Momentum dependence of the topological susceptibility with overlap fermions

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Introduction

- **Topological structure of the QCD vacuum**
  - ⇒ **chiral symmetry breaking / confinement**
  - ⇒ $U_A(1)$ **anomaly**
  - ⇒ θ **vacuum**
  - ⇒ **proton spin problem**
  - ⇒ **...**
- **Nonperturbative studies needed**
  - ⇒ **effective models**
  - ⇒ **QCD sum rules**
  - ⇒ **lattice QCD**

Question

- **Topological susceptibility and its momentum dependence**

$$\Rightarrow \chi(k^2) = \int d^4r\, e^{ikr} C_q(r)\ , \qquad C_q(r) = \langle q(r)q(0)\rangle$$

$\Rightarrow$ **at zero momentum** $\quad \chi(0) = \dfrac{\langle Q^2\rangle}{V}\ , \qquad Q \in \mathbb{Z}$

$U_A(1)$ anomaly, Witten-Veneziano formula ($N_c \to \infty$)

$$m^2_{\eta'} = \frac{2n_f}{f^2_\pi}\chi(0) \qquad \left(\ |\chi'(0)m^2_{\eta'}| \ll \chi(0)\ \textbf{?}\ \right)$$

Large η' mass compared to other NG bosons

[cf. Di Giacomo etal.('92)]

Strategy

- Gauge invariant way, less systematic errors
  - $\Rightarrow$ cooling gauge field configurations ×
  - $\Rightarrow$ fermionic approach (no destruction) ○
- Better to use fermions with chiral symmetry
  - $\Rightarrow$ Wilson △, KS △
  - $\Rightarrow$ Overlap ○

Use eigenvalues / eigenmodes of Dirac operator

Atiyah-Singer index theorem Index$[D] = Q \in \mathbb{Z}$

Overlap fermions

- **Dirac operator** [Neuberger ('98)]

$$D = \frac{\rho}{a}\left(1 + \frac{X}{\sqrt{X^\dagger X}}\right), \qquad X = D_W - \frac{\rho}{a}$$

D_W is Wilson-Dirac operator, a is lattice spacing

ρ controls the locality of the Dirac operator

$\Rightarrow$ Exact chiral symmetry on lattice [Lüscher ('98)]

Ginsparg-Wilson relation : $D\gamma_5 + \gamma_5 D = aD\gamma_5 D$

$\Rightarrow$ Chiral zero modes

$$D\psi_n^- = 0, \quad \gamma_5\psi_n^- = -\psi_n^- \quad (n = 1, \ldots, n_-)$$
$$D\psi_n^+ = 0, \quad \gamma_5\psi_n^+ = +\psi_n^+ \quad (n = 1, \ldots, n_+)$$

$\Rightarrow$ non-zero modes

$$D\psi_\lambda = \lambda\psi_\lambda, \quad \mathrm{Tr}(\psi_\lambda^\dagger \gamma_5 \psi_\lambda) = 0$$

$q(x)$

- **Topological charge density** [Hasenfratz, Laliena, Niedermayer ('98)]

$$q(x) \equiv -\mathrm{tr}\left[\gamma_5(1 - \frac{a}{2}D(x,x))\right] \quad \Leftrightarrow \quad \frac{g^2}{64\pi^2}\mathrm{tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right]$$

$$\Rightarrow \quad Q = \sum_{x \in V} q(x) \in \mathbb{Z}$$

- **Eigenmode expansion of** $q(x)$ [Horváth etal. ('03)]

$$q_\lambda(x) = -\sum_{\lambda \le \lambda_{\mathrm{cut}}} \left(1 - \frac{\lambda}{2}\right) \psi^*_\lambda(x)\gamma_5\psi_\lambda(x)$$

$$\Rightarrow \quad Q = \sum_{x \in V} q_\lambda(x) \text{ even if } \lambda_{\mathrm{cut}} \text{ is finite}$$

$\chi(k^2)$

- **Topological susceptibility**

$$\chi(k^2) = \sum_{r \in V} e^{ikr} C_q(r)$$

$$\text{where } C_q(r) = \frac{1}{V} \langle \sum_{x \in V} q(x+r) q(x) \rangle$$

