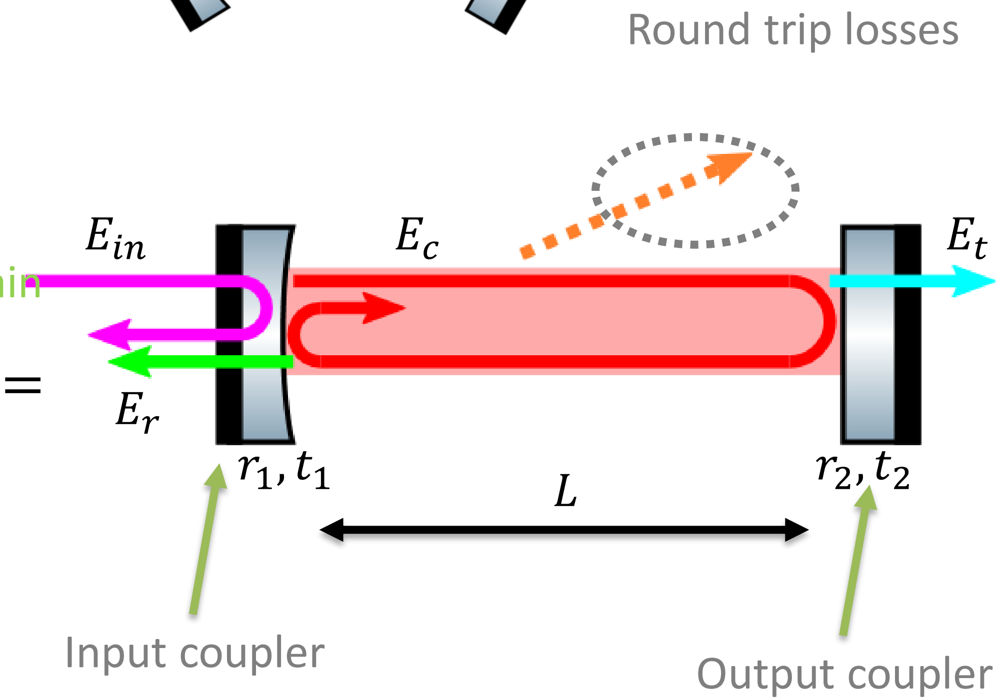
- Reflected field:

$$E_r = -r_1 E_{in} + r_2 t_1 E_c e^{-ik2l} = E_{in}\left(-r_1 + \frac{r_2 t_1^2 e^{-ik2l}}{1 - r_1 r_2 e^{-ik2L}}\right) =$$

$$= E_{in} \frac{-r_1 + r_1^2 r_2 e^{-ik2L} + r_2 t_1^2 e^{-ik2l}}{1 - r_1 r_2 e^{-ik2L}} = -E_{in} \frac{r_1 - r_2 e^{-ik2L}}{1 - r_1 r_2 e^{-ik2L}}$$

- Transmitted field:

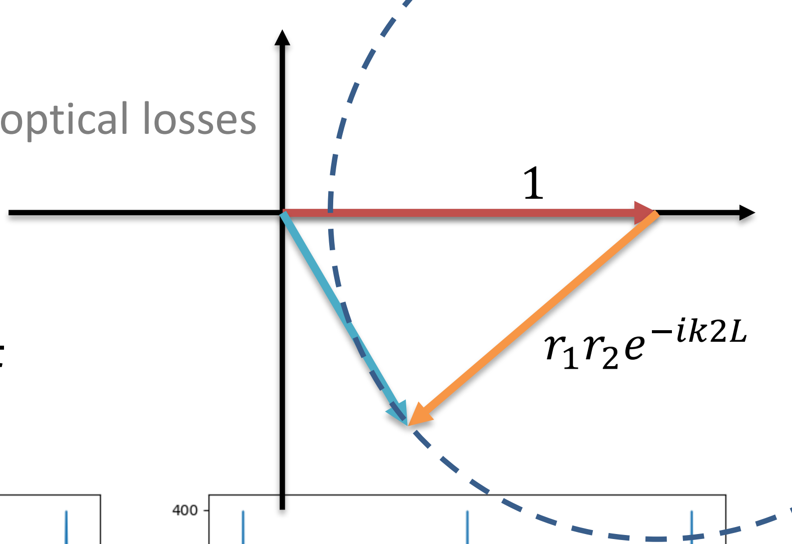
$$E_t = t_2 E_c = E_{in} \frac{t_1 t_2}{1 - r_1 r_2 e^{-ik2L}}$$



