The Glueball Spectrum in the Large–N Limit

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[work in collaboration with B. Lucini and A. Rago]

Why Large N?

Consider QCD$_N$ with arbitrarily large number of colours N.

- colour–singlet states do not mix and have $\Gamma = 0$
- sensible expansion parameter $\frac{1}{N}$ ['t Hooft 1974]
- AdS/CFT correspondence [Maldacena 1998]

Open questions:

1. the analytical approaches based on large–N ideas can give us helpful insights into the non–perturbative regime of QCD:
   we need to compare these predictions to observables derived from first principles
2. in order for these approaches to describe correctly the real world, physical observables in QCD should be close to their SU(∞) values

Why the Glueball Spectrum?

Study the glueball spectrum in SU(N) pure gauge theory with $N \in [3, 8]$

- genuine non–perturbative phenomenon
- one of the easiest observables to compare with analytical, AdS/CFT inspired, predictions
- neglect the fermionic contribution because the SU(∞) theory is dynamically quenched

$$am(N) \;=\; am(\infty) + \frac{c}{N^2}$$

is a reliable ansatz down to $N = 3$ for the 0^{++} and 2^{++} glueball states.

[Lucini, Teper, Wenger 2004]

Aim of this work:

- compute, for the first time, the leading term and the corrections for the low–lying spectrum in all the symmetry channels J^{PC}
- identify spurious contributions such as scattering states or torelon excitations that can affect the measured spectrum on the Lattice

Lattice Spectroscopy

- Symmetry Channels

  at finite Lattice spacing, Lattice energy eigenstates belong to the 5 irreps. R of the cubic symmetry group.
  Adding parity and charge,
  we get 20 Lattice symmetry channels R^{PC}

J	A_1	A_2	E	T_1	T_2
0	1	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	1
3	0	1	0	1	1
4	1	0	1	1	1

- Euclidean Correlators

  use gauge–invariant, zero–momentum operators $\mathcal{O}(t)$ with the same symmetries of the Lattice states

$$C(an_t) \equiv \left\langle \mathcal{O}^\dagger(t = an_t)\mathcal{O}(0) \right\rangle = \sum_n |\langle n| \mathcal{O} |0\rangle|^2 e^{-am_n n_t}$$

- Variational Ansatz

  find the best linear combination of operators within a variational set and extract the mass from fitting its correlator

$$\hat{\Phi}(t) = \sum_\alpha v_\alpha \mathcal{O}_\alpha(t) \; ; \qquad \left\langle \hat{\Phi}^\dagger(t)\hat{\Phi}(0) \right\rangle = |c_0|^2 \cosh\left(m_0 t - N_t/2\right)$$

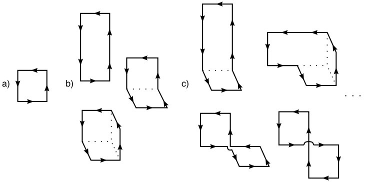
Constructed Operators

General gauge–invariant, vacuum–subtracted operator, projected into a single irrep. R:

$$\bar{\mathcal{O}}(t) = \mathcal{O}(t) - \langle 0| \mathcal{O}(t) |0\rangle \quad \rightarrow \quad \Phi^{(R)}(t) = \sum_i c_i^{(R)} \mathcal{R}_i(\bar{\mathcal{O}}(t))$$

- single–glueball operators:

$$\mathcal{O}_G(t) = \frac{1}{N_L^3} \sum_{\vec{x}} \mathrm{Tr} \prod_{l \in \mathcal{C}(\vec{x})} U_l$$



- multi–glueball operators:

$$\mathcal{O}_S(t) = (\mathcal{O}_G(t) - \langle \mathcal{O}_G \rangle)^2$$

- torelon operators:

$$\mathcal{O}_T(t) = \frac{1}{2N_L^2} \sum_{\mu \neq i} \sum_x L_p^{(i)}(x,t) L_p^{\dagger(i)}(x+\mu a, t)$$

Variational Set

Number of operators Φ_i included in the variational basis for each symmetry channel

	type G				type S				type T			
	$++$	$-+$	$+-$	$--$	$++$	$-+$	$+-$	$--$	$++$	$-+$	$+-$	$--$
A_1	32	8	4	12	32	8	4	12	8	4	–	–
A_2	12	4	12	12	12	4	12	12	2	–	4	4
E	88	28	28	56	88	28	28	56	28	12	12	12
T_1	76	96	192	108	76	96	192	108	12	12	56	36
T_2	176	132	132	156	176	132	132	156	36	36	32	12

