How the Quark Number fluctuates in QCD at small chemical potential

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What: The distribution $\langle \delta(n - n') \rangle$ of n

Why: Understand how $\langle n \rangle$ builds up

How: Analytically in Chiral Perturbation Theory

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Why: Understand how $\langle n\rangle$ builds up

How: Analytically in Chiral Perturbation Theory

Shows: How Complex Langevin solves sign problems

Pions have zero baryon charge

- so how can CPT teach us about n *?*

Certainly in CPT

$$\langle n \rangle = 0$$

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But $\langle \delta(n - n') \rangle$ is non trivial in CPT

Warning: $\langle\delta(n - n')\rangle$ is the distribution of n over A_ν

$$n \equiv \frac{d}{d\mu} \log \det(D + \mu\gamma_0 + m)$$

The average quark number, $\langle n_q \rangle$, is

$$\langle n_q \rangle \equiv \langle n \rangle$$

The average of the square of the quark number

$$\langle n_q^2 \rangle = \frac{1}{Z}\,\frac{d^2}{d\mu^2}\,Z = \langle n^2 \rangle + \langle \left(\frac{dn}{d\mu}\right) \rangle$$

The average of the square of n

$$\langle n^2 \rangle = \frac{1}{Z}\,\frac{d}{d\mu_u}\frac{d}{d\mu_d}\,Z\bigg|_{\mu_u=\mu_d=\mu}$$

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$\langle n_q^2 \rangle$ not the second moment of $\langle \delta(n - n') \rangle$

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$$\langle n^2 \rangle = \frac{d^2}{d\mu_1 d\mu_2} G_0(\mu_1, \mu_2)\Big|_{\mu_1=\mu_2=\mu} \neq 0$$

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1-loop free energy

$$G_0(\mu_1, \mu_2) = V \frac{m_\pi^2 T^2}{\pi^2} \sum_{n=1}^{\infty} \frac{K_2(\frac{m_\pi n}{T})}{n^2} \cosh(\frac{\mu_1 - \mu_2}{T} n)$$

Fluctuations in the complex n plane

$$n(\mu)^* = \left(\mathrm{Tr}\frac{\gamma_0}{D + \mu\gamma_0 + m}\right)^* = -n(-\mu)$$