$\Rightarrow$ $C_q(r)$ **depends on** λ_{cut}

$\Rightarrow$ $\chi(k^2 > 0)$ **depends on** λ_{cut}

$$\Rightarrow \chi(0) = \sum_{r \in V} C_q(r) = \frac{\langle Q^2 \rangle}{V}$$

is determined only by the number of zero modes

$\chi'(0)$

- **Derivative of topological susceptibility**

$$\chi'(0) = \frac{d\chi(k^2)}{dk^2}\bigg|_{k^2=0} = -\frac{1}{8}\sum_{r\in V} r^2 C_q(r)$$

$\Rightarrow$ **Does** $\chi'(0)$ **depend on** λ_{cut} **?**

$\Rightarrow$ **Is** $|\chi'(0)m_{\eta'}^2| \ll \chi(0)$ **realized ?**

Simulation details

- **Lüscher-Weisz gauge action**

  $\Rightarrow$ **supress unphysical zero modes**

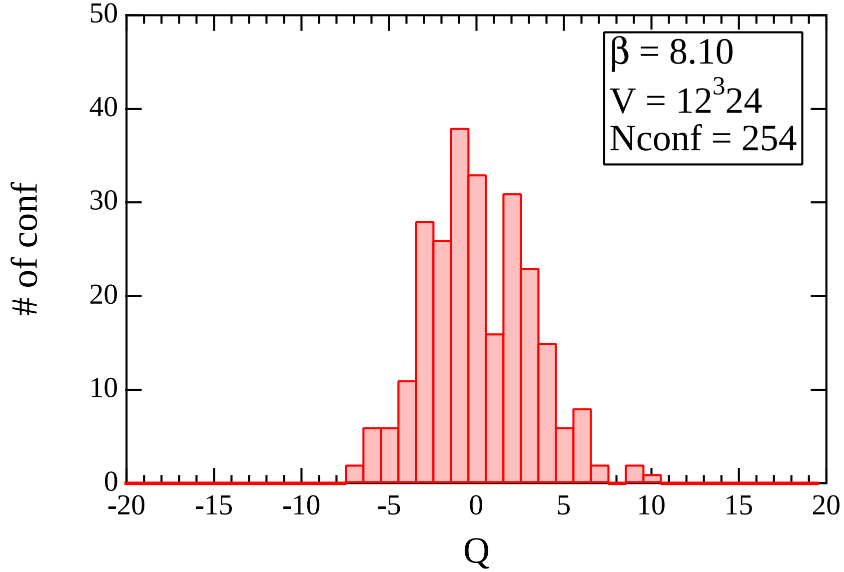
β	a [fm]	$V = L^3T$	[fm^4]	$N_{\rm conf}$	$N_{\rm modes}$
8.45	0.105 (fine)	$16^3 32$	(10.6)	267	$O(140)$
8.10	0.142 (coarse)	$12^3 24$	(10.1)	254	$O(140)$

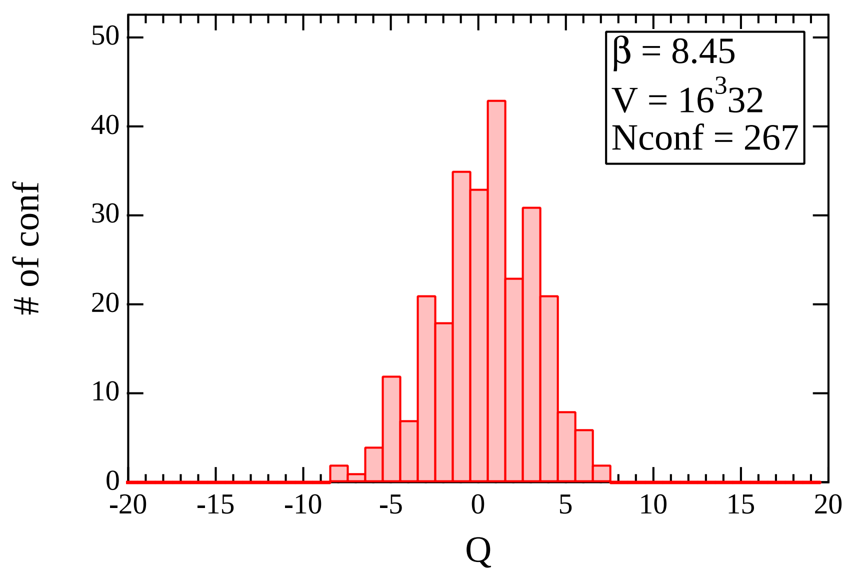
[QCDSF-UKQCD ('03, '07)] (lattice spacing a is fixed by $f_\pi = 92$ MeV)