Define the relative projection (mix_A) of a variational state, onto basic operators of the 3 distinct subsets G, S, T

$$\hat{\Phi} = \sum_i v_i \Phi_i(t) \equiv \alpha_G \Phi_G + \alpha_S \Phi_S + \alpha_T \Phi_T \; ; \qquad \text{mix}_A = \frac{|\alpha_A|^2}{\sum_i |\alpha_i|^2} \; ; \quad A \in \{G, S, T\}$$

Masses extracted from correlators of $\hat{\Phi}$ with $\text{mix}_S, \text{mix}_T \geq 20\%$ can not be reliably interpreted as pure single–glueball resonances.

Setup of Lattice Simulations

$$\mathcal{S}_{LAT} = \beta \sum_x \left[1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x)\right]; \qquad \beta = \frac{2N}{g_0^2}$$

The β coupling is the only free parameter and it is used to set the Lattice spacing a.

To fix the same physical scale for all the gauge groups SU(N) choose the value

$$\beta = \beta_c(N_t = 6)$$

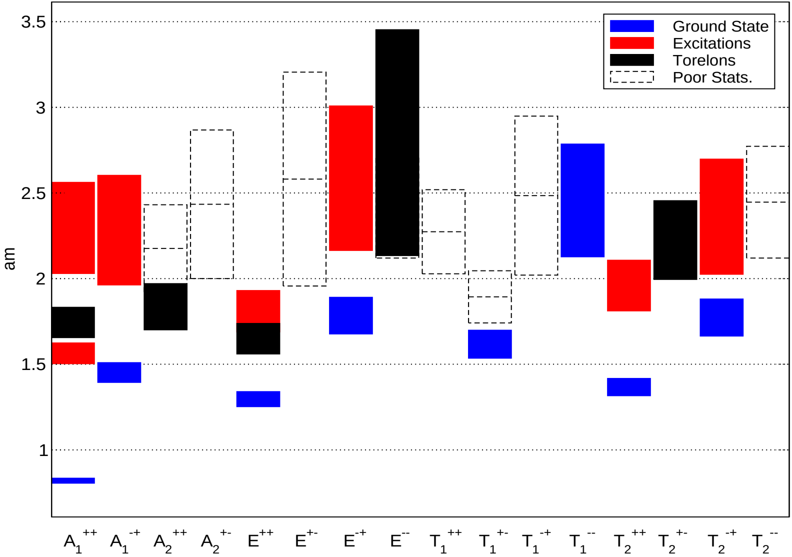
corresponding to the deconfining transition on lattices with $N_t = 6$.

This Lattice spacing ($a \sim 0.122$ fm) is in the scaling region. [Gupta, Datta 2009]

N	$\beta_c(N_t = 6)$	β	L	$N_{measure}$	$N_{compound}$
3	5.8941(12)	5.8945	12	10000	200
4	10.7893(23)	10.789	12	10000	200
5	17.1068(30)	17.107	12	10000	200
6	24.8458(33)	24.845	12	10000	200
7	33.9995(37)	33.995	12	10400	250
8	44.4960(30)	44.496	12	8000	250

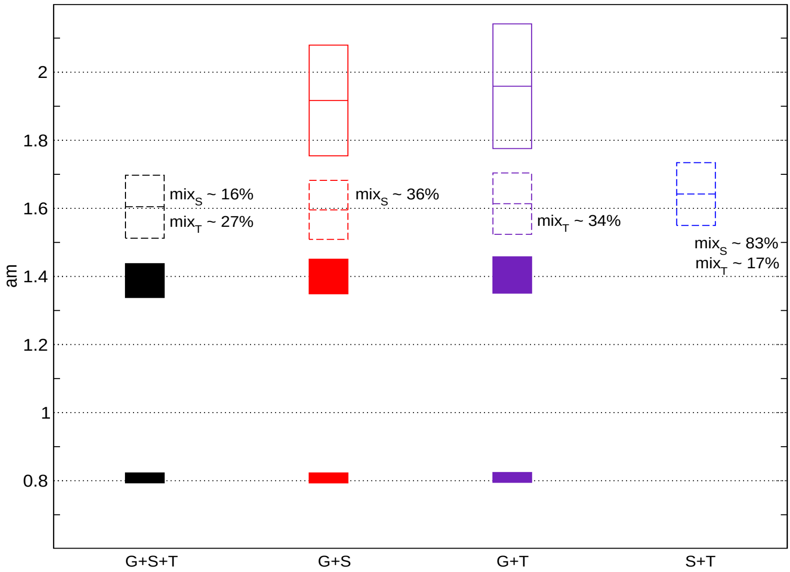
A Typical Glueball Spectrum

Spectrum of SU(7) using only type G and type T operators.