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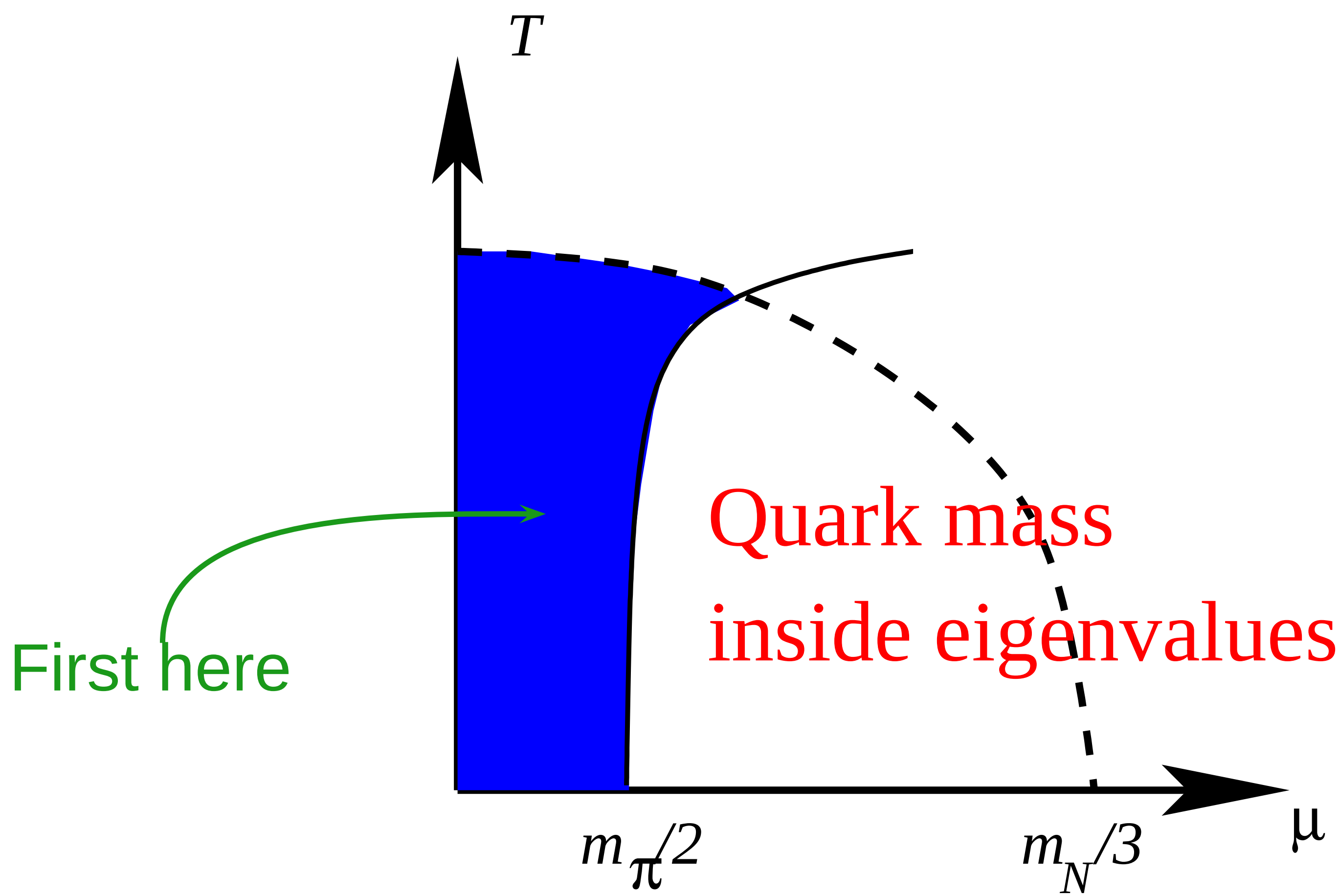
$$P_n(x,y) \equiv \langle \delta\left(x - \mathrm{Re}[n]\right)\delta\left(y - \mathrm{Im}[n]\right)\rangle$$

Compute all moments $\langle \text{Re}[n]^k \text{Im}[n]^j \rangle$ in CPT


T
Quark mass
inside eigenvalues
First here
μ
$m_\pi/2$
$m_N/3$

The n distribution for $\mu < m_\pi/2$ $N_f = 2$

Factorization at 1-loop

$$P_n(x, y) = P_{\mathrm{Re}[n]}(x) P_{\mathrm{Im}[n]}(y)$$

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The n distribution for $\mu < m_\pi/2$ $\quad N_f = 2$

Factorization at 1-loop

$$P_n(x,y) = P_{\mathrm{Re}[n]}(x) P_{\mathrm{Im}[n]}(y)$$

$$P_{\mathrm{Re}[n]}(x) \simeq e^{-(x-\nu_I)^2/(\chi^B_{ud}+\chi^I_{ud})} \qquad P_{\mathrm{Im}[n]}(y) \simeq e^{(iy+\nu_I)^2/(\chi^I_{ud}-\chi^B_{ud})}$$

Note that $P_{\mathrm{Im}[n]}(y)$ takes complex values (the sign problem)

