Non abelian Bianchi identities, monopoles and gauge invariance

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Outline

't Hooft tensor

Definition

't Hooft tensor $F_{\mu\nu}$ is a gauge invariant tensor that in the unitary gauge coincides with the Abelian field strength of the residual $U(1)$

$SU(N)$ case

A different 't Hooft tensor $F^a_{\mu\nu}$ can be associated to each of the fundamental weights ϕ^a_0, $a = 1, \ldots, r \quad (r = N-1)$

$$F^a_{\mu\nu} = \mathrm{Tr}(\phi^a G_{\mu\nu}) - \frac{i}{e}\mathrm{Tr}(\phi^a [D_\mu \phi^a, D_\nu \phi^a])$$

where $\phi^a = U(x)\phi^a_0 U^\dagger(x)$ Del Debbio et al. hep-lat/0203023

't Hooft tensor

General group case

$$F^a_{\mu\nu} = \mathrm{Tr}(\phi^a G_{\mu\nu}) - \frac{i}{e} \sum_I{}' \frac{1}{\lambda^a_I} \mathrm{Tr}(\phi^a [D_\mu \phi^a, D_\nu \phi^a]) + \\ -\frac{i}{e} \sum_{I \neq J}{}' \frac{1}{\lambda^a_I \lambda^a_J} \mathrm{Tr}(\phi^a [[D_\mu \phi^a, \phi^a], [D_\nu \phi^a, \phi^a]]) + \\ - \cdots$$

$$\phi^a_0 = \vec{c}^a \cdot \vec{H}, \quad \vec{c}^a \cdot \vec{\alpha}^b = \delta^{ab}, \quad \vec{\alpha}^b = \text{simple root}$$
$$\lambda^a_I = \{(\vec{c}^a \cdot \vec{\alpha})^2 | \vec{\alpha} \in \text{root system}\}$$

Di Giacomo et al. JHEP10(2008) 096

Fundamental property

The 't Hooft tensor is always linear in the gauge field

Non Abelian Bianchi Identities

Definition

A violation of Non Abelian Bianchi Identities is by definition a non zero value of $J_\nu = D_\mu \tilde{G}_{\mu\nu}$

Gauge invariant content

By Coleman-Mandula theorem we can gauge-diagonalize all J_ν and fundamental weights ϕ_0^a are a basis for diagonal operators.

$$\mathrm{Tr}(\phi_0^a [D_\mu \tilde{G}_{\mu\nu}]_{\mathrm{diag}}) = \mathrm{Tr}(\phi_0^a [J_\nu]_{\mathrm{diag}})$$

In a generic gauge $\phi^a = U(x)\phi_0^a U^\dagger(x)$ and

$$\mathrm{Tr}(\phi^a D_\mu \tilde{G}_{\mu\nu}) = \mathrm{Tr}(\phi^a J_\nu)$$

Abelian Bianchi Identities

Definition

A violation of Abelian Bianchi Identities is by definition a non zero value of the magnetic current $j^a_\nu = \partial_\mu \tilde{F}^a_{\mu\nu}$

The current j^a_ν strongly depends on the specific Abelian Projection used, that is on ϕ^a, as clearly seen on the lattice.

NABI $\Leftrightarrow$ ABI

$$j^a_\nu = \mathrm{Tr}(\phi^a J_\nu)$$

Proof for general groups in Bonati et al. Phys. Rev. D **81**, 085022 (2010)

An explicit example: 't Hooft-Polyakov monopole (1)

The hedgehog gauge (in cartesian coord.)

$$A_0^a = 0 \quad A_i^a = \epsilon_{aji} \frac{r_j}{er^2}(1-K)$$

$$K \equiv K(evr);\; v = \text{Higgs v.e.v.}$$
$$K(x) \overset{x\to 0}{\approx} 1 - x^2;\; K(x) \overset{x\to\infty}{\approx} e^{-x}$$

$$V(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}$$

The unitary gauge (in polar coord.)