Inclusion of Scattering Operators

Masses of A_1^{++} states in SU(4) obtained using different variational sets

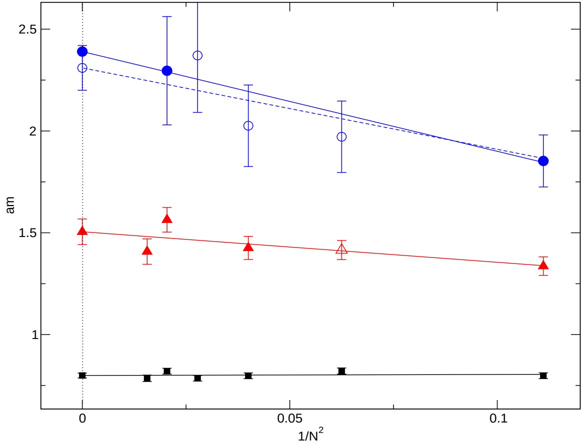


Extrapolation at $N = \infty$

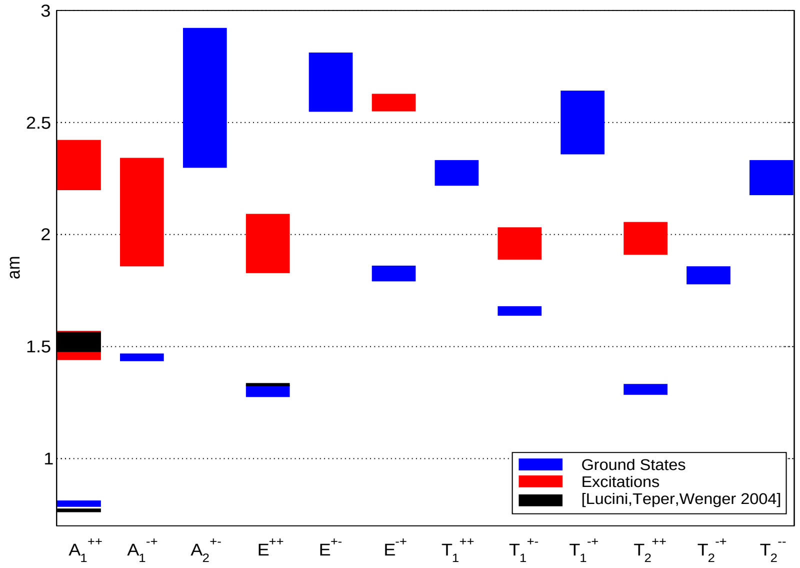
We fit our 6 measured spectra with the ansatz

$$am(N) \; = \; am(\infty) + \frac{c}{N^2}$$

Ground state and excitations in the A_1^{++} channel



The SU(∞) Glueball Spectrum



Conclusions

Summary:

- we studied the large–N spectrum of the four dimensional pure Yang–Mills theory in all the symmetry channels
- we are able to identify scattering and torelon contributions together with some excited glueball states, using a large variational basis of operators
- we confirm the result that SU(3) $\approx$ SU(∞)

Some future developments:

- finite volume investigation of scattering and torelons states to check our preliminary results
- extrapolation of the large–N spectrum in the continuum limit, possibly using anisotropic lattices, to make contact with AdS/CFT predictions
- large–N glueball spectrum at finite temperature