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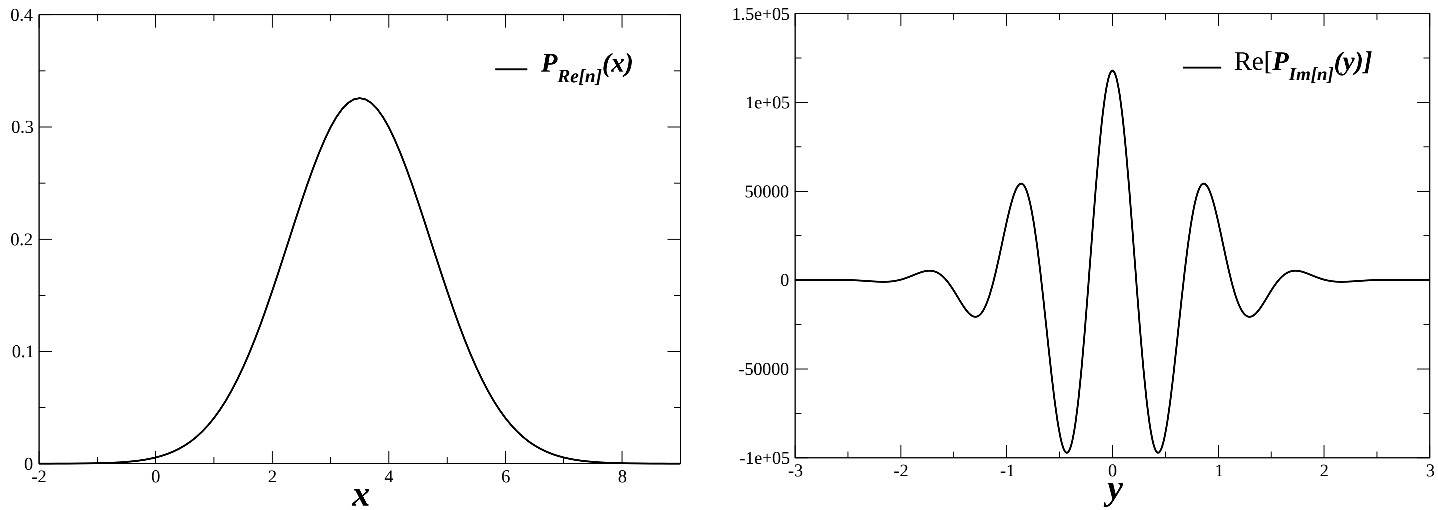

$P_{\mathrm{Im}[n]}(y)$: Amplitude$\sim e^V$ Width$\sim \sqrt{V}$ Period~ 1

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The expectation value of the quark number is zero in CPT

$$
\begin{aligned}
\langle n \rangle &= \int dxdy\ (x+iy)P_n(x,y) \\
&= \int dx\ xP_{\mathrm{Re}[n]}(x) + i\int dy\ yP_{\mathrm{Im}[n]}(y) \\
&= \nu_I + ii\nu_I = 0
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Detailed cancellation between the contribution from the real part and the imaginary part

Complex Langevin

The CL action for $y = Im[n]$ ($N_f = 2$)

$$S = -\log[P_{\mathrm{Im}[n]}(y)]$$

Complexify $\mathrm{Im}[n]$: $y = a + ib$

Aarts personal correspondence (2009)
de Forcrand PoS (LAT2009) 10, arXiv:1005.0539

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CL works perfectly !

