Quantum entanglement in SU(3) lattice Yang-Mills theory at zero and finite temperatures

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— Outline —

- Entanglement entropy
- Replica method
- Some examples
- Numerical results
- Summary and outlook

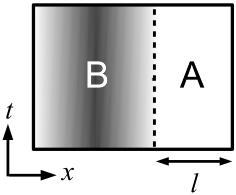
— Entanglement entropy : introduction —

- Measures how much a given quantum state is entangled quantum mechanically.
- A non-local quantity like the Wilson loop as opposed to correlation functions
- We can probe the quantum properties of the ground state for a quantum system (quantum spin system, quantum Hall liquid,. . .).
- Proportional to the degrees of freedom
- The entanglement entropy can be used as an order parameter
- Applications: quantum information and computing, condensed matter physics, ...

— Entanglement entropy : in quantum field theory —

- Divide spacetime into two regions A and B
- The reduced density matrix of the ground state

$$\rho_A = \mathrm{Tr}_B\, \rho = \mathrm{Tr}_B\, |0\rangle\langle 0|$$



- ρ_A can be regarded as the density matrix for an observer who can only access to the subsystem A.
- The entanglement entropy as the von Neumann entropy

$$S_A(l) = -\,\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

- How to calculate the entanglement entropy?
  1. replica method [Carabrese and Cardy, 2004]
  2. holography [Ryu and Takayanagi, 2006]

— Replica method —

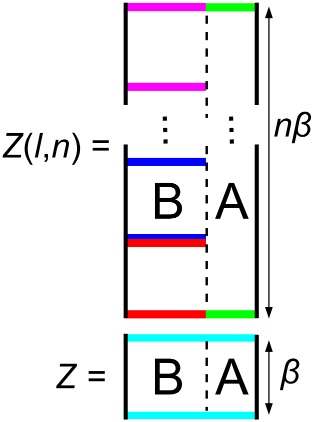
- The entanglement entropy $(\mathrm{Tr}_A\, \rho_A = 1)$

$$S_A(l) = -\mathrm{Tr}_A(\rho_A \ln \rho_A) = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \mathrm{Tr}_A\, \rho_A^n$$

- Entanglement entropy can be expressed as

$$S_A(l) = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \left(\frac{Z(l,n)}{Z^n} \right) = -\lim_{n\to 1} \frac{\partial}{\partial n} F[l,n] + F$$

where $Z(l,n)$ is the partition function on the n-sheeted Riemann surface.



— Some examples —

- For $(1+1)$-dimensional model at the critical point (CFT),

  [Holzhey, Larsen and Wilczek, 1994, Calabrese and Cardy, 2004]

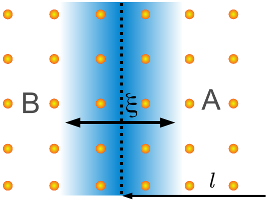
$$S_A(l) = \frac{c}{3} \log \frac{l}{a} + c_1',$$

  where c is the central charge of CFT, a the UV cutoff.

- Not in the critical regime,

$$S_A(l) \underset{l \gg \xi}{\longrightarrow} \frac{c}{3} \log \frac{\xi}{a},$$

  where ξ is the correlation length of the system.



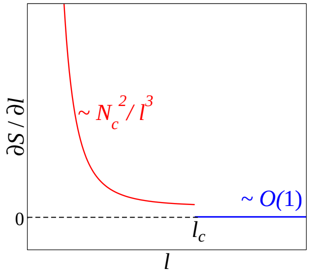
- S_A is the amount of quantum correlations between subregions.
- S_A can serve as an order parameter for a quantum phase transition.

— Entanglement entropy : holographic approach —

- In $(3+1)$-dimensional $\mathcal{N}=4$ SYM [Ryu and Takayanagi, 2006]

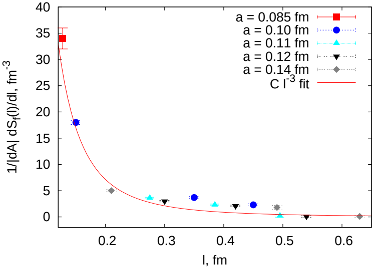
$$\frac{1}{|\partial A|}\frac{\partial S_A(l)}{\partial l} = 2N_c^2 \frac{c}{l^3}, \qquad c \simeq \begin{cases} 0.051 & \text{AdS result} \\ 0.078 & \text{free field} \end{cases}$$

