Let's go dynamic : a quick journey through 2+1 simulations to understand χSB

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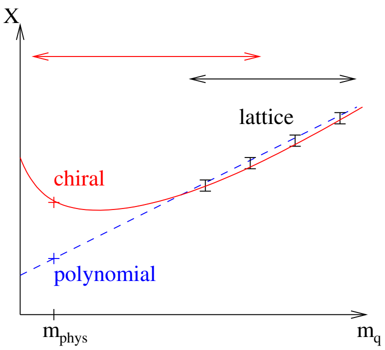
χPT and lattice

χPT : structure of $\pi, K\eta$ interactions but not values of couplings

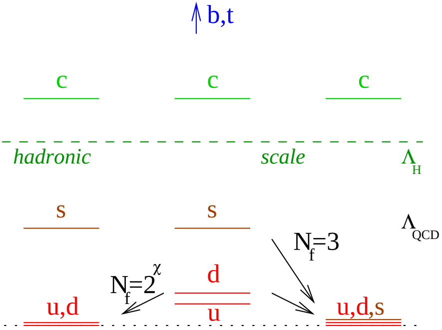
Overlap with lattice ?

- Light masses for χPT, but unknown constants
- Heavier masses for lattice, but extrapolation



- In which region can we use χPT ?
- Can we learn from the lattice on chiral symmetry breaking ?

Three chiral limits of interest



$m_u, m_d \to 0$

$N_f = 3$: $m_s \to 0$

$N_f = 2$: m_s physical

$N_f = 2^{\mathrm{lat}}$: no dynamical s

Two versions of χPT	$N_f = 2$:	π only d.o.f	(few param. & processes)
	$N_f = 3$:	π, K, η d.o.f	(more param. & processes)

From 2 to 3 massless flavours

$$\Sigma(2; m_s) = \lim_{m_u, m_d \to 0} -\langle 0|\bar{u}u|0\rangle$$

$$\begin{cases} \Sigma(3) & = & \Sigma(2; 0) \\ \Sigma(2) & = & \Sigma(2; m_s^{\mathrm{phys}}) \\ \Sigma(2^{\mathrm{lat}}) & = & \Sigma(2; \infty) \end{cases}$$

From 2 to 3 massless flavours

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$$\Sigma(2; m_s) = \Sigma(2; 0) + m_s \frac{\partial \Sigma(2; m_s)}{\partial m_s} + O(m_s^2)$$

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$$\Sigma(2) = \Sigma(3) + m_s^{\mathrm{phys}} \lim_{m_u, m_d \to 0} i \int d^4x \, \langle 0|\bar{u}u(x)\,\bar{s}s(0)|0\rangle + O(m_s^2)$$

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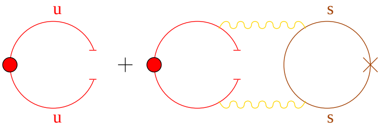
$\Sigma(2)$ contains

- A “genuine” condensate $\Sigma(3)$
- An “induced” condensate $m_s \times$ (scalar $1/N_c$-suppressed)
  effect from sea $s\bar{s}$-pairs

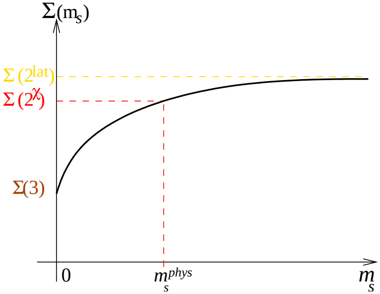


(similar analysis with $F^2(N_f) = \lim_{N_f} F_\pi^2$)

Which scenario for $N_f = 2$ and $N_f = 3$?

$$\underbrace{\langle \bar{u}u \rangle|_{m_{u,d}=0, m_s \text{ phys}}}_{\text{sizeable } \Sigma(2)} = \underbrace{\langle \bar{u}u \rangle|_{m_{u,d,s}=0}}_{\Sigma(3)} + \underbrace{m_s \langle (\bar{u}u)(\bar{s}s) \rangle}_{m_s L_6} + \ldots$$