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Complex Langevin

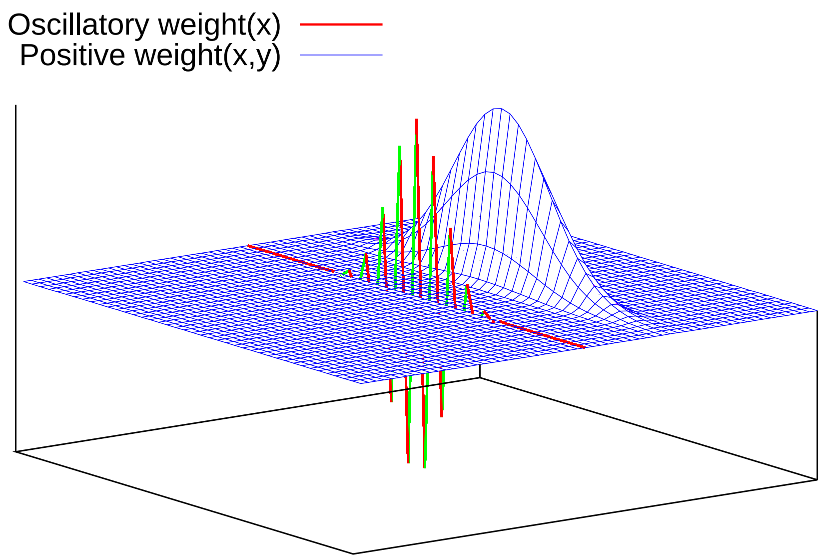

Illustration by Philippe de Forcrand

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Complex Langevin

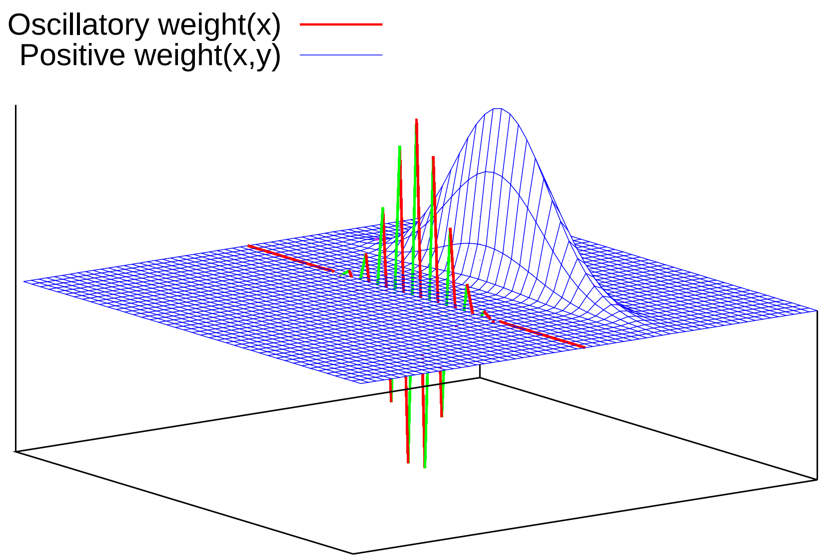

Illustration by Philippe de Forcrand

CPT tells us:

1) Amplitude: real axis $\mathcal{O}(\exp(V))$; complex plane $\mathcal{O}(1)$
2) shift by $\mathcal{O}(V)$ in imaginary direction

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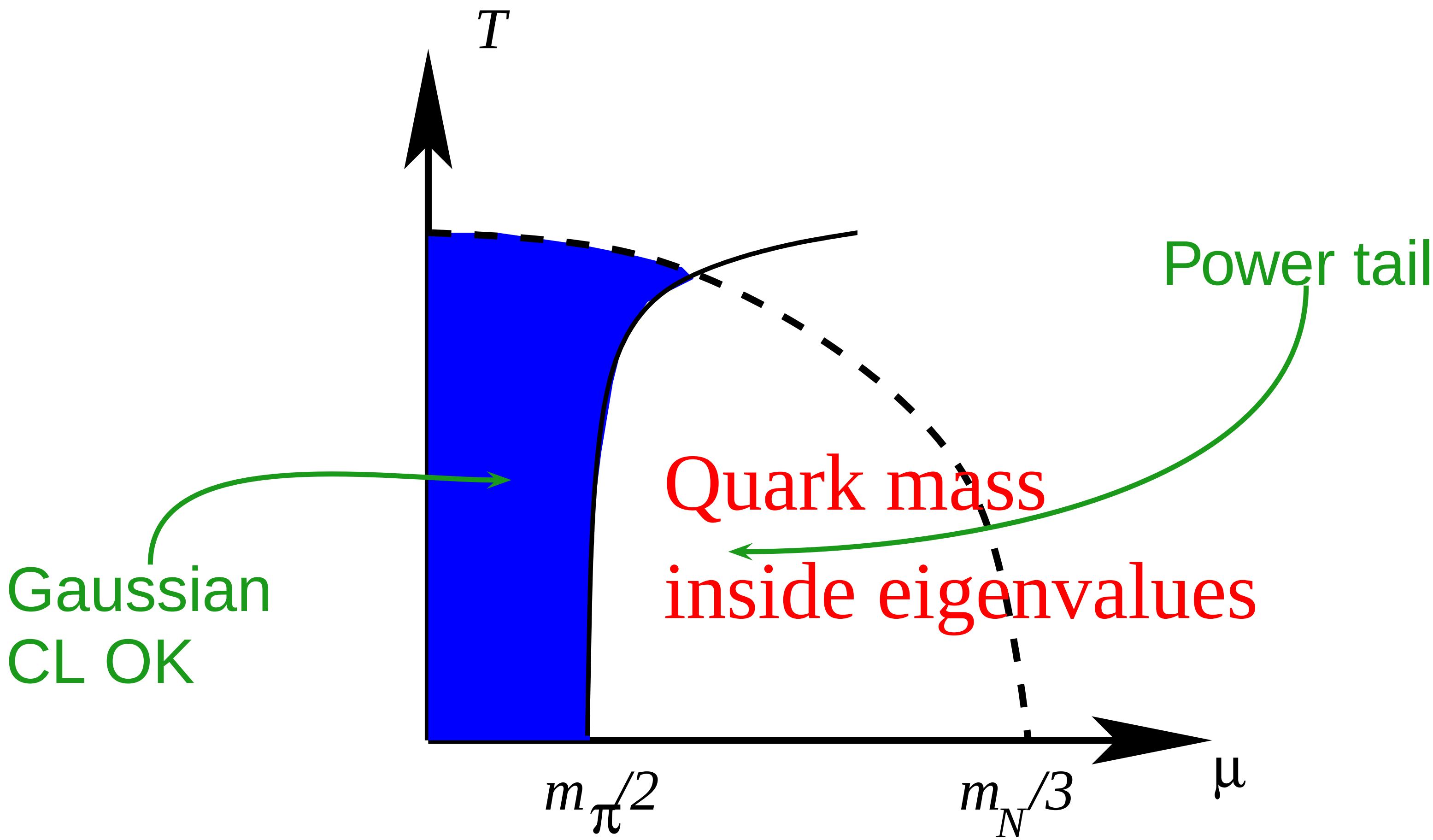
n-distribution for $\mu > m_\pi/2$



