Microscopic Wilson Dirac Operator Spectrum for $N_f = 1$

Talk by P.H. Damgaard at Lattice 2010

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Overview

- Effect of dynamical quarks on the microscopic spectrum of the Wilson Dirac operator at finite lattice spacing a
- There *are* important differences between the quenched and unquenched theories
- The $N_f = 1$ theory is special already in the continuum
- In contrast to the $N_f = 2$ case (**hard!**), the $N_f = 1$ calculation has been carried through analytically
- Detailed results for spectrum of the Hermitian Wilson Dirac operator and the spectrum of real modes

The ϵ-regime: Continuum

Spontaneous breaking of chiral symmetry $\Sigma = \lim_{m\to 0} \lim_{V\to\infty} \langle\bar{\psi}\psi\rangle$

Finite-volume scaling in the ϵ-regime:

$$V \to \infty \quad , \qquad m_\pi L \ll 1$$

Low-energy QCD depends only on the scaling variable

$$\hat{m} \equiv m\Sigma V$$

Rescale Dirac operator eigenvalues λ , $\xi \equiv \lambda\Sigma V$

Universal distributions and spectral correlation functions.

Effective theory for $N_f = 1$ (continuum)

No spontaneous chiral symmetry breaking

Chiral condensate $\Sigma \ = \ \lim_{m\to 0} \lim_{V\to\infty} \langle \bar{\psi}\psi \rangle$ because of the anomaly

Leading-order effective theory [Leutwyler-Smilga]:

$$\mathcal{Z} \ = \ \exp\left[m\Sigma V\right]$$

Fix topology

$$\mathcal{Z}_\nu \ = \ \int_{U(1)} dU (\det U)^\nu \exp\left[\frac{1}{2} m\Sigma V [U + U^\dagger]\right]$$

Looks like the leading-order chiral Lagrangian for a $U(1)$ theory!

Effective theory for $N_f = 1$ (Wilson)

Include explicit chiral symmetry breaking of Wilson fermions

Expand as power series in a^2 [Sharpe, Singleton, Rupak, Shoresh, Bär]

Leading-order effective theory:

$$\mathcal{Z} \;=\; \exp\left[m\Sigma V - W_8 V a^2\right]$$

The coefficient W_8 is a low-energy Wilson constant

Corresponding operators $\sim a^2(\bar{\psi}\psi)^2$

Counting rule

Different countings for the effective theory [Shindler, Bär, Necco, Schaefer]

We choose to keep

$$\hat{m} \equiv m\Sigma V \qquad \text{and} \qquad \hat{a}^2 \equiv a^2 W_8 V$$

fixed as $V \to \infty$

This is the regime where there is maximal competition between m and a^2 effects

Chiral rotation

In the continuum a chiral rotation α shifts the vacuum angle $\theta \to \theta + \alpha$

For Wilson fermions define

$$\mathcal{Z} \;=\; \exp\left[m\cos(\theta)\Sigma V - W_8 V a^2 \cos(2\theta)\right]$$

and do the Fourier transform

Then

$$\mathcal{Z} \;=\; \sum_{\nu=-\infty}^{\infty} \mathcal{Z}_\nu$$

Observation: The partition function is a sum over sectors with $|\nu|$ real modes of the Wilson Dirac operator

Partial Quenching

To get spectral information we need either

* Pairs of extra species with opposite statistics or
* Replicas

Here we use the graded method (add a boson and a fermion)

$$Z_{2|1}(\hat{\mathcal{M}}, \hat{\mathcal{Z}}) = \int_{Gl(2|1)} dU \mathrm{Sdet}(U)^{\nu} e^{i\frac{1}{2}\mathrm{Str}(\hat{\mathcal{M}}[U-U^{-1}])+i\frac{1}{2}\mathrm{Str}(\hat{\mathcal{Z}}[U+U^{-1}])+\hat{a}^2\mathrm{Str}(U^2+U^{-2})}$$

where

$$\hat{\mathcal{M}} = \begin{pmatrix} \hat{m}_f & 0 & 0 \\ 0 & \hat{m} & 0 \\ 0 & 0 & \hat{m}' \end{pmatrix} \quad \hat{\mathcal{Z}} = \begin{pmatrix} \hat{z}_f & 0 & 0 \\ 0 & \hat{z} & 0 \\ 0 & 0 & \hat{z}' \end{pmatrix}$$