Analysis of fermion det in terms of Dirac eigenvalues: $\Sigma(3) \leq \Sigma(2)$



$\Sigma(3) \simeq \Sigma(2)$ and $\langle (\bar{u}u)(\bar{s}s) \rangle$ small
Zweig rule OK for scalars
No impact of strange sea quarks

or

$\Sigma(3) < \Sigma(2)$ and $\langle (\bar{u}u)(\bar{s}s) \rangle$ large
Large Zweig-rule violation
Strange sea quarks important

In the scalar sector, Zweig rule and large-N_c badly violated
described in $O(p^4)$ LECs L_4 and L_6

Indication from non-perturbative methods

Same analysis for two main order parameters of χSB

$$\Sigma(2) - \Sigma(3) \propto m_s L_6 \qquad F^2(2) - F^2(3) \propto m_s L_4$$

- Large dispersive estimates of $\langle(\bar{u}u)(\bar{s}s)\rangle$: $\Sigma(3)/\Sigma(2) \simeq 1/2$
- Low-energy πK scattering from dispersive analysis of data yields $10^3 L_4(M_\rho) = 0.53 \pm 0.39$ *B.Moussallam,SDG,P.Büttiker*
- Recent fits for NNLO $N_f = 3$ χPT *I. Jemos, Euroflavour 09*
  $10^3 L_4^r(M_\rho) = 0.86 \pm 0.86 \qquad 10^3 L_6^r(M_\rho) = 0.36 \pm 0.95$
  with issues in the convergence of chiral series ($F(3)$=62.4 MeV)
- Lattice with 2+1 dynamical flavours
  - MILC: $\Sigma(2)/\Sigma(3) \simeq 1.52(17)(^{+38}_{-15})$
  - PACS-CS: Large NLO contributions in $N_f = 3$ ChPT due to m_s
  - UKQCD-RBC: Hard to fit $K_{\ell 3}$ with $N_f = 3$ ChPT, use only $N_f = 2$

Consequences for three-flavour chiral series

$$
\begin{aligned}
F_\pi^2 &= F(3)^2 + 16(m_s+2m)B_0\Delta L_4 + 16mB_0\Delta L_5 + O(m_q^2) \\
F_\pi^2 M_\pi^2 &= \Sigma(3) + 64m[(m_s+2m)B_0^2\Delta L_6 + mB_0^2\Delta L_8] + O(m_q^2)
\end{aligned}
$$

- $B_0 = -\lim_{m_u,m_d,m_s\to 0}\langle\bar{u}u\rangle/F_\pi^2 = \Sigma(3)/F^2(3)$ $m = m_u = m_d$
- $\Delta L_i = L_i^r(M_\rho) + \chi$ log scale-independent

If m_s-enhanced L_4^r, L_6^r are "large" (Zweig-rule violation in 0^{++})

- Numerical competition between LO and NLO in F_P^2, $F_P^2 M_P^2$
- Chiral series not saturated by LO: $F_\pi \not\approx F_0$, $M_\pi^2 \not\approx 2mB_0$ ...

$$
\frac{1}{X_{LO}+X_{NLO}} \not\approx \frac{1}{X_{LO}} - \frac{X_{NLO}}{X_{LO}^2} \qquad \sqrt{X_{LO}+X_{NLO}} \not\approx \sqrt{X_{LO}} + \frac{X_{NLO}}{2\sqrt{X_{LO}}}
$$

Choose/determine carefully observables with good convergence
and how how write down/use their expansion

Resummed χPT

Large effect of $s\bar{s}$ pairs $\Longrightarrow$ weak convergence $\Longrightarrow$ $2mB_0 \not\approx M_\pi^2$

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- Assume overall convergence for a subset of observables
  vector/axial correlators and derivatives away from singularities
- Leave open a numerical competition between LO and NLO
  while keeping track of (small) NNLO remainders
- Compute observables in terms of chiral LECs (F_0, B_0, L_i, C_i)
- Reexpress LECs in terms of $M_\pi^2, F_\pi^2 \ldots$
  only if physical motivation (nonanalytic poles, cuts, unitarity. . .)