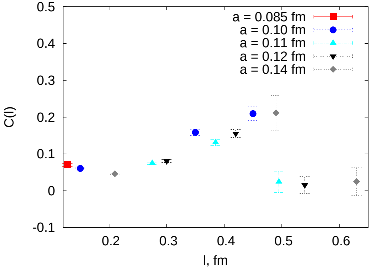
- nonanalytic behavior for confining backgrounds:
  [Klebanov, Kutasov and Murugan, 2008]
  - Klebanov-Strassler
  - D4-branes on a circle
  - D3-branes on a circle ($(2+1)$-dim. theory)
  - Soft wall model (e^{-z^2} dilaton)
- Other behavior for backgrounds:
  - Maldacena-Nunez
  - Hard wall model
  - Soft wall model (e^{-z^n}, $n<2$)



— Entanglement entropy : in lattice gauge theory —

- (Exactly solvable) $(1+1)$-dimensional $SU(N_c)$ lattice gauge theory analytic behavior (independent of l) [Velytsky, 2008]
- $(3+1)$-dimensional $SU(2)$ lattice gauge theory treated within Migdal-Kadanoff approximation
  $\Rightarrow$ nonanalytic change at $l_c T_c \in (1.56, 1.66)$ [Velytsky, 2008]
- SU(2) quenched simulation [Buividovich and Polikarpov, 2008]





— Lattice QCD simulations —

- SU(3) quenched lattice simulations

  5000 sweeps for thermalization, measurement every 100 sweeps,

  $3000 \sim 10000$ confs.

- Pseudo heat-bath MC update: Wilson plaquette action

$$S_W = \beta \sum_p \left(1 - \frac{1}{2N_c} \mathrm{Tr}(U_p + U_p^\dagger) \right)$$

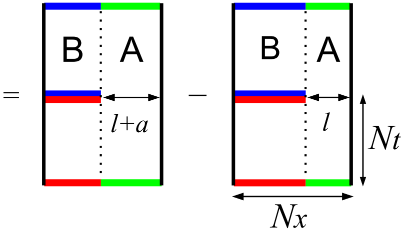
- Mesure the derivative of $S_A(l)$ with respect to l

$$\begin{aligned} \frac{\partial S_A(l)}{dl} &= \frac{\partial}{\partial l}\left[- \lim_{n\to 1} \frac{\partial}{\partial n} \ln \left(\frac{Z(l,n,T)}{Z^n(T)} \right) \right] \\ &= \lim_{n\to 1} \frac{\partial}{\partial l} \frac{\partial}{\partial n} F[l,n,T] \end{aligned}$$

— Lattice QCD simulations —

- Estimate the derivative by

$$
\begin{aligned}
\frac{\partial S_A(l)}{dl} &= \lim_{n\to 1} \frac{\partial}{\partial l}\frac{\partial}{\partial n} F[A,n] \\
&\to \frac{\partial}{\partial l} \lim_{n\to 1} \left(F[l,n+1] - F[l,n]\right) \\
&\to \frac{F[l+a,2] - F[l,2]}{a}
\end{aligned}
$$



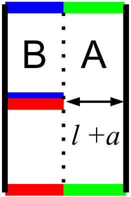
— Lattice QCD simulations —

- Differneces of free energies [Endrodi et al., PoS LAT2007]

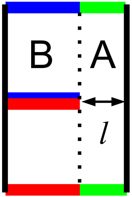
$$Z(\alpha) = \int \mathscr{D}\phi \exp\left(-(1-\alpha)S_1[\phi] - \alpha S_2[\phi]\right)$$

$$F_2 - F_1 = -\int_0^1 d\alpha \frac{\partial}{\partial\alpha} \ln Z(\alpha) = \int_0^1 d\alpha \, \langle S_2[\phi] - S_1[\phi] \rangle_\alpha$$