[Ilgenfritz et al., PRD76 ('07)]

Q

- **Distribution**



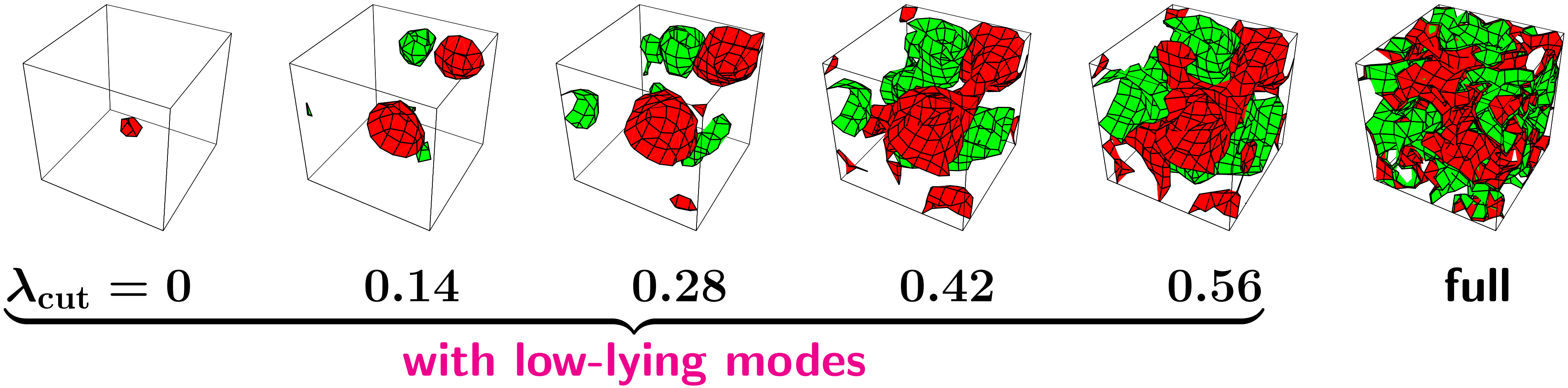


$$\Rightarrow \quad \chi(0) = \frac{\langle Q^2 \rangle}{V} = (195(5)\ \text{MeV})^4 \quad (\beta = 8.10)$$

$$= (187(5)\ \text{MeV})^4 \quad (\beta = 8.45)$$

$q(x)$

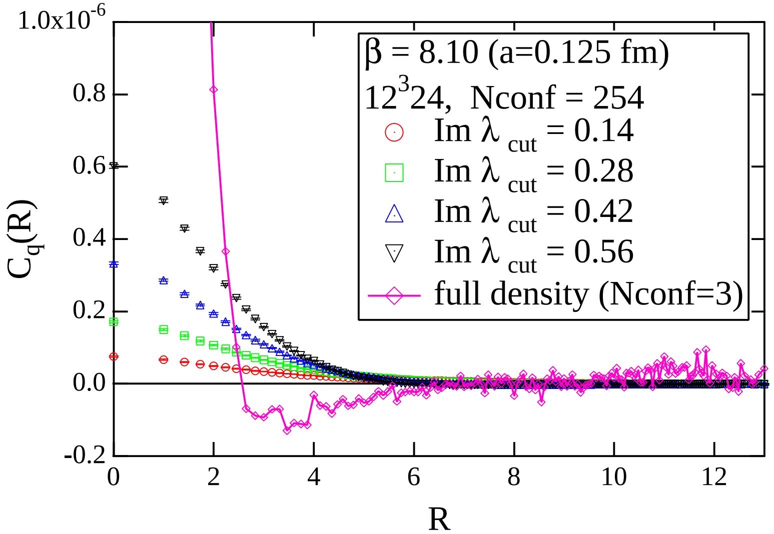
- $\beta = 8.10$ ($a = 0.125$ fm), $V = 12^3 24$
  $q(x) = +0.0005, -0.0005$