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Resummed Chiral Perturbation Theory
coping with competition between LO and NLO
(identical to usual χPT if LO almost saturates the chiral series)

SDG, Fuchs, Girlanda, Stern

Observables: masses and decay constants

From $\langle A_\mu A_\nu \rangle$ and $\langle \partial^\mu A_\mu \partial^\nu A_\nu \rangle$

F_P^2 and $F_P^2 M_P^2$ ($P = \pi, K$) expected to have small NNLO remainders

$$F_\pi^2 = F_\pi^2 Z(3) + 8Y(3)M_\pi^2[(r+2)\Delta L_4 + \Delta L_5] + F_\pi^2 e_\pi$$
$$F_\pi^2 M_\pi^2 = F_\pi^2 M_\pi^2 X(3) + 16Y^2(3)M_\pi^4[(r+2)\Delta L_6 + \Delta L_8] + F_\pi^2 M_\pi^2 d_\pi$$

$$X(3) = \frac{2m\Sigma(3)}{F_\pi^2 M_\pi^2} = \frac{LO(F_\pi^2 M_\pi^2)}{F_\pi^2 M_\pi^2} \qquad r = \frac{m_s}{m}$$
$$Z(3) = \frac{F^2(3)}{F_\pi^2} = \frac{LO(F_\pi^2)}{F_\pi^2} \qquad Y(3) = \frac{X(3)}{Z(3)} = \frac{2mB_0}{M_\pi^2}$$

- $\Delta L_i = L_i$ + chiral logs: $\frac{1}{32\pi^2} \log \frac{\overset{\circ}{M}_P^2}{\mu^2}$ $\qquad \overset{\circ}{M}_P^2 = \mathrm{LO}[M_P^2]$
- e_P and d_P NNLO remainders $O(m_s^2)$ expected of order 10%
- Can be inverted: $L_{4,5,6,8} = \mathcal{F}[F_\pi, F_K, M_\pi, M_K, r, X(3), Z(3), 4\ \mathrm{rem.}]$

Observables: pion em and $K_{\ell 3}$ form factors

$$\langle\pi^+|j_\mu|\pi^+\rangle = (p+p')^\mu F_V^\pi(t)$$
$$\sqrt{2}\langle K^+|\bar{u}\gamma_\mu s|\pi^0\rangle = (p'+p)^\mu f_+^{K\pi}(t) + (p'-p)^\mu f_-^{K\pi}(t)$$

From LSZ reduction to $\langle A_\nu V_\mu A_\rho\rangle$,
$F_\pi^2 F_V^\pi$, $F_\pi F_K f_+(t)$, $F_\pi F_K f_0(t)$ expected to have small NNLO remainders

$$F_\pi F_K f_+(t) = \frac{F_\pi^2+F_K^2}{2} + \frac{3}{2}[tM^r_{K\pi}(t) + tM^r_{K\eta}(t) - L_{K\pi}(t) - L_{K\eta}(t)] + 2tL_9^r + F_\pi F_K d_+ + te_+$$

- Ambiguity on F_0^2 at NLO fixed (replaced by $F_\pi F_K$)
- M, L one-loop scalar integrals, with cuts set at physical masses
- Similar expansion for f_0 (fulfilling explicitly Callan-Treiman)
- Similar expansion for F_π^V which can be inverted

$$L_9 = \mathcal{F}\left[\langle r^2\rangle_\pi^V, r, X(3), Z(3), 1\ \mathrm{rem.}\right]$$