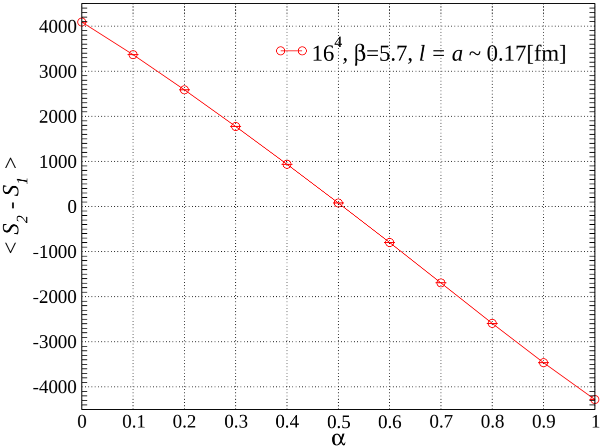
S_2 = action on



S_1 = action on

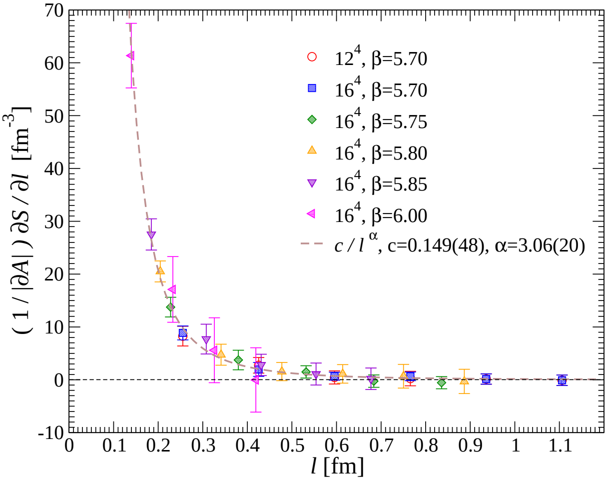


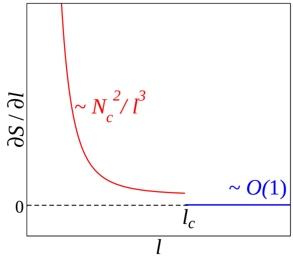
— Action difference —



- Contribution from $\alpha > 0.5$ and $\alpha < 0.5$ almost cancel.

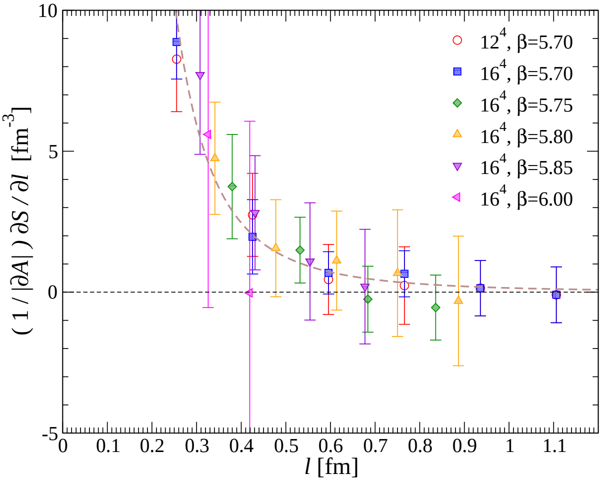
— Results: derivative of $S_A(l)$ wrt l —



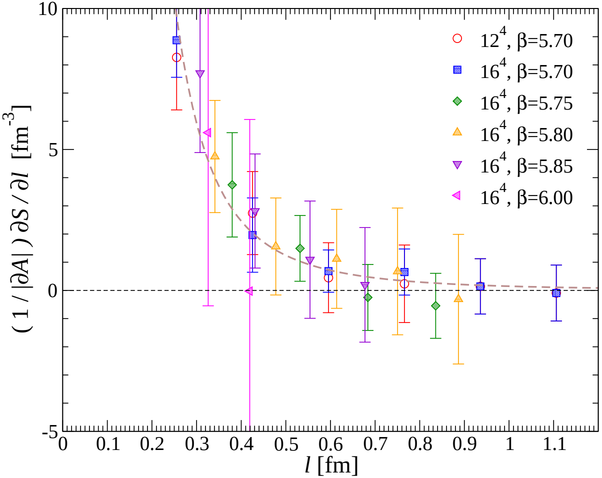


- $\partial S_A / \partial l \sim |\partial A| N_c^2 / l^3$ for $\mathcal{N} = 4$ SYM in(3+1)-dim.

— Results: derivative of $S_A(l)$ wrt l —



— Results: derivative of $S_A(l)$ wrt l —



- No clear discontinuity has been observed.