Explicit representation:

$$U = \begin{pmatrix} e^{it+iu}\cos(\theta) & ie^{it+i\phi}\sin(\theta) & 0 \\ ie^{it-i\phi}\sin(\theta) & e^{it-iu}\cos(\theta) & 0 \\ 0 & 0 & e^{s} \end{pmatrix} \exp\begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ \beta_1 & \beta_2 & 0 \end{pmatrix}$$

where $\theta, t, u \in [-\pi, \pi]$ and $\phi \in [0, \pi]$.

The boson is integrated over the real line

The α's and β's are Grassmann variables

Other terms in the action like $\sim (\mathrm{Str}[U + U^\dagger])^2$ can be added without fundamental difficulty

We have done the Grassmann and one angular integrations explicity

Three compact and one non-compact integrations left

These integrations are done numerically

Eigenvalues of the Wilson Dirac operator

Wilson Dirac operator D is only γ_5-Hermitian: $D^\dagger \; = \; \gamma_5 D \gamma_5$

Easier to look at the Hermitian Wilson Dirac operator $D_5 = \gamma_5(D + m)$

The resolvent

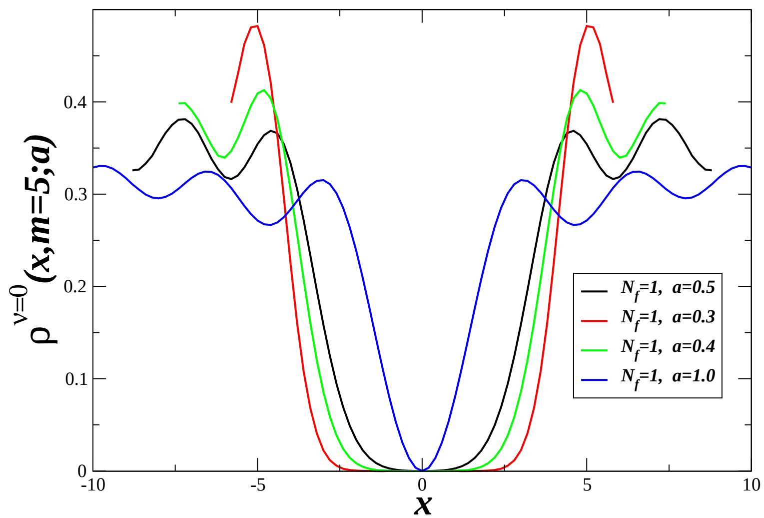
$$G^\nu(\hat{z}, \hat{m}) \; \equiv \; \lim_{\hat{z}' \to \hat{z}} \frac{\partial}{\partial \hat{z}} \ln Z^\nu_{2|1}(\hat{m}, \hat{m}, \hat{m}, \hat{z}, \hat{z}') \; = \; \left\langle \mathrm{Tr} \left(\frac{1}{D_5 + \hat{z}} \right) \right\rangle$$

Spectral density of D_5:

$$\rho^\nu_5(\hat{x}) \; = \; \frac{1}{\pi} \mathrm{Im}[G^\nu(\hat{x}, \hat{m})]$$

