A study of $\mathcal{N}$=2 Landau-Ginzburg model by lattice simulation based on a Nicolai map

based on arXiv:1005.4671

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1 Purpose

2d $\mathcal{N}$=2 Landau-Ginzburg model (LG model)

$$S = \int d^2x d^4\theta\, K(\Phi,\bar{\Phi}) + \left(\int d^2x d^2\theta\, W(\Phi) + c.c. \right)$$

$\Phi \cdots$ chiral superfield

At the IR fixed point, $W(\Phi) = \lambda\Phi^k$ is believed to describe...

$\lambda_{\text{eff}} \to \infty$, lattice !

$\left\{ \begin{array}{l} \mathcal{N}=2 \text{ minimal model} \\ \hookrightarrow \text{Gepner model (compactified string), ...} \end{array} \right.$ $\leftarrow$ check for $K(\Phi,\bar{\Phi}) = \bar{\Phi}\Phi$ (WZ model)

Why it is believed that LG models describe CFTs ?

- 2d bosonic case '86 A.B.Zamolodchikov

In the $c = 1 - \frac{6}{p(p+1)}$ minimal model, the fusion rule implies $\cdots\ \phi_{(2,2)}^{2p-3} \propto \partial^2\phi_{(2,2)}$

In the 2d bosonic LG model $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + g\phi^{2p-2}$, EOM is $\cdots\ \phi^{2p-3} \propto \partial^2\phi$

$\overset{\text{conjecture}}{\Rightarrow}$ $\phi = \phi_{(2,2)}$ at the IR fixed point.

How to check the conjecture

- early studies

RG flow of c-functions	→ '89 Kastor, Martinec and Shenker
catastrophe theory	→ '89 Vafa and Warner
ϵ-expansion	→ '89 Howe and West
elliptic genus, SCA	→ '93 Witten
...	

→ For $W(\Phi) = \lambda\Phi^k$,

$$\begin{cases} c = 3(1 - \frac{2}{k}) \\ \Phi : (h, \bar{h}) = (\frac{1}{2k}, \frac{1}{2k}) \\ \Phi^2 : (h, \bar{h}) = (\frac{2}{2k}, \frac{2}{2k}) \\ \vdots \\ \Phi^{k-2} : (h, \bar{h}) = (\frac{k-2}{2k}, \frac{k-2}{2k}) \end{cases}$$

- We computed **correlation functions** non-perturbatively for $W(\Phi) \propto \Phi^3$.

susceptibility of CFT:

$$\chi \equiv \int d^2x \langle \phi(x)\phi^*(0) \rangle \xrightarrow{\text{finite volume}} \int_V d^2x \frac{1}{|x|^{2h+2\bar{h}}} \propto V^{1-h-\bar{h}}$$

$$\Rightarrow \log \chi = (1 - h - \bar{h}) \log \mathrm{V} + \text{const.}$$

For the present $W(\Phi) \propto \Phi^3$, $1 - h - \bar{h} = 1 - \frac{1}{6} - \frac{1}{6} =$ **0.666...**

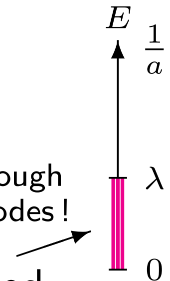
2 Lattice Formulation of WZ model

Relying on the existence of the Nicolai map as the guiding principle,

'83 Sakai and Sakamoto
'02 Catterall and Karamov
'02 Kikukawa and Nakayama

$$\mathcal{S} = \sum \left\{ \phi^* T \phi + W^* (1 - \frac{a^2}{4} T) W + \left(W'(-S_1 + iS_2)\phi + c.c. \right) + \bar{\psi} \left(D + \frac{1+\gamma_3}{2} W'' \frac{1+\hat{\gamma}_3}{2} + \frac{1-\gamma_3}{2} W''^* \frac{1-\hat{\gamma}_3}{2} \right) \psi \right\}$$

where $D = \frac{1}{2}\left[1 + \frac{X}{\sqrt{X^\dagger X}} \right] = T + \gamma_1 S_1 + \gamma_2 S_2, \quad W = \frac{\lambda}{3}\Phi^3$



- λ is the unique mass parameter (besides a) $\Rightarrow$ { continuum limit : $a\lambda \to 0$; To see CFT, $L \gg (a\lambda)^{-1}$ is needed. }

- no extra fine-tunings $\Leftarrow$ { ◎ one SUSY Q $\leftarrow$ Nicolai map; ◎ Z_3 R-symmetry $\leftarrow$ overlap fermion }