QCD and lattice

Actual QCD: chiral expansions for obs. X with quark masses (m_s, m)

$$
\begin{aligned}
F_\pi^2, F_K^2 &: \quad L_{4,5} = \mathcal{F}\,[r, X(3), Z(3), 2 \text{ rem.}] \\
F_\pi^2 M_\pi^2, F_K^2 M_K^2 &: \quad L_{6,8} = \mathcal{F}\,[r, X(3), Z(3), 2 \text{ rem.}] \\
\langle r^2 \rangle_\pi^V &: \quad L_9 = \mathcal{F}\,[r, X(3), Z(3), 1 \text{ rem.}]
\end{aligned}
$$

$K_{\ell 3}$ form factors are functions of $t, r, X(3), Z(3), L_9, 4$ rem.

Lattice: same expansions for $\tilde{X}$ with quark masses $(\tilde{m}_s, \tilde{m})$

- Previous relations used to remove $L_{4,5,6,8,9}$
- Chiral expansions for $\tilde{X}$ depending only on

$$r, X(3), Z(3), \qquad p = \tilde{m}_s/m_s \qquad q = \tilde{m}/\tilde{m}_s$$

- and on rescaled NNLO remainders

$$d = O(m_s^2) \to \tilde{d} = O(\tilde{m}_s^2) = p^2 d$$

Fit to the lattice data

Fit to RBC/UKQCD and PACS-CS lattice data

[difficulties to fit NLO $N_f = 3$ chiral expansions]

- 2+1 simulations with observables as function of quark masses
- Observables for several $q = \tilde{m}/\tilde{m}_s$
  - $F_\pi^2, F_\pi^2 M_\pi^2$ (both)
  - $F_\pi F_K f_+(t)$ and $F_\pi F_K f_0(t)$ (RBC/UKQCD)
- Parameters to fit: $r, X(3), Z(3), p$, remainders and F_K/F_π
- Only statistical errors available, without correlations
  - naive χ^2 to minimise
  - no sophisticated treatment of systematics

V. Bernard, SDG, G. Toucas, in preparation

Fit to RBC/UKQCD data (1)

UKQCD/RBC data on π, K masses, decay csts and $K_{\ell 3}$ form factors
Allton et al. 2008, Boyle et al. 2007, Boyle et al. 2010

- Domain-wall fermions, 1 spacing, 2 volumes, only statistical errors
- Take only unitary pts (unquenched), non-degenerate π, K masses
- Form factors with $t \geq -0.2\ \mathrm{GeV}^{-2}$ (pions light enough, momenta small enough)

r	24.6 ± 2.1
$X(3)$	0.28 ± 0.10
$Y(3)$	0.56 ± 0.20
$Z(3)$	0.49 ± 0.05
F_K/F_π	1.18 ± 0.03
χ^2/N	$3.4/6$

- Good fit to data (stat errors only)
- 14 params (9 remainders small)
- LO do not saturate $N_f = 3$ series
- Ratio of decay constants lower than Allton et al.: $F_K/F_\pi = 1.205 \pm 0.018 \pm 0.062$

Fit to RBC/UKQCD data (2)

$m_s(2\ \mathrm{GeV})[\mathrm{MeV}]$	114.0 ± 4.5
$m(2\ \mathrm{GeV})[\mathrm{MeV}]$	4.7 ± 0.3
$B_0(2\ \mathrm{GeV})[\mathrm{GeV}]$	1.19 ± 0.44
$F_0[\mathrm{MeV}]$	64.7 ± 3.3
$\Sigma(2)/\Sigma(3)$	3.25 ± 1.12
$B(2)/B(3)$	1.78 ± 0.57
$F(2)/F(3)$	1.35 ± 0.07
$f_0(0)$	0.984 ± 0.006
F_π^2	$0.49 + 0.62 - 0.11$
F_K^2	$0.35 + 0.73 - 0.08$
$F_\pi^2 M_\pi^2$	$0.28 + 0.61 + 0.11$
$F_K^2 M_K^2$	$0.20 + 0.71 + 0.09$