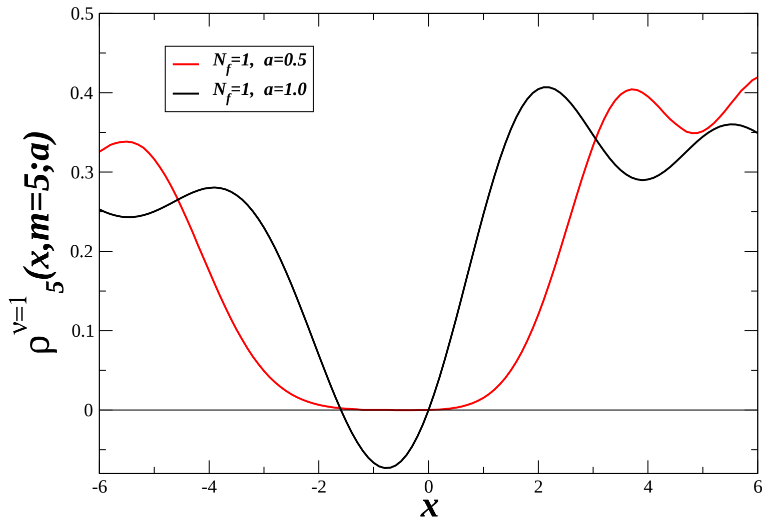
Spectrum of the Hermitian Wilson Dirac operator

Spectral density of D_5 for $\nu = 0$

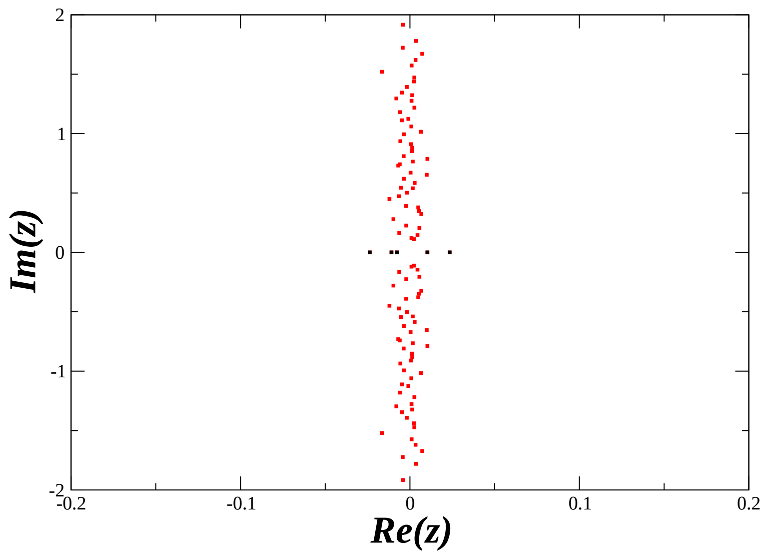


In contrast to $N_f = 0$, ρ_5^ν always vanishes at $x = 0$

A mild sign problem when m is small



Microscopic spectrum of the real modes I



Microscopic spectrum of the real modes II

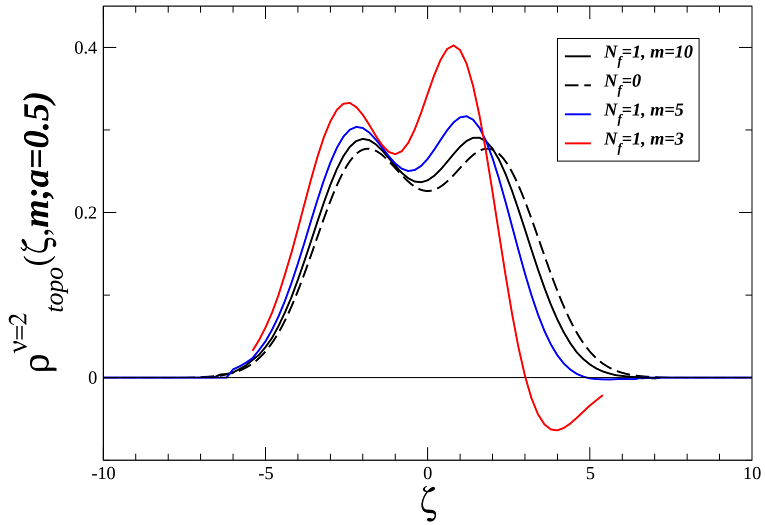
Define the usual resolvent (and put $\hat{z} = \hat{z}' = 0$)

$$\Sigma^\nu(\hat{m}_f, \hat{m}) \equiv \lim_{\hat{m}' \to \hat{m}} \frac{\partial}{\partial \hat{m}} \ln Z^\nu_{2|1}(\hat{m}_f, \hat{m}, \hat{m}')$$