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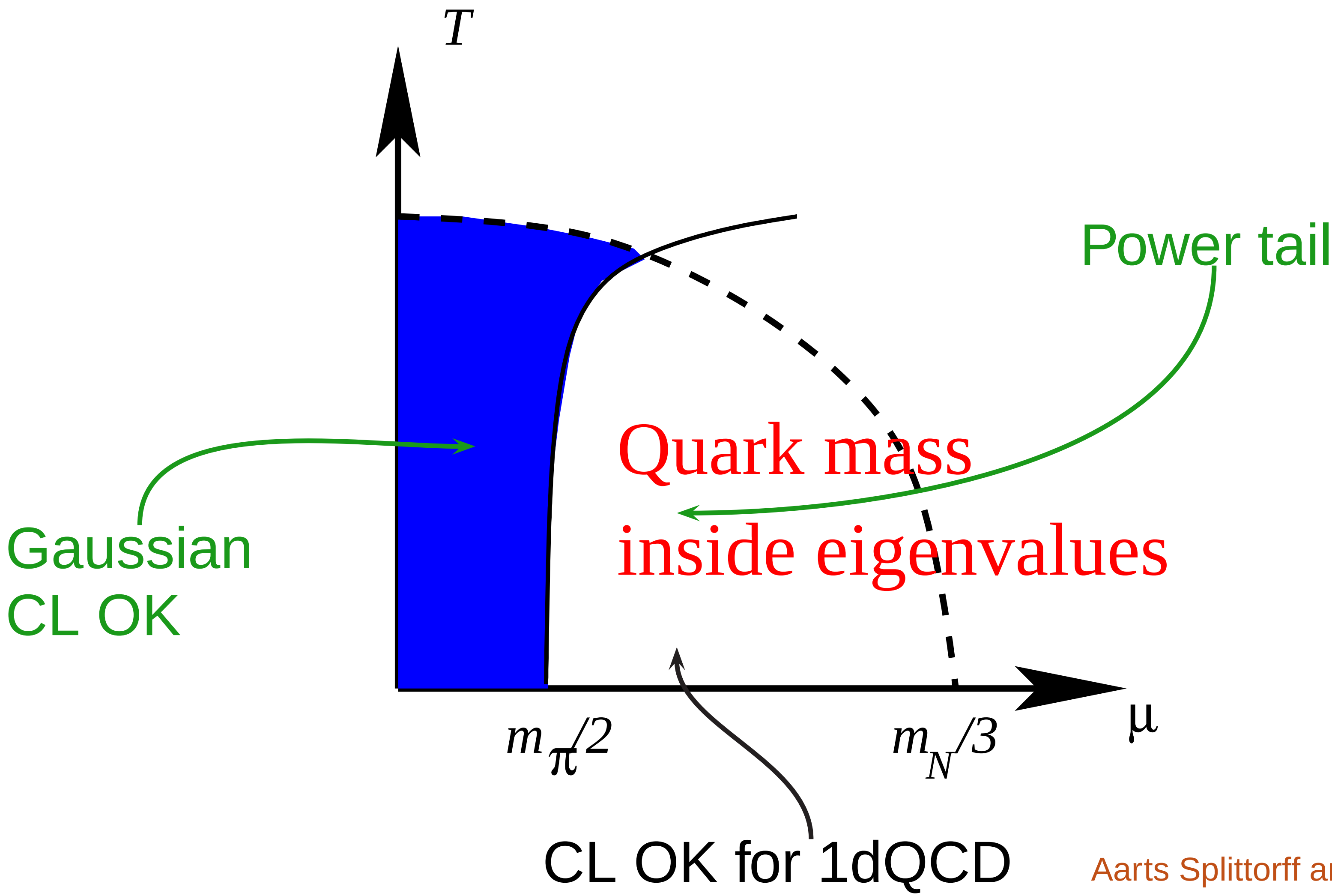