- Only statistical errors !
- Significant decrease of order parameters from $N_f = 2$ to $N_f = 3$ chiral limits [hence troubles with $N_f = 3\ \chi$PT]
- $f_0(0)$ higher than value in Boyle et al. $f_0(0) = 0.960(^{+5}_{-6})$
- Convergence at χ^2_{min} NNLO $\ll$ LO + NLO but LO $\sim$ NLO

[Similar fit with all data, with $\chi^2/N = 21.1/19$]

Fit to PACS-CS data (1)

PACS-CS data on π, K masses and decay constants

Aoki et al. 2008

- $O(a)$-improved Wilson, 1 spacing, 1 volume, only statistical errors
- One-loop perturbative renormalisation (30% underestimation of quark masses compared to non-perturbative renormalisation)

  Aoki et al. 2009

- Take only 3 lightest values of the pion masses to ensure χPT valid

r	26.5 ± 2.3
$X(3)$	0.59 ± 0.20
$Y(3)$	0.90 ± 0.22
$Z(3)$	0.66 ± 0.08
F_K/F_π	1.23 ± 0.03
χ^2/N	$0.9/3$

- Good fit to data (stat errors only)
- 14 params (9 remainders small)
- LO do not saturate $N_f = 3$ series
- Ratio of decay constants higher than Aoki et al.: $F_K/F_\pi = 1.189 \pm 0.020$

Fit to PACS-CS data (2)

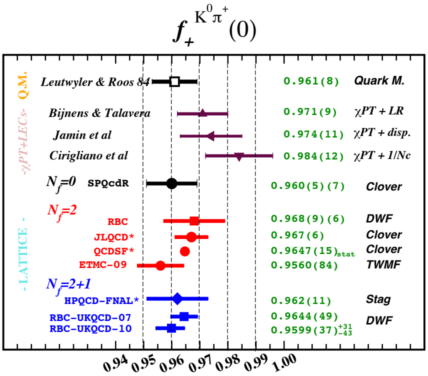
$m_s(2\,\mathrm{GeV})[\mathrm{MeV}]$	70.3 ± 4.2
$m(2\,\mathrm{GeV})[\mathrm{MeV}]$	2.7 ± 0.3
$B_0(2\,\mathrm{GeV})[\mathrm{GeV}]$	2.65 ± 0.28
$F_0[\mathrm{MeV}]$	75.1 ± 4.2
$\Sigma(2)/\Sigma(3)$	1.52 ± 0.49
$B(2)/B(3)$	1.16 ± 0.26
$F(2)/F(3)$	1.15 ± 0.07
$f_0(0)$	1.004 ± 0.116
F_π^2	$0.66 + 0.22 + 0.12$
F_K^2	$0.43 + 0.49 + 0.08$
$F_\pi^2 M_\pi^2$	$0.60 + 0.30 + 0.10$
$F_K^2 M_K^2$	$0.42 + 0.50 + 0.08$

- Only statistical errors !
- Mild decrease of order parameters from $N_f = 2$ to $N_f = 3$ chiral limits [hence troubles with $N_f = 3$ χPT]
- $f_0(0)$ as an outcome of the fit (no input from $K_{\ell 3}$ form factors)
- Convergence at χ^2_{min} NNLO $\ll$ LO + NLO but LO $\sim$ NLO

[Similar fit with all data, with $\chi^2/N = 13.9/15$]

F_K/F_π and $f_+(0)$

	$f_+(0)$	F_K/F_π
Our fit to RBC/UKQCD	0.984 ± 0.006	1.18 ± 0.03
Our fit to PACS-CS	1.004 ± 0.116	1.23 ± 0.03
RBC/UKQCD	$0.960(^{+5}_{-6})$	$1.205 \pm 0.018 \pm 0.062$
PACS-CS	$\times$	1.189 ± 0.020