— Entanglement entropy at finite temperature —

- Entanglement entropy

$$S_A = -\,\mathrm{Tr}_A(\rho_A \ln \rho_A), \quad \rho_A = \mathrm{Tr}_B(\rho_{total}),$$

- A thermal state is a mixed state,

$$\rho_{AB} = \sum_n \exp\left(-\frac{E_n}{T}\right)|n\rangle\langle n|$$

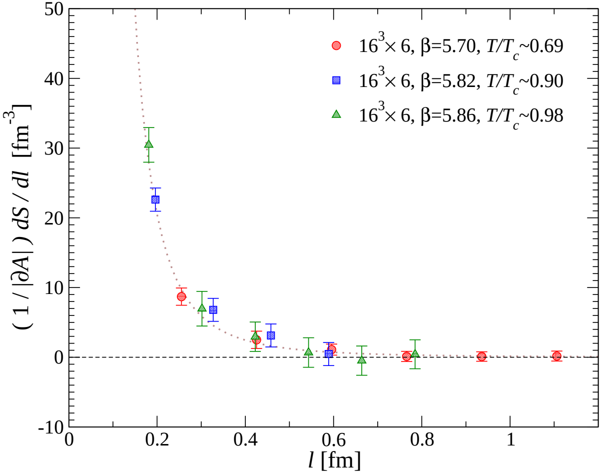
- For $(1+1)$-dimensional CFT at finite temperature $T = 1/\beta$,

$$S_A(l) = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + c_1' = \begin{cases} \frac{c}{3}\log\frac{l}{a} + c_1' & l \ll \beta, \\ \frac{\pi c}{3\beta} l + c_1' & l \gg \beta. \end{cases}$$

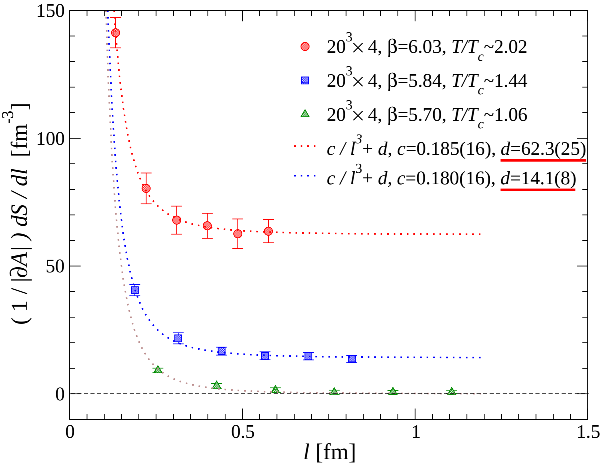
- At sufficiently large l (or in the high temperature limit),

  S_A reduces to the thermal entropy.

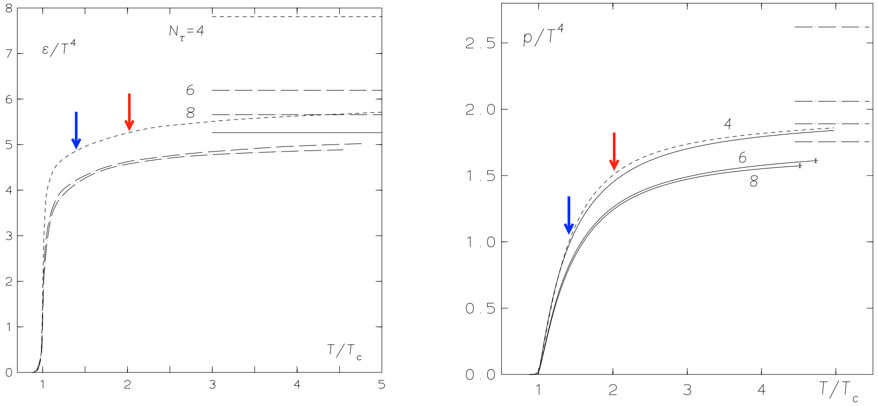
— Results: dS_A/dl at finite temperatures (below T_c) —



— Results: dS_A/dl at finite temperatures (above T_c) —



— Results: dS_A/dl at finite temperatures —



Figures taken from Boyd et al., NPB469, 419 (1996)

Rough estimates : $s \sim \begin{cases} (4.8+1.0)\times 1.44^3 \sim 17.3 & \leftrightarrow \quad 14.1(8) \\ (5.3+1.5)\times 2.02^3 \sim 56.0 & \leftrightarrow \quad 62.3(25) \end{cases}$

— Summary and outlook —

- We discussed the entanglement entropy in SU(3) pure YM theory.
- Entanglement entropy measures amount of quantum correlations between subregions
- $\partial S/\partial l$ behaves as $1/l^3$ at small l, and vanishes at large l.
- Discontinuity has not been observed at zero temperature.
- In the deconfinement phase, entanglement entropy approaches to a finite value at large l, comparable to the thermal entropy.