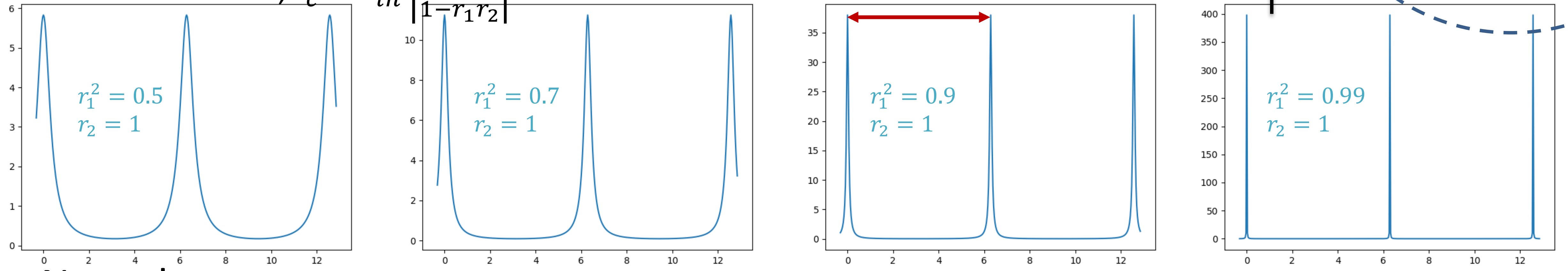
Optical resonance

In general this includes al the optical losses

- Circulating intensity: $I_c = |E_c|^2 = E_{in}^2 \left| \frac{t_1}{1 - r_1 r_2 e^{-ik2L}} \right|^2$
  - This is maximum for minimum $\left|1 - r_1 r_2 e^{-ik2L}\right| \Rightarrow L = n\frac{\pi}{k} = n\frac{\lambda}{2} = n\frac{c}{2\,f}$
  - At **resonance**, $I_c = I_{in} \left| \frac{t_1}{1 - r_1 r_2} \right|^2$



$FSR = \frac{c}{2L}$



- Note that:
  - frequency gap between the peaks (**free spectral range**) only depends on cavity length
  - Width of peaks decreases with increased reflectivity
    - **Finesse**: $\mathcal{F} \equiv \frac{\pi\sqrt{r_1 r_2}}{1 - r_1 r_2} \approx \frac{FSR}{FWHM} \approx \frac{2\pi}{losses}$

      Valid for small cavity losses

      Full-width at half maximum
    - The finesse determines how "selective" the cavity is

Storage time: $\tau_s \approx \frac{L\,\mathcal{F}}{c\,\pi}$

Example: $r_1^2 = 0.99, r_2 = 1, L = 3\ km \Rightarrow$
$\mathcal{F} \approx 625, \qquad \tau_s \approx 2\ ms$

Fabry-Perot cavities to increase the effective length

- We found that to maximize IFO response to GW: $L_{arm} = \frac{\lambda_{gw}}{4} \approx 750\ km\ \left(\frac{100\ Hz}{f_{gw}}\right)$
- Modern GW interferometer have $L_{arm} \approx 3\ km$
- We need to increase the effective length by a factor ~ 100.