Aarts Splittorff arXiv:1006.0332

Conclusions

Interplay between lattice QCD and analytic QCD is essential to understand QCD at $\mu \neq 0$

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Interplay between lattice QCD and analytic QCD is essential to understand QCD at $\mu \neq 0$

Here:

- Derived the distribution of n from CPT
- Studied how $\langle n \rangle$ becomes zero (cancellations)
- Directly linked Complex Langevin

Additional slides

The sign problem

$$Z_{1+1} = \int dA \, \det{}^2(D + \mu\gamma_0 + m) \, e^{-S_{\rm YM}}$$

Anti Hermitian

Hermitian

$$\det{}^2(D + \mu\gamma_0 + m) = \left|\det(D + \mu\gamma_0 + m)\right|^2 e^{2i\theta}$$

The measure is not real and positive

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The measure is not real and positive

No Monte Carlo sampling of A_η *at* $\mu \neq 0$

In terms of the eigenvalues, z_k, of $\gamma_0(D+m)$

$$
\begin{aligned}
n_q &= n = \sum_k \frac{1}{z_k+\mu} \\
n_q^2 &= \sum_{k\neq l} \frac{1}{z_k+\mu}\frac{1}{z_l+\mu} \\
n^2 &= \sum_{k,l} \frac{1}{z_k+\mu}\frac{1}{z_l+\mu} = [\sum_k \frac{1}{z_k+\mu}]^2
\end{aligned}
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$\langle n_q^2\rangle$ not the average of a square

not the second moment of a distribution

How large should y_{max} be in order that

$$\int_{-y_{max}}^{y_{max}} dy\ iy P^{1+1}_{\mathrm{Im}[n]}(y) \sim -\nu_I$$

The answer is:

$$y_{max} \sim \nu_I \sim V$$

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The answer is:

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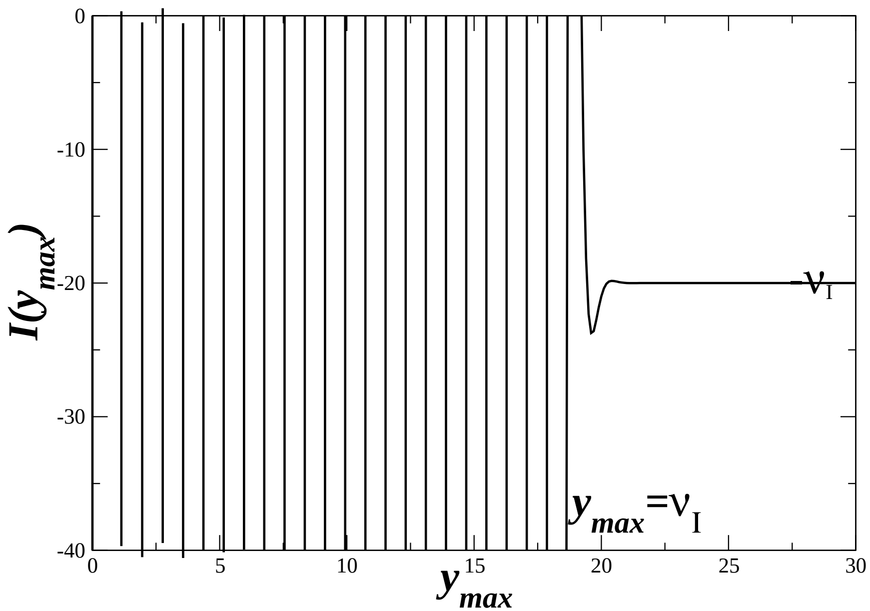
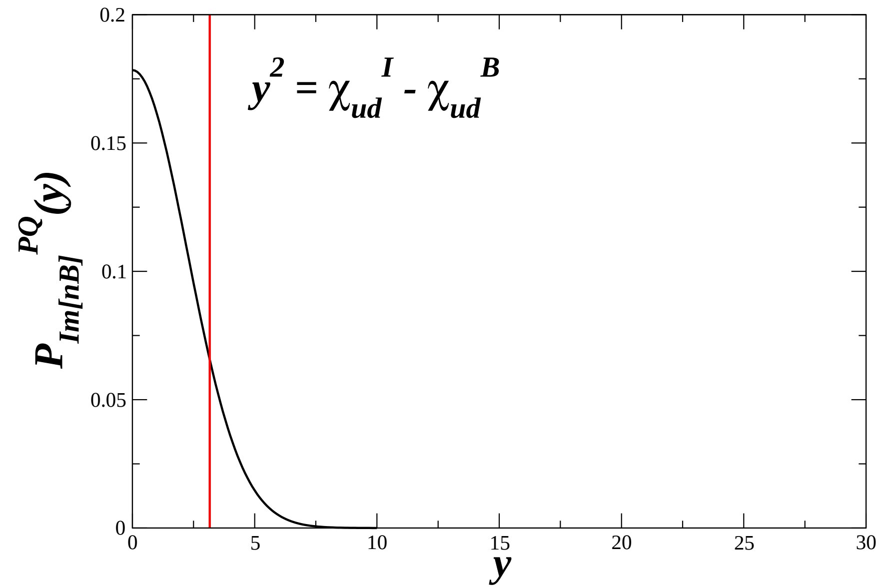
Observation: We must integrate over V periods of the oscillations in order to obtain the density

The range needed and the $\mathrm{Im}[n]$ generated ($\mu < m_\pi/2$)





$$\int_{-y_{max}}^{y_{max}} dy\ iy P^{1+1}_{\mathrm{Im}[n]}(y) \sim -\nu_I$$

$\langle\delta(n-n')\rangle$ shows the pion noise

$$\langle n\rangle = \int dn'\; n'\; \langle\delta(n-n')\rangle$$

$\langle \delta(n - n') \rangle$ shows the pion noise

$$\langle n \rangle = \int dn' \, n' \, \langle \delta(n - n') \rangle$$

The sign problem and Complex Langevin

Notation

$$\nu_I \equiv \frac{d}{d\mu_1}\Delta G_0(\mu_1, -\mu)\Big|_{\mu_1=\mu}$$

$$\chi^B_{ud} \equiv \frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(\mu_1, \mu_2)\Big|_{\mu_1=\mu_2=\mu}$$

$$\chi^I_{ud} \equiv \frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(-\mu_1, \mu_2)\Big|_{\mu_1=\mu_2=\mu}$$

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$$\Delta G_0(\mu_1,\mu_2) = V\frac{m_\pi^2 T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\left[\cosh(\frac{\mu_1-\mu_2}{T}n)-1\right]$$