$$A_0 = 0 \quad A_r = 0$$
$$A_\theta = \frac{1}{2er}\begin{pmatrix} 0 & iKe^{-i\phi} \\ -iKe^{i\phi} & 0 \end{pmatrix}$$
$$A_\phi = \frac{1}{2er}\begin{pmatrix} \frac{1-\cos\theta}{\sin\theta} & Ke^{-i\phi} \\ Ke^{i\phi} & -\frac{1-\cos\theta}{\sin\theta} \end{pmatrix}$$

$$A_\mu^{\mathrm{unit}} = V^\dagger A_\mu^{\mathrm{hed}} V - \frac{i}{e} V^\dagger \partial_\mu V$$

see *e.g.* Y. Shnir "Magnetic Monopoles", Springer 2005

An explicit example: 't Hooft-Polyakov monopole (2)

When there is a Higgs field the natural AP is $\phi^a = \phi^a_0 \equiv \frac{\sigma^3}{2}$

NABI: $J_0 = \vec{D}\cdot\vec{B} = \frac{2\pi}{e}\delta^3(\vec{r})\sigma^3$ where $B_i \equiv \frac{1}{2}\epsilon_{ijk}G_{jk}$

ABI: $j_0 = \vec{\nabla}\cdot\vec{b} = 4\pi g\delta^3(\vec{r})$ where $b_i \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}$

$$\boxed{g = \frac{1}{2e}}$$

Pure gauge theory ('t Hooft) $\phi^a(r) = U(r)\frac{\sigma^3}{2}U(r)^\dagger$

$$U(0) = \exp(i\alpha\sigma^3/2)\exp(i\beta\sigma^2/2)\exp(i\gamma\sigma^3/2)$$

$$\boxed{g = \frac{1}{2e}\mathrm{Tr}\left(\frac{\sigma^3}{2}\phi^a\right) = \frac{\cos\beta}{2e}}$$

Coleman observation

Theorem Coleman "The magnetic monopole 50 years later"

Every field configuration such that

- it is time-independent and time reversal invariant
- it is solution of equations of motion
- it behaves asymptotically as $\vec{A} = \frac{\vec{a}(\theta,\phi)}{r} + O\left(\frac{1}{r^2}\right)$

is gauge equivalent to $A_\phi = eQ(1 - \cos\theta)$, with Q constant matrix in the algebra. By a global gauge transformation $Q = g\sigma^3$ and Dirac condition becomes $\exp\left(4\pi i e g \sigma^3\right) = 1$, $\boxed{g = \frac{n}{2e}}$

Consequence (apparently never really appreciated)

Every monopole field selects *its own* Abelian Projection.

How to select the correct Abelian Projections?

General semi-classical method

Let A_M be a general monopole configuration. Then

- let $\widetilde{A_M}$ be a configuration within the same homotopy class of A_M satisfying Coleman assumptions
- gauge transform $\widetilde{A_M}$ to Coleman form A^C_M
- let A_{tHP} be the 't Hooft-Polyakov solution in the unitary gauge with the appropriate charge, then

$$A^C_M = \underbrace{A^C_M - A_{tHP}}_{A_{Fluct}} + A_{tHP}$$

By construction A_{Fluct} is sub-leading and because of the linearity of 't Hooft tensor it has no magnetic charge.

The correct Abelian Projections

How to select the unitary gauge in 't Hooft-Polyakov monopole?

It is simple to show that 't Hooft-Polyakov solution in the unitary gauge exactly satisfies the equation

$$\partial_\mu A_\mu^+ + ie[A_\mu^3, A_\mu^+] = 0$$

It is just the Maximal Abelian Gauge!

The result

To have a magnetic charge obeying Dirac constraint the legitimate Abelian Projections are those which asymptotically coincide with the Maximal Abelian Gauge.