The Fitted SU(∞) Glueball Spectrum

R^{PC}	$am(\sigma)$	c	range N	$\tilde{\chi}^2$
A_1^{++}	0.799(12)	0.05(0.22)	(3,4,5,6,7,8)	1.43
$A_1^{++\star}$	1.505(63)	-1.5(0.9)	(3,5,7,8)	1.84
$A_1^{++\star\star}$	2.39	-4.9	(3,7)	–
A_1^{-+}	1.452(15)	-0.3(0.2)	(3,4,7)	0.05
$A_1^{-+\star}$	2.10(24)	6(6)	(4,5,7)	0.18
A_2^{+-}	2.61(31)	-9(6)	(4,5,6)	0.29
E^{++}	1.302(25)	-0.01(0.41)	(3,4,5,6,7,8)	0.64
$E^{++\star}$	1.96(13)	-3(3)	(4,5,6,7)	0.68
E^{-+}	1.826(33)	-1.6(0.5)	(3,4,5,6,7,8)	0.18
$E^{-+\star}$	2.589(37)	-1.5(0.6)	(3,5,6,7)	0.01
E^{+-}	2.683	-1.0	(3,8)	–
T_1^{++}	2.796	-17.7	(5,6)	–
T_1^{+-}	1.659(19)	-0.4(0.3)	(3,4,5,6,7,8)	0.11
$T_1^{+-\star}$	1.96(7)	0.4(2.0)	(5,6,8)	0.06
T_1^{-+}	2.50(14)	-3(3)	(4,5,6,8)	0.06
T_2^{++}	1.354(42)	-0.5(0.7)	(3,4,5,6,7,8)	1.58
$T_2^{++\star}$	1.983(71)	-2(1)	(3,4,5,6,7)	0.38
T_2^{-+}	1.818(38)	-1.4(0.6)	(3,5,6,7,8)	0.23
T_2^{--}	2.254(76)	0.1(1.0)	(3,6,8)	0.07

Some References

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- Glueball Regge trajectories, *Meyer, H. B.*, hep-lat/0508002
- The glueball spectrum from an anisotropic lattice study, *Morningstar, C. J. and Peardon, M. J.*, hep-lat/9901004

Variational Calculation

1. start with a set of basic operators Φ_α, all with the same quantum numbers R^{PC}
2. measure their correlators on a set of gauge configurations

$$\tilde{C}_{\alpha\beta}(t) \;=\; \sum_{\tau} \langle 0|\, \Phi^{\dagger}_{\alpha}(t+\tau)\Phi_{\beta}(\tau)\, |0\rangle$$

3. construct a linear combination of the basic operators

$$\hat{\Phi}(t) \;=\; \sum_{\alpha} v_\alpha \Phi_\alpha(t)$$

   that minimises the mass extracted from the decay of $C_{\hat{\Phi}\hat{\Phi}}(t)$

4. The minimisation problem is turned to a generalised eigenvalue problem
   where v_α are the components of eigenvectors $\vec{v}$ of the equation

$$\tilde{C}(\bar{t})\,\vec{v} \;=\; e^{-m\,\bar{t}}\tilde{C}(0)\,\vec{v}$$

   (with fixed $\bar{t} = 1$)

This is a variational procedure and the results strongly depend on the operators included in the initial basis.

1/N Expansion

Besides the usual perturbative expansion in g_0, it is possible to expand the theory in $1/N$ at fixed $\lambda = g_0^2 N$.

Counting rules for diagrams:

- vertices $\propto N$
- propagators $\propto 1/N$
- index loops $\propto N$

Genus expansion for vacuum connected diagrams:

diagram $\propto N^{2-2h-b}$, h: handles, b: boundaries; N^2 is the leading order at large N

Boundaries are possible only if fermions in the fundamental representation are included (QCD).

AdS/CFT correspondence

Maldacena conjecture:

a super–string theory on $AdS_{d+1} \times \mathbf{X}$ is equivalent to the large–N limit of the strong–coupling regime of super–symmetric Yang–Mills theories.

By suitably choosing the internal manifold $\mathbf{X}$ different internal symmetries can be reproduced on the field theory.

- well–established correspondence in some limits (the field theory side is always a conformal field theory on the boundary of the AdS manifold)
- super–symmetry and conformality can be broken by the introduction of one (or more) compactified dimension and using suitable boundary conditions (Witten)
- lattice results for $2+1$ dimensional pure gauge theory in the large–N limit are in qualitative agreement with string perturbative analytical calculations

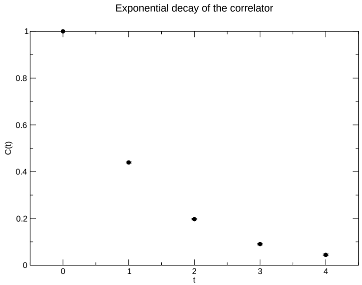
We are confident that our results in $3+1$ dimensions, can be compared to string calculations given an evidence of the validity of the AdS/CFT correspondence.
The hope is to describe QCD ($N = 3$!) by this correspondence.