Our fits include only stat errors and neglect correlations among observables

Flavianet kaon working group 2010

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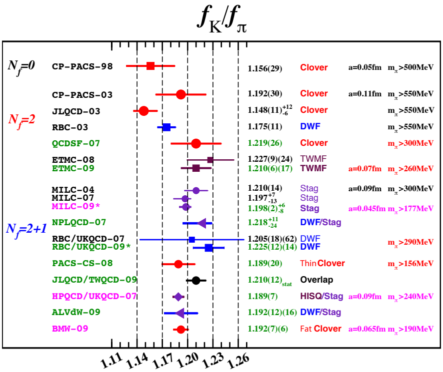
PACS-CS	$\times$	1.189 ± 0.020



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Comments and conclusions

- Two chiral limits of interest $N_f = 3 : m_u, m_d, m_s \to 0$
  $N_f = 2 : m_u, m_d \to 0, m_s$ physical

$$\Sigma(2) = \Sigma(3) + m_s \langle (\bar{u}u)(\bar{s}s) \rangle + O(m_s^2)$$

- Role of sea $s\bar{s}$-pairs $\leftrightarrow$ N_f-dependence of order parameters
  $\leftrightarrow$ Zweig rule violation in scalar sector
- Weak convergence of chiral series: NNLO $\ll$ LO+NLO, LO $\sim$ NLO

Resummed Chiral Perturbation Theory to applied lattice data

- Good fits with a limited number of parameters
- Provide a decent alternative to the ansätze inspired by $N_f = 2$ χPT used to extract $f_+(0)$ from the lattice
- $f_+(0)$ larger than usual lattice estimates, but closer to χPT ones
- Only stat errors, before $a \to 0$, $L \to \infty$, so no firm conclusions

Maybe worth having a try on your favourite 2+1 data ?

Backup

One-loop resummed χPT (1)

Green functions in one-loop generating functional

$$Z = Z_t + Z_u + Z_A$$

"Bare" expansion in terms of LECs $F_0, B_0, L_i \ldots$ with LO masses

$$\overset{\circ}{M}{}_\pi^2 = Y(3) M_\pi^2, \quad \overset{\circ}{M}{}_K^2 = \frac{r+1}{2} Y(3) M_\pi^2 \qquad r = \frac{m_s}{m}, \; Y(3) = \frac{2mB_0}{M_\pi^2}$$

Where $\overset{\circ}{M}{}_P^2 \to M_P^2$ in bare expansion ? Only if physically supported !

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Green functions in one-loop generating functional

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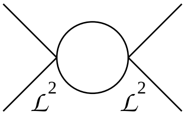
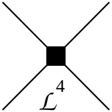


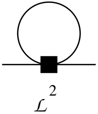
- Z_u one-loop graphs with two $O(p^2)$ vertices

Unitarity cuts at $(\overset{\circ}{M}_P + \overset{\circ}{M}_Q)^2$
converging to $(M_P + M_Q)^2$
when higher orders into account

$\Longrightarrow$ Replace $\overset{\circ}{M}{}_P^2 \to M_P^2$ for the position of the cuts Z_u [i.e. in $\bar{J}_{PQ}$]

- Z_A purely topological, no chiral couplings





- Z_t tree and tadpole graphs

- $O(p^2)$ and $O(p^4)$ tree graphs : chiral couplings
- tadpoles : factors of log modified by higher orders so keep

$$\frac{\overset{\circ}{M}{}_P^2}{32\pi^2} \log \frac{\overset{\circ}{M}{}_P^2}{\mu^2}$$

Why is it a resummation ?

$$X(3) = \frac{2m\Sigma(3)}{F_\pi^2 M_\pi^2}\,, \qquad Z(3) = \frac{F^2(3)}{F_\pi^2}\,, \qquad r = \frac{m_s}{m}$$

$$\begin{aligned} F_\pi^2 &= F_\pi^2 Z(3) + 8Y(3)M_\pi^2[(r+2)\Delta L_4 + \Delta L_5] + F_\pi^2 e_\pi \\ F_\pi^2 M_\pi^2 &= F_\pi^2 M_\pi^2 X(3) + 16Y^2(3)M_\pi^4[(r+2)\Delta L_6 + \Delta L_8] + F_\pi^2 M_\pi^2 d_\pi \end{aligned}$$