— Summary and outlook —

- We have to examine
  - discretization errors (with an improved gauge action)
  - systematic errors ($n > 2$ cut simulations)
- Interesting to calculate the entanglement entropy in
  - compact QED (confinement phase and Coulomb phase)
  - full QCD

— Entanglement entropy : definition —

- Divide a system into two subsystems A and B
- The density matrix on A is defined by

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}),$$

i.e., by tracing over the states of the subsystem B.

- ρ_A can be regarded as the density matrix for an observer who can only access to the subsystem A.
- Entanglement entropy as the von Neumann entropy

$$S_A = -\,\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

— Entanglement entropy : a simple example —

- Two spin 1/2 particles
- For a separable (product) state, e.g.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\!\uparrow\rangle_A + |\!\downarrow\rangle_A \Big) \otimes |\!\uparrow\rangle_B \quad \Longrightarrow \quad S_A = 0$$

- For an entangled state, e.g.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\!\uparrow\rangle_A \otimes |\!\downarrow\rangle_B + |\!\downarrow\rangle_A \otimes |\!\uparrow\rangle_B \Big) \quad \Longrightarrow \quad S_A = \ln 2$$

— Entanglement entropy : a simple example —

- Consider two spin 1/2 particles ($|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$)

$$|\psi\rangle = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B + b \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B + c \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B$$

- The density matrix $\rho_{AB} = |\psi\rangle\langle\psi|$ is given by

$$\rho_{AB} = aa^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_B + ab^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_B + \cdots$$

- The reduced density matrix $\rho_A =$ is

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ ca^* + db^* & cc^* + dd^* \end{pmatrix}$$

— Entanglement entropy : a simple example —

- The eigenvalues of the density matrix ρ_A are

$$\lambda_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4|ad - bc|^2}\right)$$

- The entanglement entropy is

$$S_A = -\operatorname{Tr}_A(\rho_A \ln \rho_A) = -\sum_i \lambda_i \ln \lambda_i$$

- For a separable (product) state, e.g.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A + |\downarrow\rangle_A\Big) \otimes |\uparrow\rangle_B \quad \Longrightarrow \quad S_A = 0$$

- For an entangled state, e.g.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\Big) \quad \Longrightarrow \quad S_A = \ln 2$$

— Analyticity of Tr ρ_A —

See [Calabrese and Cardy, quant-ph/0505193]

$$\mathrm{Tr}\,\rho_A^n = \frac{Z(l,n,T)}{Z^n(T)},$$

$$\mathrm{Tr}\,\rho_A^n = \sum_i \lambda_i, \qquad 0 \le \lambda_i < 1$$

- Tr ρ_A^n is absolutely convergent and analytic for all $\mathfrak{Re}\, n > 1$.
- The derivative with respect to n exists and analytic in the region.
- If $\rho_A = -\sum_i \lambda_i \ln \lambda_i$ is finite, then the limit as $n \to 1^+$ of the first derivative coverges to ρ_A.
- $Z(l,n,T)/Z^n(T)$ has a unique analytic continuation to $\mathfrak{Re}\, n > 1$.

— Replica method —

- The entanglement entropy

$$S_A(l) = -\,\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

can be written as ($\mathrm{Tr}_A\,\rho_A = 1$)

$$S_A(l) = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \mathrm{Tr}_A\,\rho_A^n$$

- The total density matrix ρ is ($Z(T) = \mathrm{Tr}\exp(-\beta H)$)

$$\rho[\phi''(\vec{x}), \phi'(\vec{x})] = Z^{-1}(T)\,\langle \phi''(\vec{x})|\exp(-\beta H)|\phi'(\vec{x})\rangle\,,$$

or, in the path integral expression,

$$\rho[\phi''(\vec{x}), \phi'(\vec{x})] = Z^{-1}(T)\int_{\phi(\vec{x},t=0)=\phi'(\vec{x})}^{\phi(\vec{x},t=\beta)=\phi''(\vec{x})} \mathscr{D}\phi\,\exp\left(-S_E\right)$$