Density of real eigenvalues of the Wilson Dirac operator:

$$\rho^\nu_{\rm topo}(\hat{\zeta}) = \frac{1}{\pi} {\rm Im}[\Sigma^\nu(\hat{m}_f, \hat{\zeta})]$$

$\rho_{\rm topo}(\hat{\zeta})$ changes sign at $\hat{m}$



The density is automatically normalized to $\int d\hat{\zeta}\rho^{\nu}_{\rm topo}(\hat{\zeta}) = \nu$

The $N_f = 1$ theory is quite special

In each sector of ν real modes $\langle\bar{\psi}\psi\rangle_\nu$ is a non-trivial function

When we *sum over all sectors* the effective theory trivializes leading to $\langle\bar{\psi}\psi\rangle = \Sigma$

This is *constant* and independent of a^2 (Recall: $\mathcal{Z} = e^{m\Sigma V - a^2 W_8 V}$)

We have checked this by explicit summation over ν

Wilson chiral Random Matrix Theory

The matrix

$$D = \begin{pmatrix} aA & iW \\ iW^\dagger & aB \end{pmatrix}$$

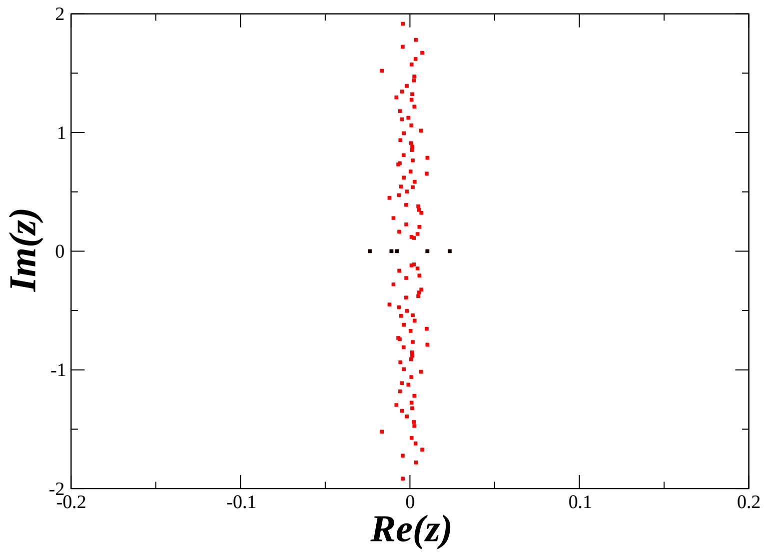
can be used to define Wilson chiral Random Matrix Theory

$$\mathcal{Z}_\nu = \int dA dB dW \det[D + m_f] e^{-N(WW^\dagger + \frac{1}{2}A^2 + \frac{1}{2}B^2)}$$

with A, B Hermitian and W complex

In the large-N limit we recover the same effective theory and same eigenvalue densities [PHD, Splittorff, Verbaarschot]

A typical scatter plot of Wilson chiral RMT eigenvalues



Here $\nu = 5$

The 5 real modes are clearly visible

The sign of W_8

We have assumed that W_8 is *positive*

We are only able to make sense of the spectrum if $W_8 > 0$

(Same story as in the quenched case – c.f. Jac Verbaarschot's talk)

Conclusions

- Wilson chiral RMT equivalent to leading-order Wilson ChPT
- The $N_f = 1$ theory has been described in sectors of ν real modes
- We have computed the eigenvalue densities of D_5 real modes of D
- Still a jump in complexity up to $N_f = 2$ (but it's doable)
- Once established, a new way to extract W_i's