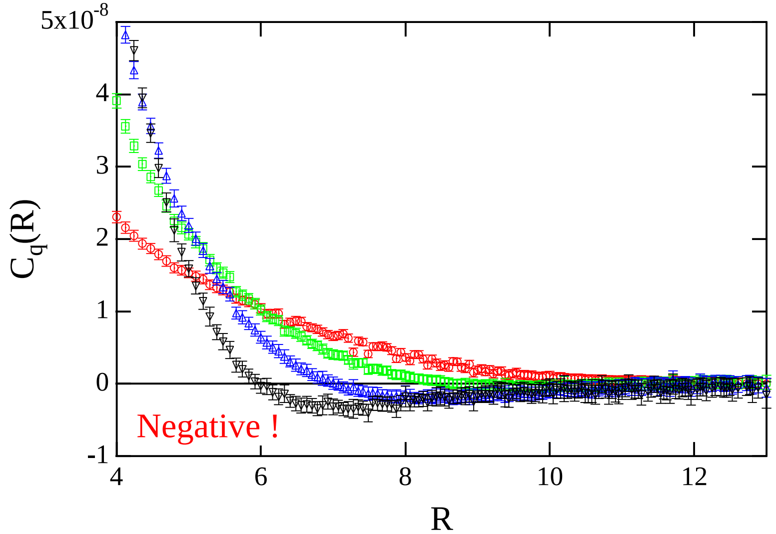


$\Rightarrow$ **sign coherent structure**

$C_q(r)$

- **Dependence on the number of eigenmodes**



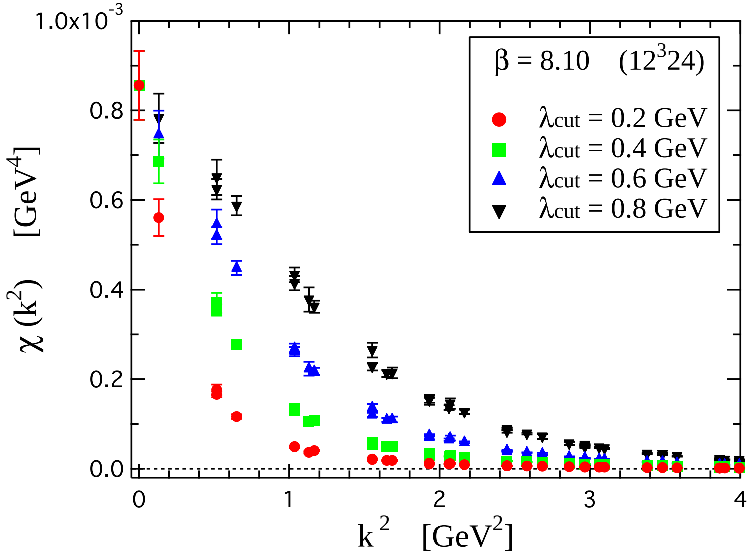


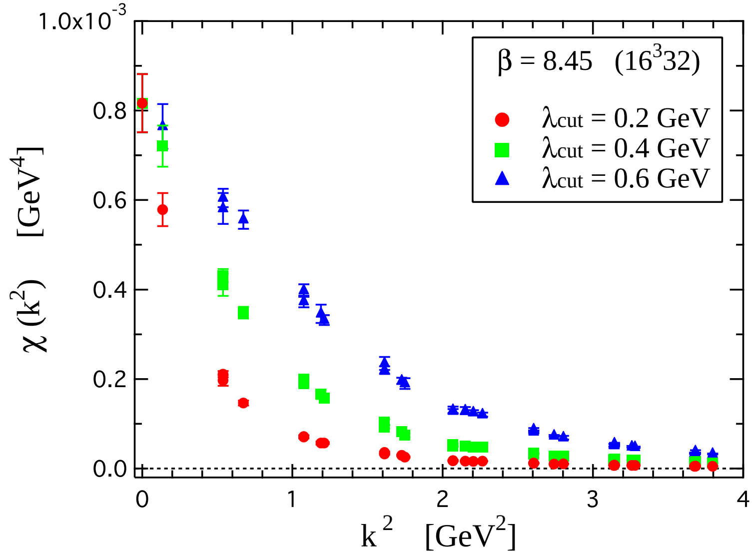
$\Rightarrow$ **Positive core**

$\Rightarrow$ **Negativity** $C_q(r) < 0$ **for** $r > 0$

$\chi(k^2)$

- **Dependence on the number of eigenmodes**





$\Rightarrow$ $\chi(k^2 = 0)$ **does not depend on the number of eigenmodes**

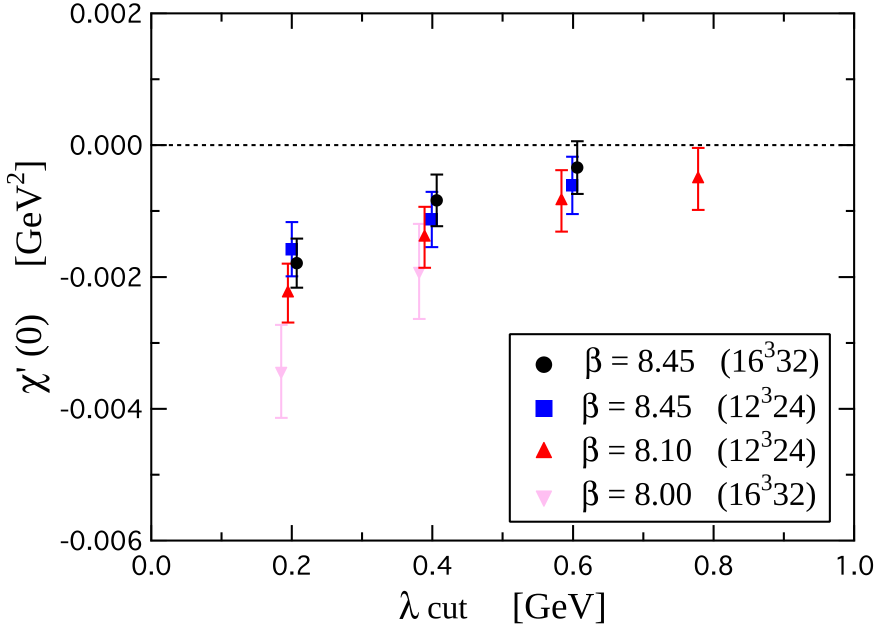
$\Rightarrow$ **The slope around** $k^2 = 0$ **is negative,**

becomes moderate with increasing the number of eigenmodes

$\Rightarrow$ **(The slope becomes positive if zero modes are suppressed)**

$\chi'(0)$

- **Dependence on the number of eigenmodes**



$\Rightarrow \chi'(0) < 0$

$\Rightarrow \chi'(0) \to 0$ **with increasing the number of eigenmodes ?**

$\Rightarrow |\chi'(0)| \simeq 0.001\ \mathrm{GeV}^2$ **only with low-lying modes**

Summary & discussions

- **Topological structure of the QCD vacuum with overlap fermions**
  - $\Rightarrow$ **gauge invariance, chiral symmetry**
  - $\Rightarrow$ Q, $q(x)$, $C_q(r)$, $\chi(k^2)$, $\chi'(0)$
  - $\Rightarrow$ $\chi(0)$ **is determined only by zero modes (known)**

