Matching the lattice coupling to the continuum for the tree level Symanzik improved gauge action

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Motivations

In order to compute three loop renormalization constants we can rely on continuum computation of anomalous dimension, provided we have the matching of the lattice coupling constant to the one in the continuum. Unfortunately, for Symanzik action is known only at one loop (we need two loops).

Power series in the bare coupling of physical quantities are poorly convergent. Using a renormalized coupling constant enhances the predictive power of lattice perturbation theory.

More generally, this is an example of matching between different renormalization schemes which entails determination of β-function, a topical interest nowadays.

Matching different renormalization schemes requires a matching of coupling constants. Two loop form of the expansion is familiar:

$$\alpha(s\mu) = \alpha'(\mu) + c_1(s)\alpha'(\mu)^2 + c_2(s)\alpha'(\mu)^3 + \dots$$

with

$$\begin{aligned} c_1(s) &= -2b_0 \log s + 2b_0 \log \frac{\Lambda}{\Lambda'} \\ c_2(s) &= c_1(s)^2 - 2b_1 \log s + 2b_1 \log \frac{\Lambda}{\Lambda'} + \frac{b_2' - b_2}{b_0} \end{aligned}$$

While b_0 and b_1 are universal, b_2 and the Λ's are scheme dependent!

This is the link with the β-function: knowledge of these quantities in different schemes allow to change from one to another.

Tree Level Symanzik improved action

We are interested in computing the matching between $\alpha_{\overline{MS}}$ and α_0 for TL Symanzik improved gauge action/Wilson $n_f = 2$ fermions.
We recall that this gauge action is defined as

$$S_G = \frac{1}{g^2}\left[C_0 \sum_{plaquette} \mathrm{Tr}U_{pl} + C_1 \sum_{rectangle} \mathrm{Tr}U_{rtg}\right]$$

where $C_1 = -\frac{1}{12}$ and $C_0 = 1 - 8c_1$.

Our strategy: we exploit the knowledge of the matching between α_V and $\alpha_{\overline{MS}}$, we numerically compute the matching between α_V and α_0 and we finally perform the matching between $\alpha_{\overline{MS}}$ and α_0:

$$\begin{aligned}
\alpha_V &= \alpha_0 + c_1\alpha_0^2 + c_2\alpha_0^3 \\
&= \alpha_{\overline{MS}} + d_1\alpha_{\overline{MS}}^2 + d_2\alpha_{\overline{MS}}^3 \\
\alpha_{\overline{MS}} &= \alpha_0 + (d_1 - c_1)\alpha_0^2 + (2c_1^2 - c_2 - 2c_1d_1 + d_2)\alpha_0^3.
\end{aligned}$$

Static quark potential

Static quark potential is the quantity which describes the interaction energy of an infinitely heavy $q\bar{q}$ pair.
It consist of 2 terms, a short distance Coulomb potential and a long distance term responsible for confinement.

$$V_{QCD}(R) = -C_F \frac{\alpha(R)}{R} + \sigma R$$

If we restrict to the perturbation theory, only the Coulomb term survives

$$V(R) = -C_F \frac{\alpha_V(R)}{R}$$

In doing this, we have defined a new renormalized coupling $\alpha_V(R)$.

One can show that $\alpha_V(R)$ is a good expansion parameter for physical quantities, that is, high order perturbative coefficients are small.

The static quark potential is related to Wilson loop $W(R,T) = C \exp^{-TV(R)}$. To avoid corner divergencies we extract the potential according to

$$\begin{aligned} V(R) &= lim_{T\to\infty} \left[\log \left(\frac{W(R,T-1)}{W(R,T)} \right) \right] \\ &= 2\delta m - C_F \frac{\alpha_V(R)}{R} \end{aligned}$$

where δm is the residual mass of the heavy quark.
We can the express the potential in terms of the lattice coupling. This reads

$$V(R) = 2\delta m - \frac{C_F}{R} \left(\alpha_0 + c_1(R)\alpha_0^2 + c_2(R)\alpha_0^3 + \ldots \right)$$