- Delay lines: make the laser beam bounce back and forth between two mirrors, hitting each time a different part of it.
  - Very sensitive to back-scattered light
  - Need very big mirrors to avoid beams overlap
  - Need a special (and challenging) mirror machining/coating to allow the beam to enter/escape the line

- Use optical cavities: each photon bounces back and forth between two mirrors (back on its own path), effectively traveling a much longer distance before going back to the beamsplitter
  - Length increased by the average number of bounces before escaping, $N \sim \frac{\mathcal{F}}{2\pi}$

$L_{eff} = 2L\frac{\mathcal{F}}{2\pi} \sim 300\ km$ for $\mathcal{F} \sim 500$, *L=3km*

FP cavities to increase the interaction time

- Remember than in the TT gauge we saw that the light picks-up a propagation delay due to the fact that it travels in the spacetime curved by GW

- Different but equivalent perspective: by making the photon bounce back and forth, the interaction time with the GW is increased from $\tau_s \approx \frac{2L}{c}$ to $\tau_s \approx \frac{L}{c}\frac{\mathcal{F}}{\pi}$.

- This brings about the idea that a cavity is a low pass-filter: everything happening on a timescale much faster than $\tau_s \approx \frac{L}{c}\frac{\mathcal{F}}{\pi}$ is averaged out.

- An intuitive view: make τ_s as long as possible to increase the storage time, but smaller than the oscillation period of your GW: $\tau_s \approx \frac{L}{c}\frac{\mathcal{F}}{\pi} \sim \frac{1}{2\, f_{gw}} \Rightarrow \mathcal{F} \sim \frac{\pi\, c}{2\, L\, f_{gw}} \sim 1000 \left(\frac{100\ Hz}{f_{gw}}\right)$

Response of Fabry-Perot arm cavities

- A more accurate view: FP reflected phase sensitivity to cavity length displacement

$$\mathcal{F} = \frac{\pi\sqrt{r_1 r_2}}{1 - r_1 r_2} \Rightarrow r_1 r_2 \approx 1 - \frac{\pi}{\mathcal{F}} + O\left(\frac{\pi^2}{\mathcal{F}^2}\right)$$

- Let's define the total power losses p: $(1-p_1)r_2^2 = (1-p)$, $\quad 1 - p_i = r_i^2 + t_i^2$
- Then:

$$r_1^2 = 1 - p_1 - t_1^2 < 1 - p_1 \Rightarrow r_1^2 r_2^2 < 1 - p \;\; \Rightarrow \;\; 1 - \frac{\pi}{\mathcal{F}} = r_1 r_2 < 1 - \frac{p}{2}$$

- For the **coupling factor** $\sigma \equiv \frac{p\mathcal{F}}{\pi}$ it holds $0 < \sigma < 2$

The cavity is: **undercoupled** for cavity for $1 < \sigma < 2$
overcoupled $0 < \sigma < 1$
optimally coupled (impedance matched) for $\sigma = 1$

- The phase of the reflected field $-E_{in}\frac{r_1 - r_2 e^{-ik2L}}{1 - r_1 r_2 e^{-ik2L}}$ for a displacement ϵ from resonance can be written as:

$$\phi = \pi + \arctan\left(\frac{\mathcal{F}\epsilon}{\pi}\frac{1}{1-\sigma}\right) + \arctan\left(\frac{\mathcal{F}\epsilon}{\pi}\right)$$

If $\sigma > 1$, the two contributions have opposite sign and partially cancel each other, which we don't want

- For $r_2 = 1, r_1 \sim 1, p_1 = 0$ we obtain:

$$p = 1 - r_2^2 = 0 \Rightarrow \sigma = 0 \Rightarrow \frac{\partial\phi}{\partial\epsilon} \approx \frac{2\mathcal{F}}{\pi}$$

Compare with $\frac{\partial\phi}{\partial\epsilon} = 1$ for a simple Michelson

Response of Fabry-Perot arm cavities

- For an Michelson with FP cavities, invested by a GW (plus-polarized, propagating perpendicular to the plane of the IFO), in the detector frame:

The effect in the two arms has opposite sign

Laser wave-vector

Enhancement factor due to FB cavities

$$\Delta\phi_{FP} = \Delta\phi_x - \Delta\phi_y = 2\,2k_l\,\frac{1}{2}Lh_0\cos(\omega_{gw}t)\,\frac{2\mathcal{F}}{\pi} \Rightarrow |\Delta\phi_{FP}| = \frac{4\mathcal{F}}{\pi}k_l L h_0$$

Light travels back and forth

Armlength distortion due to GW

- Let's move to the TT gauge, and use the sidebands picture.
  - The field is composed by 3 contributions (carrier at ω_l and sidebands at $\omega_l \pm \omega_{gw}$)
  - In the presence of GW, at each round-trip some of the carrier power is moved into the sidebands
  - Effects of the GW on the sidebands can be neglected, as it is second order in h_0

- One eventually finds:

$$|\Delta\phi_x| = h_0 k_l L\,\mathrm{sinc}\left(\frac{\omega_{gw}L}{c}\right)\frac{r_2(1-r_1^2-p)}{r_2(1-p)-r_1}\frac{1}{\left|e^{2i\omega_{gw}\frac{L}{c}} - r_1 r_2\right|}$$

Cavity pole

$$|\Delta\phi_x| = h_0 k_l L \,\mathrm{sinc}\left(\frac{\omega_{gw} L}{c}\right) \frac{r_2\left(1-r_1^2-p\right)}{r_2(1-p)-r_1} \frac{1}{\sqrt{1+(r_1 r_2)^2 - 2 r_1 r_2 \cos\left(\frac{2\omega_{gw} L}{c}\right)}}$$

- One can make a few simplifications:
  - Since GW IFO work at $p\sim 0, r_2 \sim 0$, we can write: $2 \approx \frac{r_2\left(1-r_1^2-p\right)}{r_2(1-p)-r_1} = 2\left(1+\epsilon(r_1,r_2,p)\right)$
  - GW IFO use large $\mathcal{F}$ and operate at $\frac{\mathcal{F}L}{c} \sim \frac{1}{\omega_{gw}} \Rightarrow \frac{\omega_{gw} L}{c} \ll 1 \Rightarrow \mathrm{sinc}\left(\frac{\omega_{gw} L}{c}\right) \sim 1, \cos\left(\frac{2\omega_{gw} L}{c}\right) \approx 1 - \frac{1}{2}\left(\frac{2\omega_{gw} L}{c}\right)^2$

$$|\Delta\phi_x| \approx 2 h_0 k_l L \frac{1+\epsilon(r_1,r_2,p)}{1-r_1 r_2} \frac{1}{\sqrt{1+\frac{r_1 r_2}{(1-r_1 r_2)^2} - \left(\frac{2\omega_{gw} L}{c}\right)^2}} \approx 2 h_0 k_l L \frac{\mathcal{F}}{\pi} \frac{1}{\sqrt{\left(1+4\pi f_{gw} \tau_s\right)^2}}$$

Includes a factor 2 for the effects of the two arms

- In terms of the **cavity pole** $f_p \equiv \frac{1}{4\pi\tau_s} \approx \frac{c}{4\mathcal{F}L}$: $|\Delta\phi_{FP}| \approx h_0 \frac{4\mathcal{F}}{\pi} k_l L \frac{1}{\sqrt{1+\left(\frac{f_{gw}}{f_p}\right)^2}}$

As for a simple Michelson, the TT gauge includes the averaging effect for $f_{gw} > 1/\tau_2$ and reproduce the detector frame result otherwise

- This is the transfer function of the interferometer: $T_{FP}\left(f_{gw}\right) = \frac{8\mathcal{F}L}{\lambda_l} \frac{1}{\sqrt{1+\left(\frac{f_{gw}}{f_p}\right)^2}}$

Transfer function of a GW IFO with FP cavities