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$$Y(3)=\frac{2mB_0}{M_\pi^2}=\frac{2[1-\epsilon(r)-d]}{[1-\eta(r)-e]+\sqrt{[1-\eta(r)-e]^2+k\times[2\Delta L_6-\Delta L_4]}}$$

$$k \simeq 32(r+2)\frac{M_\pi^2}{F_\pi^2} \qquad\qquad e,d \leftrightarrow e_{\pi,K}, d_{\pi,K}$$

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$$X(3) = \frac{2m\Sigma(3)}{F_\pi^2 M_\pi^2}\,, \qquad Z(3) = \frac{F^2(3)}{F_\pi^2}\,, \qquad r = \frac{m_s}{m}$$

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- But $k \simeq 1900$: $\Delta L_6, \Delta L_4 = O(10^{-3})$ yields shift of $Y(3)$ from 1, $\Longrightarrow$ resummation of $k \times [2\Delta L_6 - \Delta L_4]$ needed

How big (or small) should be L_4 and L_6 ?

$$F_\pi^2 = F(3)^2 + 16(m_s + 2m)B_0 \Delta L_4 + 16 m B_0 \Delta L_5 + O(m_q^2) = F^2(3) + O(m_q)$$

If NLO < LO, taking $L_4(M_\rho) = 0.5 \cdot 10^{-3}$, $L_5(M_\rho) = 1.4 \cdot 10^{-3}$,

$$\begin{aligned} \frac{F(3)^2}{F_\pi^2} &= \frac{F^2(3)}{F^2(3) + O(m_q)} \\ &\to 1 - 8\frac{2M_K^2 + M_\pi^2}{F_\pi^2}\Delta L_4 - 8\frac{M_\pi^2}{F_\pi^2}\Delta L_5 + \ldots \\ &= 1 - 0.51(s\bar{s}\ \text{pairs}) - 0.04(\text{other}) + O(p^4) \end{aligned}$$

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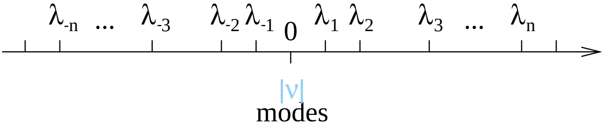
Positive $O(10^{-3})$ value of $L_4^r(M_\rho)$ (id for L_6) yields LO $\sim$ NLO

Safer to impose only weak convergence NNLO$\ll$LO+NLO

The Dirac operator

In Euclidean QCD on a torus L^4, Dirac operator can be diagonalised

$$H[G] \equiv \not{D} = \gamma_\mu(\partial_\mu + iG_\mu) \quad H\phi_n = \lambda_n[G]\phi_n \qquad |\lambda_n[G]| < C\frac{n^{1/4}}{L} \equiv \omega_n$$



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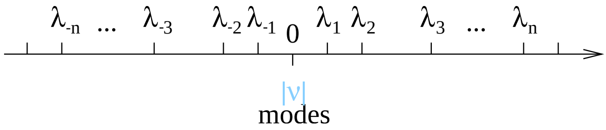

After integration over fermionic variables, a correlation function expressed as a statistical average over G

$$\ll\Gamma\gg \propto \int dG\, e^{-S_{YM}[G]} \prod_j \Delta(m_j|G)\, \hat{\Gamma}$$

with the fermionic determinant $\quad \Delta(m_j|G) \propto m^{|\nu[G]|} \prod_{n>0} (m_j^2 + \lambda_n{}^2)$

Order parameters and IR Dirac spectrum

In the limit where $L \to \infty$ then $m \to 0$

$$\Sigma(N_f) = \lim \frac{1}{L^4} \ll \int dx \; TrS_D(x, x|G) \gg = \lim \frac{1}{L^4} \ll \sum_n \frac{m}{m^2 + \lambda_n^2} \gg$$