where the coefficients c_1 and c_2 are given by

$$\begin{aligned} c_1(R) &= 2b_0 \log R + 2b_0 \log \frac{\Lambda_V}{\Lambda_0} \\ c_2(R) &= c_1(R)^2 + 2b_1 \log R + 2b_1 \log \frac{\Lambda_V}{\Lambda_0} + \frac{b_2^{(V)} - b_2^{(0)}}{b_0} \end{aligned}$$

and δm is expanded in α_0 as well:

$$\delta m = \sum_{n\geq 0} \delta m^{(n)} \alpha_0^n$$

Numerical Stochastic Perturbation Theory

Numerical Stochastic Perturbation Theory is numerica implementation of Stochastic Quantization.
Introducing a new degree of freedom (t), a quantum field described by the action S evolves according to Langevin equation

$$\frac{\partial}{\partial t}\phi_\eta(x,t) = -\frac{\delta S[\phi]}{\delta \phi_\eta(x,t)} + \eta(x,t).$$

One recovers canonical quantization in the limit $t \to \infty$:

$$\lim_{t\to\infty} \langle \phi(x_1,t)\dots\phi(x_n,t)\rangle_\eta = \langle \phi(x_1)\dots\phi(x_n)\rangle.$$

If $\phi(x,t)$ is expanded in series, $\phi_\eta(x,t) = \phi_\eta^{(0)}(x,t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x,t)$ one obtain a hierarchy of equations to numerically integrate.
Any observable is in turn expanded as a power series

$$O\left[\sum_n g^n \phi_\eta^{(n)}(x,t)\right] = \sum_n g^n O^{(n)}(x,t).$$

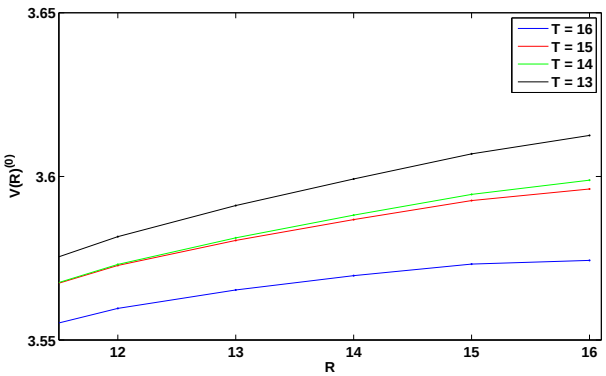
Since the Langevin equation is a continous time equation and the numerical integration requires a discrete timestep τ, an extrapolation is required $\tau \to 0$.

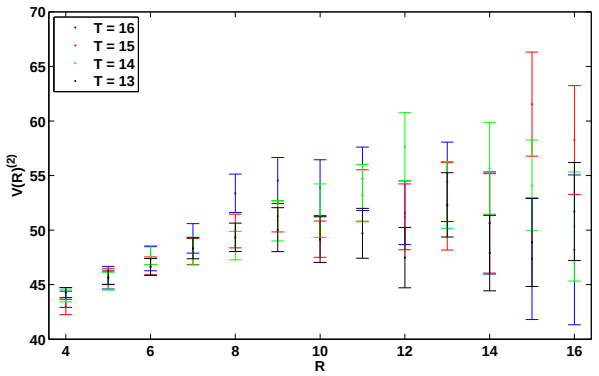
Details of the computation

To extract the potential we computed all $W(R,T)$ for $R, T \leq 16$ over ~ 150 configurations 32^4 at three different stochastic time steps. The extrapolation errors at $\tau = 0$ are included in the indetermination of the results.

Given the finite lattice size nature of our computation, we can not actually take the $T \to \infty$ limit and the results are distorted by lattice artifacts. We can anyway consider an interval of R such that $T > R$ ($T/R \sim 2.5$) but R not too small ($R \geq 3$).