— Replica method —

- The total density matrix

$$\rho[\phi''(\vec{x}),\phi'(\vec{x})] = Z^{-1}(T)\left\langle \phi''(\vec{x})\right|\exp(-\beta H)\left|\phi'(\vec{x})\right\rangle = Z^{-1}(T)\int_{\phi(\vec{x},t=0)=\phi'(\vec{x})}^{\phi(\vec{x},t=\beta)=\phi''(\vec{x})}\mathscr{D}\phi\,\exp\left(-S_E\right)$$

- $Z(T) = \mathrm{Tr}\exp(-\beta H)$ is found by setting $\phi'(\vec{x}) = \phi''(\vec{x})$ and integrating over $\phi'(\vec{x})$.
- The reduced density matrix $\rho_A[\phi''(\vec{x}),\phi'(\vec{x})] = \mathrm{Tr}_B\,\rho$ can be obtained by imposing the boundary conditions

$$\phi(\vec{x},t=0)=\phi'(\vec{x}) \quad \text{and} \quad \phi(\vec{x},t=\beta)=\phi''(\vec{x}) \quad \text{if} \quad \vec{x}\in A$$
$$\phi(\vec{x},t=0)=\phi(\vec{x},t=\beta) \quad \text{if} \quad \vec{x}\in B$$

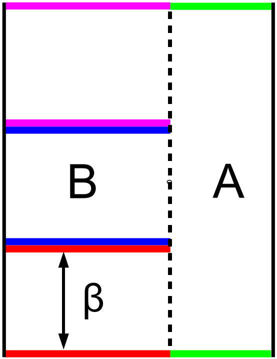
— Replica method —

- n-th power of the reduced density matrix

$$\rho_A^n[\phi''(\vec{x}), \phi'(\vec{x})] = \int_{x \in A} \mathscr{D}\phi_1 \cdots \phi_{n-1}$$
$$\rho_A[\phi''(\vec{x}), \phi_1(\vec{x})]\rho_A[\phi_1(\vec{x}), \phi_2(\vec{x})] \cdots$$
$$\times \rho_A[\phi_{n-1}(\vec{x}), \phi'(\vec{x})]$$

- The trace of ρ_A^n is found to be

$$\mathrm{Tr}\, \rho_A^n = \int_{x \in A} \mathscr{D}\phi_1 \cdots \phi_n$$
$$\rho_A[\phi_1(\vec{x}), \phi_2(\vec{x})]\rho_A[\phi_2(\vec{x}), \phi_3(\vec{x})] \cdots$$
$$\times \rho_A[\phi_n(\vec{x}), \phi_1(\vec{x})],$$



- $\mathrm{Tr}\, \rho_A^n$ is obtained by imposing the periodic boundary condition in time with period $n\beta$ if $x \in A$ and with period β if $x \in B$.

— Some examples —

- For $(1+1)$-dimensional CFT in a finite system of the length L, [Calabrese and Cardy, 2004, Korepin, 2004]

$$S_A(l) = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi l}{L} \right) + c_1'.$$

- For $(1+1)$-dimensional CFT at finite temperature $T = 1/\beta$,

$$S_A(l) = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi l}{\beta} \right) + c_1' = \begin{cases} \frac{c}{3} \log \frac{l}{a} + c_1' & l \ll \beta, \\ \frac{\pi c}{3\beta} l + c_1' & l \gg \beta. \end{cases}$$

- In the high temperature limit, S_A reduces to the thermal entropy.

— Entanglement entropy : holographic approach —

- AdS/CFT correspondence argues that the supergravity on $(d+2)$-dimensional anti-de Sitter space AdS_{d+2} is equivalent to a $(d+1)$-dimensional conformal field theory living on the boundary of AdS_{d+2} [Maldacena, 1998].
- It has been proposed that the entanglement entropy S_A in $(d+1)$-dimensional CFT can be computed from the 'area law' [Ryu and Takayanagi, 2006]

$$S_A(l) = \frac{\text{area of } \gamma_A}{4G_N^{(d+2)}},$$

where γ_A is the d-dimensional static minimal surface in AdS_{d+2} with $\partial\gamma_A = \partial A$, and $G_N^{(d+2)}$ is the $(d+2)$-dimensional Newton constant (cf. Bekenstein-Hawking formula for black hole entropy)