- This lattice model faces the sign problem

  $|D+F|$ is real, but can be negative. $\Leftarrow \gamma_1 (D+F) \gamma_1 = (D+F)^*$

3 Simulation Method

We utilized the Nicolai map : $\eta = W' + (\phi - \frac{a}{2}W')T + (\phi^* - \frac{a}{2}W^{*\prime})(S_1 + iS_2)$.

$$\langle \mathcal{O} \rangle = \frac{\langle \sum_{i=1}^{N(\eta)} \mathcal{O}(\phi_i) \mathrm{sgn}\,|D + F(\phi_i)| \rangle_\eta}{\langle \sum_{i=1}^{N(\eta)} \mathrm{sgn}\,|D + F(\phi_i)| \rangle_\eta} \quad \overset{a \to 0}{\longrightarrow} \quad \textbf{Witten index } \Delta = 2 \text{ (cubic potential)}$$

$$\text{where} \begin{cases} \langle X \rangle_\eta \equiv \dfrac{\int \mathcal{D}\eta \mathcal{D}\bar{\eta}\, X\, e^{-\sum_x |\eta|^2}}{\int \mathcal{D}\eta \mathcal{D}\bar{\eta}\, e^{-\sum_x |\eta|^2}} \\ N(\eta) \ \text{ counts the solutions of the Nicolai map } \phi_1, .., \phi_{N(\eta)} \end{cases}$$

1. Assigning $\{\eta,\, \eta^*\}$ as the standard normal distribution,
2. Solving the Nicolai map by the Newton-Raphson algorithm,
3. Sample the configurations of $\{\phi,\, \phi^*\}$.

- advantage $\cdots$ no autocorrelation
- difficulty $\cdots$ $N(\eta)$

Tests for the configurations

- $\langle\sum_{i=1}^{N(\eta)}\text{sgn}|D+F|\rangle_\eta \xrightarrow{a\to 0}$ **Witten index** $\Delta = 2$ (cubic potential)

  Why Witten index ?

  $\to$ P.B.C. & For $W(\Phi)=\frac{m}{2}\Phi^2\,(\Delta=1)$, $(\mathrm{Re}\,\eta,\mathrm{Im}\,\eta)=(\mathrm{Re}\,\phi,\mathrm{Im}\,\phi)\underbrace{\left(D+m(1-\frac{a}{2}D)\right)}_{\text{positive}\ \Rightarrow\ \Delta=1\ \text{is correctly reproduced}}$

  $\to$ correctly normalized

- **Ward identities for** $\left\langle\eta(x_1)\cdots\eta(x_m)\eta^*(y_1)\cdots\eta^*(y_n)\right\rangle$

  From $Q\psi_+=-\eta^*$, $Q\psi_-=-\eta$, $Q\eta=\frac{\delta}{\delta\psi_+}S$, $Q\eta^*=\frac{\delta}{\delta\psi_-}S$, $\langle Q(\cdots)\rangle=0$,
  and the Schwinger-Dyson eq. ,

$$\frac{\left\langle\eta(x_1)\cdots\eta^*(y_n)\sum_{i=1}^{N(\eta)}\text{sgn}|D+F|\right\rangle_\eta}{\left\langle\sum_{i=1}^{N(\eta)}\text{sgn}|D+F|\right\rangle_\eta}=\begin{cases}0 & m\neq n\\ \sum_\sigma\Pi_{k=1}^m\delta_{x_k,y_{\sigma(k)}} & m=n.\end{cases}$$

  For example, $m=n=1$ provides $\langle S_B\rangle=L^2$.

$\Rightarrow$ If $\sum_{i=1}^{N(\eta)}\text{sgn}|D+F|=2$ over the η, OK.

4 Numerical Results

Samples with $W(\Phi) = \frac{\lambda}{3}\Phi^3$, $a\lambda = 0.3$, $L = 18, 20, ..., 32$

(Newton iter. from 100 initial config. for each noise) × 320 noises

L	18	20	22	24	26	28	30	32
$(+,+)$	316	319	319	316	316	314	307	316
$(-,+,+,+)$	3	0	1	3	4	6	10	4
$(+)$	1	1	0	0	0	0	1	0
$(+,+,+)$	0	0	0	1	0	0	2	0
Δ	1.997	1.997	2	2.003	2	2	1.994	2
δ [%]	0.3	0.0	0.1	0.4	0.4	0.4	0.4	0.2

test …

$(+,+)$, $(-,+,+,+)$ ← $\sum \mathrm{sgn}|D+F| = 2$

$(+)$, $(+,+,+)$ ← $\sum \mathrm{sgn}|D+F| \neq 2$,but rare.