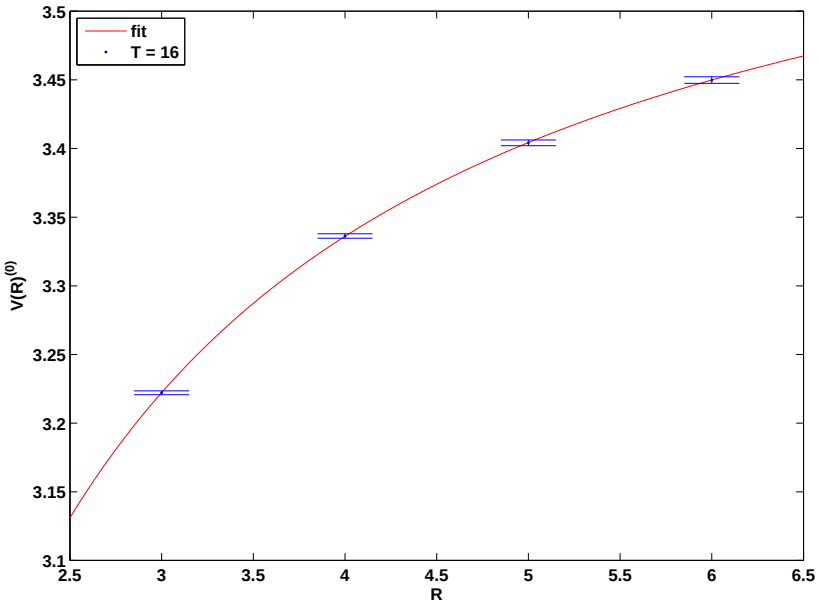
We must consider both systematic and statistical errors. When the systematic errors are clearly distinguishible from statistical ones, we consider a fixed lattice artifact approach and fit the potential for $T = 16$, otherwise we consider different values of T to improve the statistic.

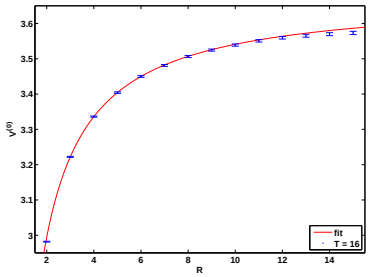




Tree level potential

We use tree level potential to verify the correctness of the tecnique and to estimate the effect of the lattice artifacts.





$$V(R)^{(0)} = \delta m^{(0)} - \frac{C_F}{R}$$

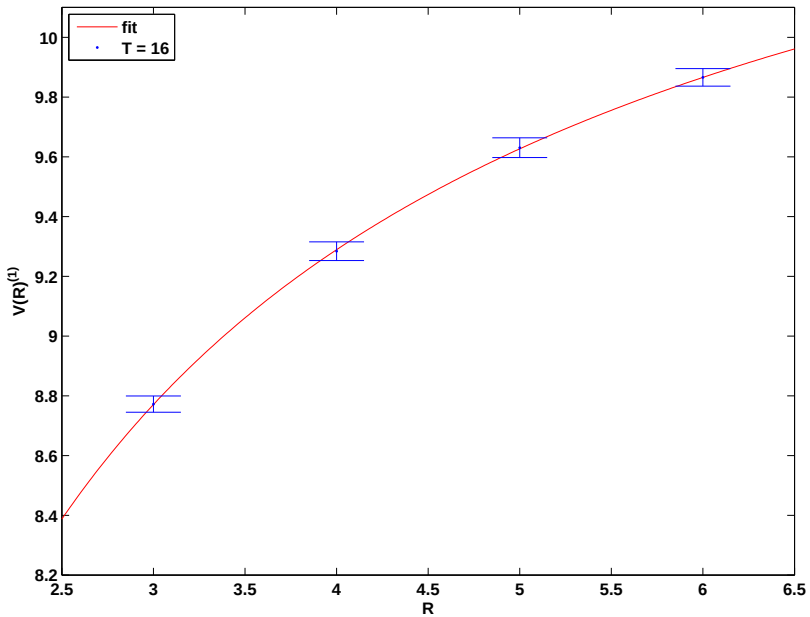
We estimated $3 \leq R \leq 6$ as best fitting interval. Precision in fitting C_F is a few percent, and

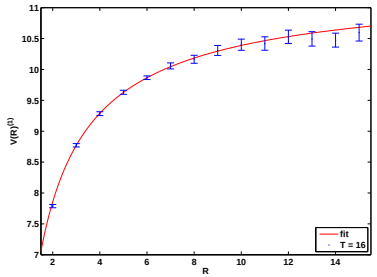
$$\delta m^{(0)} = 1.84 \pm 0.01$$

1 loop potential

At one loop we are able to extract the constant term of the matching and to compare it with analitical results.