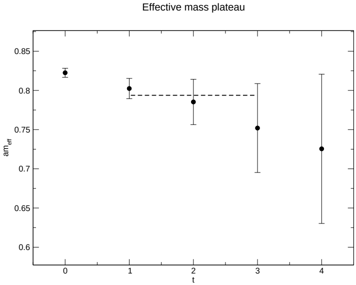
Effective Mass Plateau

Effective mass:

$$m_{eff}(t) = -\ln \frac{\left\langle \hat{\Phi}^{\dagger}(t)\hat{\Phi}(0) \right\rangle}{\left\langle \hat{\Phi}^{\dagger}(0)\hat{\Phi}(0) \right\rangle}$$

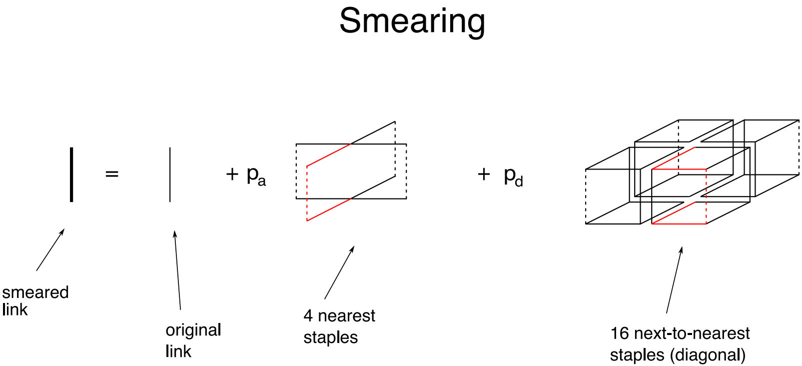
Search for an effective mass plateau where the correlator is dominated by a single exponential decay.





Smearing and Blocking

Construct the operator $\hat{\Phi}$ using modified links so that it is smooth on different length scales.



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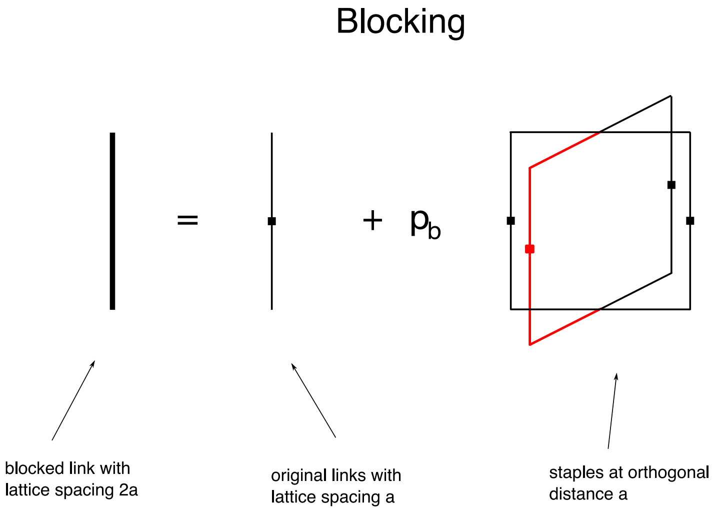
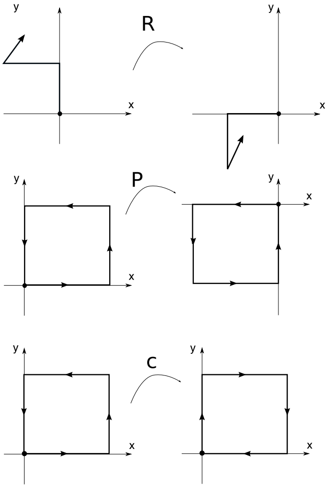


Transformations of an Operator

To understand the symmetry properties of operators, we regard them as paths, oriented collections of links.



- R: a rotation of the cubic group transforms a path rotating each link
- P: the parity transformation acts as a reflection about the origin
- C: charge conjugation reverts the direction of each link

We can construct operators in the 20 irreducible representations of the group O_h^C, the cubic group combined with reflections and charge conjugation, by finding appropriate linear combinations of paths.