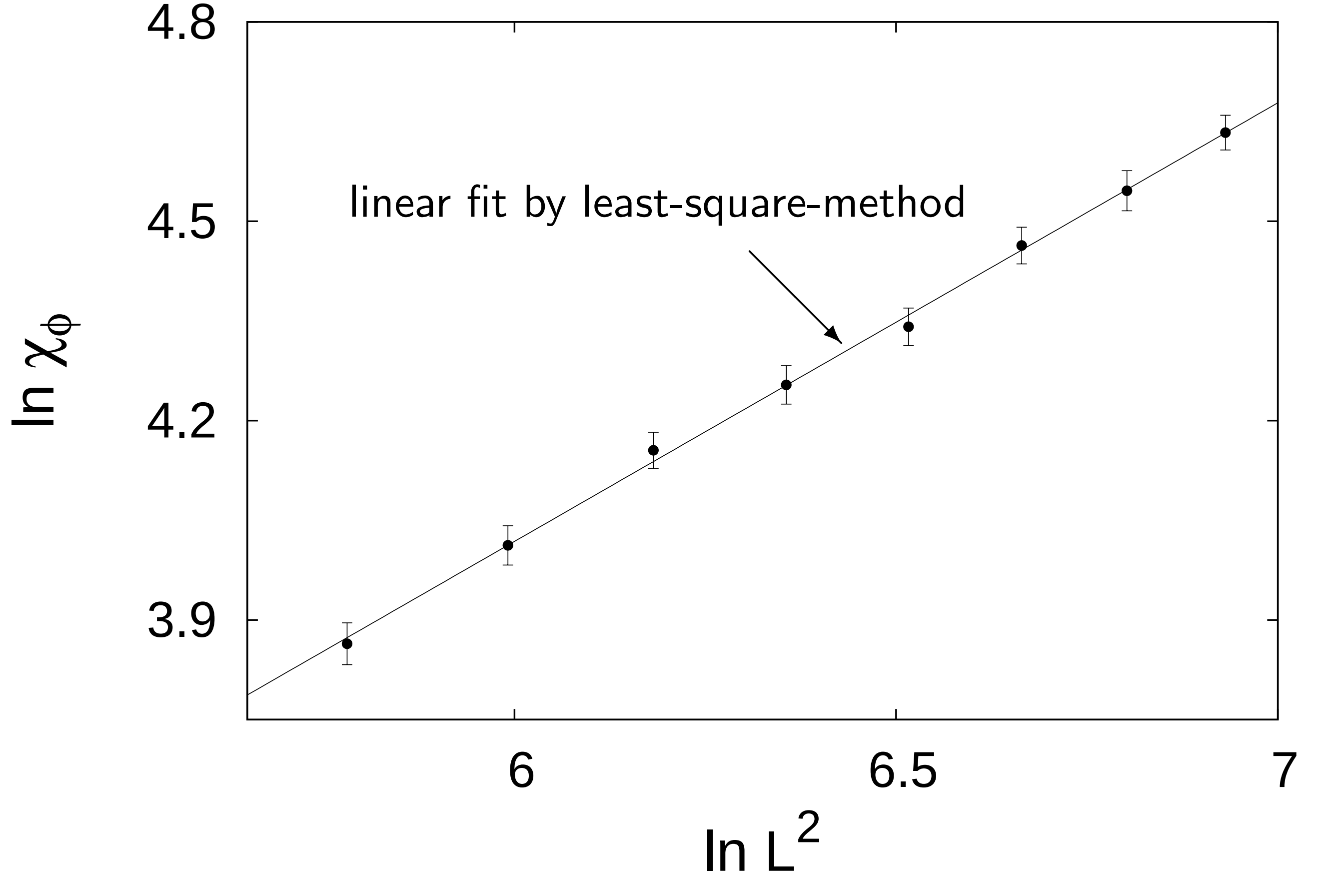
Δ ··· Witten index, δ ··· $\frac{\langle S_B \rangle - L^2}{L^2}$ (a Ward identity)

⇒ For 99% noises, $\sum_{i=1}^{N(\eta)}$ sgn $|D+F| = 2$

⇒ **Witten index $\Delta = 2$ and Ward identities are well reproduced.**

Susceptibility: $\chi_\phi \equiv \sum_{x\geq 3}\langle\phi(x)\phi(0)\rangle$

$W(\Phi)=\frac{\lambda}{3}\Phi^3,\quad a\lambda=0.3,\quad L=18,\ 20,..,\ 32$



$\Rightarrow\quad \chi_\phi \propto V^{0.660\pm0.011}$

$\Rightarrow$ consistent with the conjecture $\chi_\phi \propto V^{0.666\cdots}$ ◎