$$V(R)^{(1)} = \delta m^{(1)} - \frac{C_F}{R} 2 b_0 \left(\log R + \log \frac{\Lambda_V}{\Lambda_0} \right)$$





Analitical result gives

$$\log \frac{\Lambda_V}{\Lambda_0} = 2.8191.$$

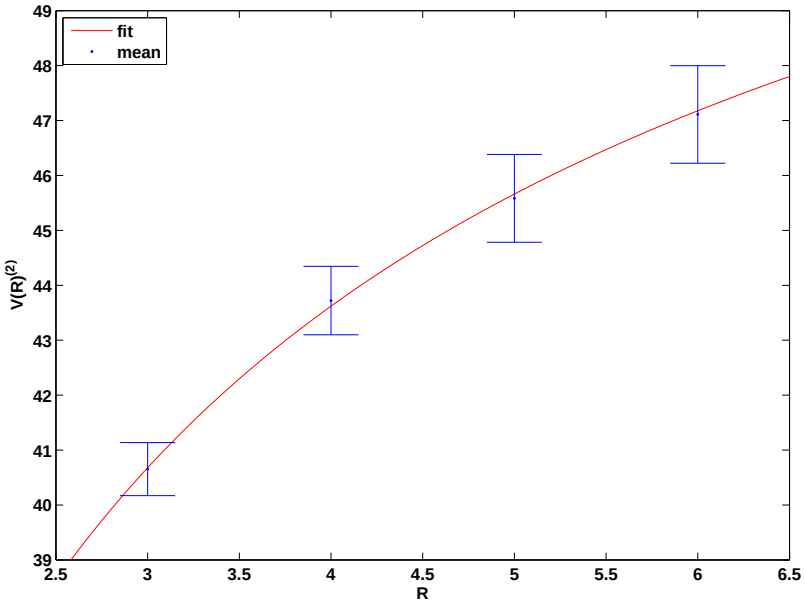
Results of the fit procedure are

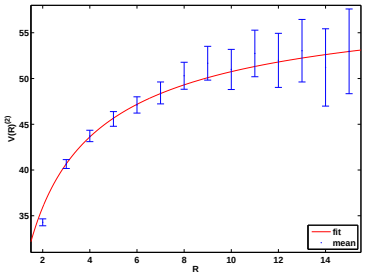
$$\delta m^{(1)} = 5.71 \pm 0.01 \qquad \log \frac{\Lambda_V}{\Lambda_0} = 2.8 \pm 0.1$$

2 loop potential

Extracting the second loop constant term requires the knowledge of $\log \frac{\Lambda_V}{\Lambda_0}$ that we get from one loop calculation, or from analitical results.

$$V(R)^{(2)} = \delta m^{(2)} - \frac{C_F}{R}\left(c_1 (R)^2 + 2 b_1 \log R + 2 b_1 \log \frac{\Lambda_V}{\Lambda_0} + \frac{b_2^{(V)} - b_2^{(0)}}{b_0} \right)$$





From the fit we obtain

$$\begin{aligned} \delta m^{(2)} &= 30 \pm 1 \\ \frac{b_2^{(V)} - b_2^{(0)}}{b_0} &= 4 \pm 1 \end{aligned}$$

Conclusions

We are interested in matching $\alpha_{\overline{MS}}$ and α_0 up to two loop for Symanzik improved gauge/Wilson $n_f = 2$ fermions. The strategy is to compute the matching between α_V and α_0 and to exploit the knowledge of the matching $\alpha_V \to \alpha_{\overline{MS}}$.
Current computation has been performed on 32^4 configurations generated years ago, and we will pass to bigger volumes.
We verified correctness of the method for known results ($\log\left(\Lambda_{\overline{MS}}/\Lambda_0\right)$) and give a rough estimate for the coefficient

$$\frac{b_2^{(\overline{MS})} - b_2^{(0)}}{b_0}.$$

Nevertheless, the relative error is not that bad when computing ($n_f = 2$)

$$\alpha_{\overline{MS}} = \alpha_0 + 2.8037\alpha_0^2 + 11.5(\pm 1.0)\alpha_0^3$$