  - $\Rightarrow$ **But** $\chi'(0)$ **is ...**

    Suppose $|\chi'| \simeq 0.001\ \mathrm{GeV}^2$, $m_{\eta'} = 0.958\ \mathrm{GeV}$

    $|\chi'(0) m_{\eta'}^2| = (0.174\ \mathrm{GeV})^4 \simeq \chi(0)$ **!!**

    Only low modes is insufficient to satisfy

    $|\chi'(0) m_{\eta'}^2| \ll \chi(0)$

Appendix: Two-point function of topological charge density

- **Definition**

$$C_q(R) = \frac{1}{V}\langle \sum_{x \in V} q(x)q(x+R) \rangle$$

$\Rightarrow$ $C_q(R) < 0$ **for** $R > 0$ [Seiler & Stamatescu ('87)]

$q(x)$ **change the sign by time reversal (anti-Hermitian operator)**

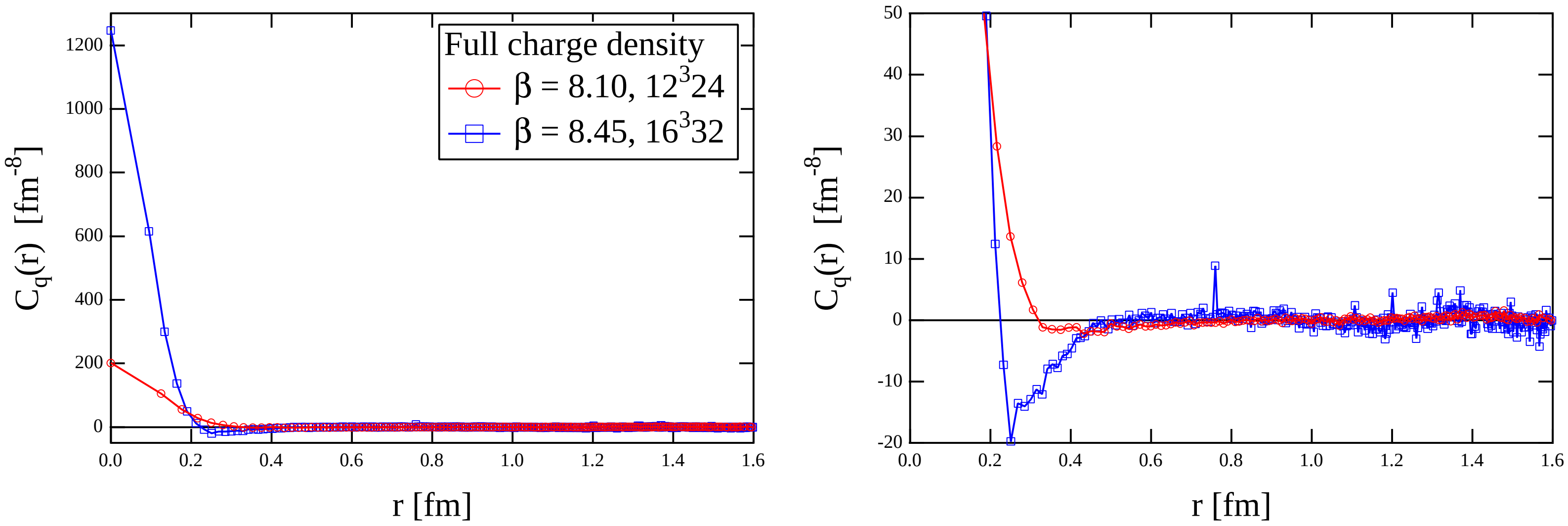
$\Rightarrow$ **Topological susceptibility** χ_{top} **is positive definite**

$$\chi_{\text{top}} = \sum_{R \in V} C_q(R) = \frac{\langle Q^2 \rangle}{V} > 0$$

Positive core around the origin

Appendix: Two-point function of topological charge density

- **Dependence on the lattice spacing**



$\Rightarrow$ **grow positive core, decrease core radius for $a \longrightarrow 0$**

$\Rightarrow$ **negative tail**

$\Rightarrow$ **expected behavior in the continuum theory** [cf. Seiler ('02)]