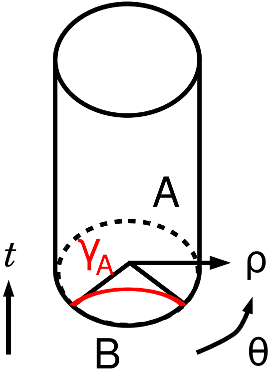
— Holographic approach : example —

- The gravitational theories on AdS_3 space of radius R are dual to $(1+1)$-dimensional CFTs with the central charge $c = 3R/2G_N^3$
- Metric of AdS_3

$$ds^2 = R^2(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\theta^2)$$

- The subsystem A is the region $0 \le \theta \le 2\pi l/L$.
- γ_A is the static geodesic which connects the boundary of A traveling inside AdS_3.
- The geodesic distance L_{γ_A} is given by

$$\cosh\left(\frac{L_{\gamma_A}}{R}\right) = 1 + 2\sinh^2\rho_0^2 \sin^2\frac{\pi l}{L}$$



— Holographic approach : example —

- Assuming $\exp(\rho_0) \gg 1$, the entanglement entropy is

$$S_A(l) \simeq \frac{c}{3} \log \left[\exp(\rho_0) \sin\left(\frac{\pi l}{L}\right) \right], \qquad \exp(\rho_0) \sim L/a,$$

which coincides with the result obtained by using the replica trick.

- In $(3+1)$-dimensional $\mathcal{N} = 4$ SYM considered to be dual to $\mathrm{AdS}_5 \times \mathrm{S}^5$ background in type IIB string theory [Ryu and Takayanagi, 2006]

$$\frac{1}{|\partial A|} S_A(l) = c \frac{N_c^2}{a^2} - c' \frac{N_c^2}{l^2},$$

$$c' \simeq \begin{cases} 0.051 & \text{AdS result} \\ \underbrace{(2+6) \times 0.0049}_{\text{gauge + real scalar}} + \underbrace{4 \times 0.0097}_{\text{Majorana}} = 0.078 & \text{free field} \end{cases}$$

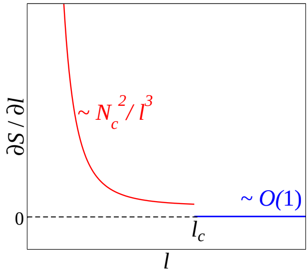
— holographic approach : confining background —

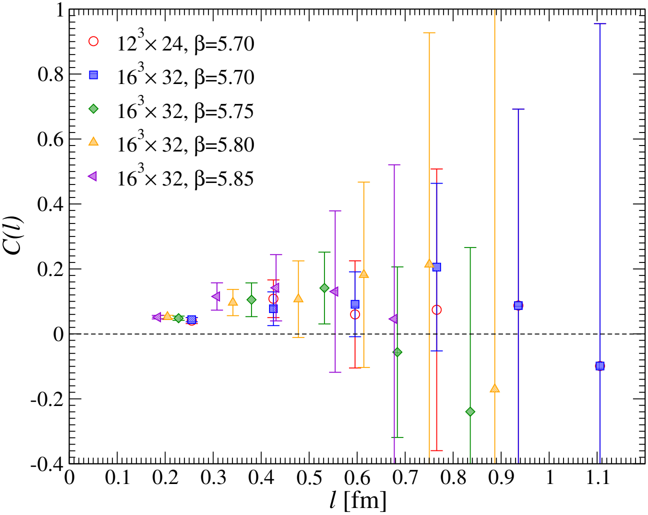
- Generalization of the 'area law' formula to non-conformal theories

  [Klebanov, Kutasov,and Murugan, 2008]

$$S_A = \frac{1}{4G_N^{(10)}} \int d^8\sigma e^{-2\phi} \sqrt{G_{\text{ind}}^{(8)}}.$$

- nonanalytic behavior for backgrounds:
  - Klebanov-Strassler
  - D4-branes on a circle
  - D3-branes on a circle ((2 + 1)-dim. theory)
  - Soft wall model (e^{-z^2} dilaton)
- Other behavior for backgrounds:
  - Maldacena-Nunez
  - Hard wall model
  - Soft wall model ($e^{-z^n}, n < 2$)





$$C(l) = \frac{l^3}{|\partial A|} \frac{\partial S_A(l)}{\partial l}$$