Scalar density $\bar{q}q$	Eigenvalue density $\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n[G])$
$\Sigma(N_f)$	Average e.v. density around 0
$\langle(\bar{u}u)(\bar{s}s)\rangle$	Fluctuation of e.v. density around 0

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For order parameters dominated by lowest Dirac e.v. like $\Sigma(N_f)$

- Dependence on m_s through fermionic determinant $\Delta(m_s|G)$
- IR end of fermionic determinant increasing function of m_s

$$\Delta_{IR}(m_s|G) = \prod_{n>0}^{K}(m_s^2 + \lambda_n{}^2)$$

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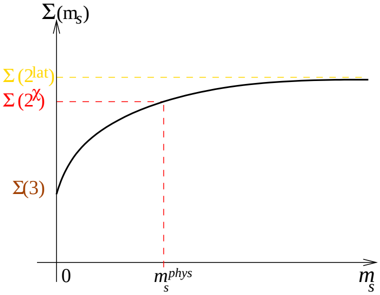
$\Sigma(3) < \Sigma(2)$ Decrease of Σ due to sea $s\bar{s}$-pairs (similar effect for $F^2 = \lim_{N_f} F_\pi^2$)

Vacuum fluctuations of $s\bar{s}$ pairs

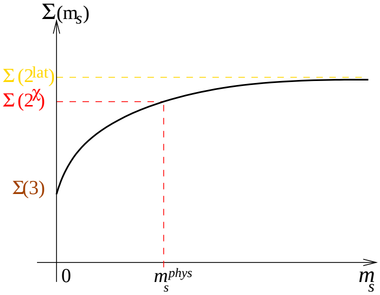


$\Sigma(2) \sim \Sigma(3) \quad + \quad \langle \bar{u}u\, \bar{s}s \rangle$

Mean & Fluctuations

of the density of Dirac eigenvalues

Vacuum fluctuations of $s\bar{s}$ pairs



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- Large average, small fluct.
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- No impact of strange sea quarks

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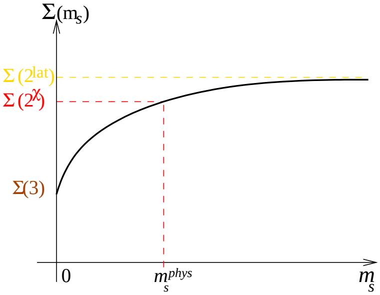

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- Large average, small fluct.
- Zweig rule not violated in 0^+
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$$\Sigma(3) \simeq \Sigma(2)$$

Near a critical point

- Small average, large fluct.
- Large violation of Zweig rule
- Strange sea quarks important

$$\Sigma(3) < \Sigma(2)$$

Dimensional estimate of NNLO remainders

- Denominator: inspired by resonance saturation $\propto 1/\Lambda_H^4$
  with Λ_H hadronic scale corresponding to exchanged resonances
- Numerator: product of $2M_\pi^2, M_K^2, F_\pi^2$ for $O(m, m_s, m_q^0)$ terms

$$d, e, d_K, e_K, d_+ = \frac{M_K^4}{\Lambda_H^4} \qquad e_+ = \frac{M_K^2 F_\pi^2}{\Lambda_H^4} \qquad e_\pi^V = \frac{6}{\langle r^2\rangle_V^\pi}\frac{M_\pi^2}{\Lambda_H^4}$$

$$d', e', d_- = \frac{2M_\pi^2 M_K^2}{\Lambda_H^4} \qquad e_- = \frac{2M_\pi^2 F_\pi^2}{\Lambda_H^4}$$

$$d_\pi = d - d' \qquad e_\pi = e - e'$$

With $\Lambda_H = 0.85$ GeV, we get

- $O(m_s^2)$ remainders of order 10%
- $O(mm_s)$ remainders of order 2%