5 Summary and future plan

Summary

- We observed $\chi = \int_V \mathrm{d}x^2 \langle \phi(x)\phi^*(0)\rangle$ in the cubic potential case, and got the consistent result with the conjecture $\chi \sim V^{0.660\pm0.011}$.
- We also extracted the effective coupling constant K of the Gaussian model, and obtained the consistent result with $K = \frac{3}{4\pi}$ which is a $\mathcal{N} = 2$ SUSY point. (see more detail in arXiv:1005.4671)

Future Plan

- further check of the A-D-E classification:

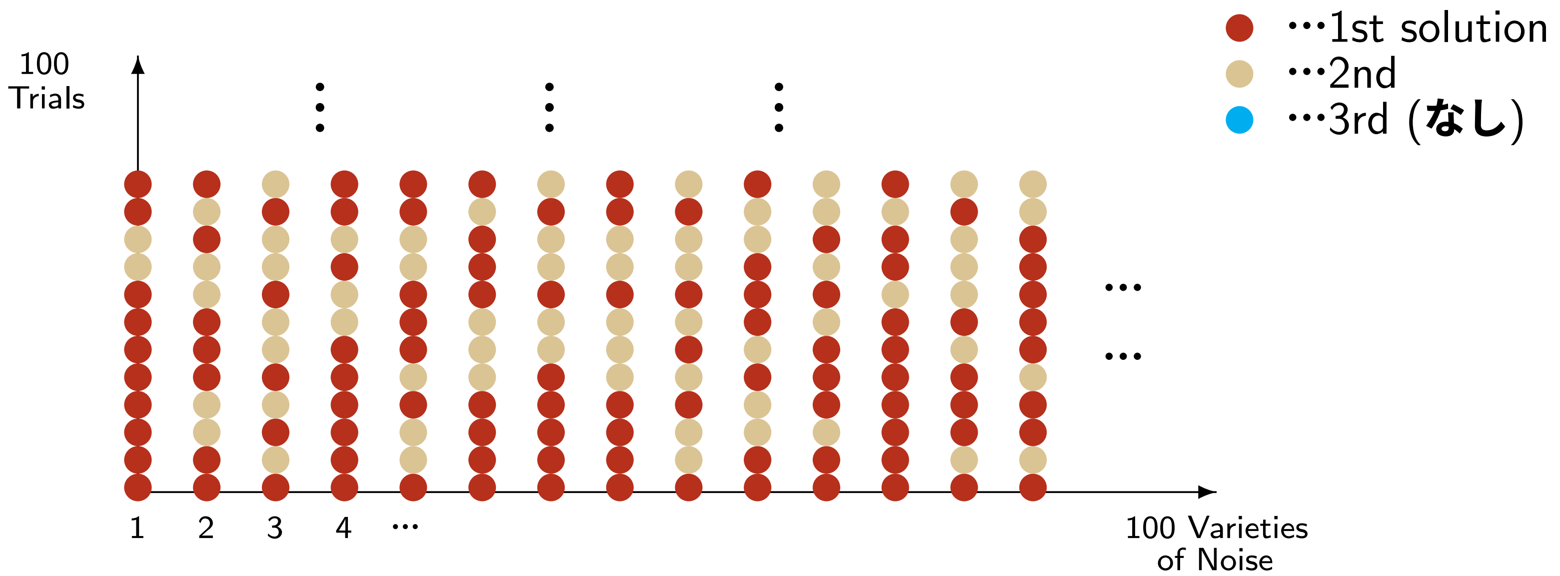
$$
\begin{aligned}
W = \quad & \Phi^4 && \longrightarrow A_3 \text{ model } \textbf{?} \\
& \Phi^3 + \Phi'^4 && \longrightarrow E_6 = A_2 \otimes A_3 \text{ model } \textbf{?} \\
& \Phi^2 + \Phi\Phi'^2 && \longrightarrow D_3 \text{ model } \textbf{?}
\end{aligned}
$$

- c-function $\longrightarrow$ central charge
c-theorem

Case1. $W(\Phi) = \frac{\lambda}{3}\Phi^3$

- (Newton method with 100 initial config. for each η) ✕ 100 set

  $\lambda = 0.3$, lattice size $= 14 \times 14$



⇒ 全 noise で解は**2個**、全て fermion **行列式は正** (Δ=2 は再現)

⇒ 4×4 〜 20×20 で、解が 2 個でない noise は 1 %もなかった。

⇒ $N(\eta)$=**2** と仮定して sample しました。