Projection Table

A linear combination of elementary closed paths $\mathcal{P}$, which transforms in one of the 5 irreducible representations (IR) of the cubic group, is obtained using the projection table $Pr(IR, R)$

$$\mathcal{P}_{IR} = \sum_{R} Pr(IR, R) R(\mathcal{P})$$

1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
0	0	1	0	0	0	−1	0	1	0	0	0	−1	0	0	0	−1	1	−1	0	0	0	0	1
−1	1	0	1	0	0	−1	0	1	0	0	0	−1	0	0	0	0	0	−1	0	0	0	1	0
−1	0	−1	1	0	1	0	0	−1	0	0	0	1	0	0	0	0	−1	0	0	0	1	0	0
0	−1	1	−1	1	0	0	0	0	−1	0	0	0	1	0	0	−1	0	0	0	1	0	0	0
0	0	1	−1	1	−1	0	0	1	−1	0	0	−1	1	0	0	−1	1	−1	1	0	0	0	0
0	1	0	1	−1	0	−1	0	0	1	−1	0	−1	0	0	1	0	0	0	0	0	0	0	0
0	1	−1	2	−2	1	−1	0	0	1	−1	0	0	−1	1	0	0	0	0	0	0	0	0	0
0	0	−1	1	−1	1	0	0	−1	1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	−1	1	−1	1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	1	−1	0	0	−1	−1	0	0	−1
0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	−1	−1	0	0	−1	−1	0
1	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	−1	0	0	−1	−1	0	0	−1
0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	−1	−1	0	0	−1	−1	0
0	0	−1	0	0	0	1	0	1	0	0	0	−1	0	0	0	−1	−1	1	0	0	0	0	1
1	1	0	−1	0	0	−1	0	−1	0	0	0	1	0	0	0	0	0	−1	0	0	0	1	0
1	0	−1	−1	0	1	0	0	1	0	0	0	−1	0	0	0	0	−1	0	0	0	1	0	0
0	1	−1	−1	1	0	0	0	0	1	0	0	0	−1	0	0	−1	0	0	0	1	0	0	0
0	0	−1	−1	1	1	0	0	1	1	0	0	−1	−1	0	0	−1	−1	1	1	0	0	0	0
0	−1	0	1	−1	0	1	0	0	−1	1	0	−1	0	0	1	0	0	0	0	0	0	0	0
0	1	−1	−2	2	1	−1	0	0	1	−1	0	0	−1	1	0	0	0	0	0	0	0	0	0
0	0	1	1	−1	−1	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
−1	−1	1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Some Numbers...

Numerical Simulations					
	β	L	configs.	measures	time
SU(3)	5.8945	12	$\sim 2 \cdot 10^6$	$\sim 10^4$	$\sim$ 4 days on 20 CPUs
SU(4)	10.789	12	$\sim 2 \cdot 10^6$	$\sim 10^4$	$\sim$ 5 days on 20 CPUs
SU(5)	17.107	12	$\sim 2 \cdot 10^6$	$\sim 10^4$	$\sim$ 6 days on 20 CPUs
SU(6)	24.845	12	$\sim 2 \cdot 10^6$	$\sim 10^4$	$\sim$ 7 days on 20 CPUs
SU(7)	33.995	12	$\sim 3 \cdot 10^6$	$\sim 10^4$	$\sim$ 8 days on 40 CPUs
SU(8)	44.496	12	$\sim 2 \cdot 10^6$	$\sim 10^4$	$\sim$ 20 days on 20 CPUs
Total of 4 months of simulations on 100 CPUs					

- the minimum correlation matrix is 8×8
- the maximum is 388×388 and occupies $2\ Gb$ of memory
- the process of diagonalisation for the biggest matrix needs more than $8\ Gb$ of memory to run and it takes more than 1 day on a single CPU

Setting the Scale: the Deconfinement Transition

We fix the lattice spacing a using the physical scale given by the deconfinement temperature.

The same energy scale must be chosen for each $\mathrm{SU}(N)$ theory in order to compare the spectra resulting from lattice numerical simulations. Indeed we are fixing the same ultraviolet cut–off for all the theories.

Deconfinement $\rightarrow$ first order phase transition $(N \geq 3)$.

Order parameter:

$$\langle \bar{L}_p \rangle = \left\langle \frac{1}{L_s^3} \sum_x \mathrm{Tr} \prod_{t=0}^{L_t-1} U_{x,x+ta\hat{0}} \right\rangle \begin{cases} =0 & \text{confinement} \\ \neq 0 & \text{deconfinement} \end{cases}$$