General consequences

- MAG has to be used if the goal is to detect a monopole. If instead one wants to create a monopole on a configuration with no magnetic charge all Abelian Projections are equivalent since there is no previous "preferred projection".
- Monopole condensation is AP independent: if $\hat{O}(\vec{x})$ is a magnetically charged operator in MAG, its magnetic charge will generically be non-vanishing also in other Abelian Projections, although it will be less than the MAG one.

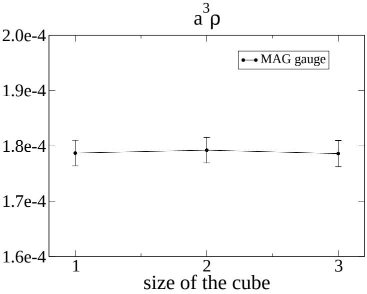
e.g. for $SU(2)$ we have $Q^{GF} = Q^{MAG} \cos\beta$

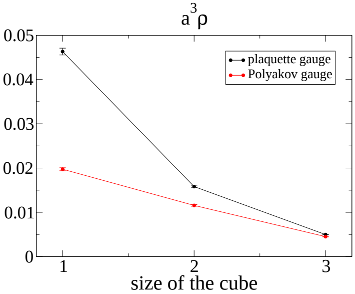
- Landau gauge corresponds to the hedgehog gauge in 't Hooft Polyakov monopole and it has $Q^{Landau} = 0$ for all configurations.

On the lattice

Since $|Q^{GF}| < |Q^{MAG}|$ the monopole density observed in a generic projection has to be less than the MAG one.

Beware of lattice artifacts!





Dapelo, D'Elia in progress

Del Debbio et al. Phys. Lett B **267**, 254 (1991)

NABI $\Leftrightarrow$ ABI: a proof for $SU(2)$

$\partial_\mu \mathrm{Tr}(\phi^a \tilde{G}_{\mu\nu}) = \mathrm{Tr}(\phi^a D_\mu \tilde{G}_{\mu\nu}) + \mathrm{Tr}(D_\mu \phi^a \tilde{G}_{\mu\nu}) \equiv \mathrm{Tr}(\phi^a J_\nu) + \mathrm{Tr}(D_\mu \phi^a \tilde{G}_{\mu\nu})$
By using $F^a_{\mu\nu} = \mathrm{Tr}(\phi^a G_{\mu\nu}) - \frac{i}{e}\mathrm{Tr}(\phi^a [D_\mu \phi^a, D_\nu \phi^a])$ we have
$\partial_\mu \tilde{F}^a_{\mu\nu} = \partial_\mu \mathrm{Tr}(\phi^a \tilde{G}_{\mu\nu}) - \frac{i}{2e}\epsilon_{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}(\phi^a [D_\rho \phi^a, D_\sigma \phi^a])$ and so

$$\boxed{\partial_\mu \tilde{F}^a_{\mu\nu} = \mathrm{Tr}(\phi^a J_\nu) + R^a_\nu}$$ where

$$\boxed{R^a_\nu \equiv \mathrm{Tr}(D_\mu \phi^a \tilde{G}_{\mu\nu}) - \frac{i}{2e}\epsilon_{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}(\phi^a [D_\rho \phi^a, D_\sigma \phi^a]) \stackrel{?}{=} 0}$$

$\phi^a = U(x)\phi^a_0 U^\dagger(x) \Rightarrow D_\mu \phi^a = ie[A_\mu + \Omega_\mu, \phi^a] \Rightarrow \mathrm{Tr}(D_\mu \phi^a \tilde{G}_{\mu\nu}) =$
$ie\mathrm{Tr}((A_\mu + \Omega_\mu)[\phi^a, \tilde{G}_{\mu\nu}]) = \mathrm{Tr}(D_\mu \phi^a P^a \tilde{G}_{\mu\nu})$ where P^a =projector on components that do not commute with ϕ^a. For $SU(2)$ $P^a = [\phi^a, [\phi^a, \cdot]]$

$$\boxed{R^a_\nu = -\frac{i}{2e}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}(D_\mu \phi^a [D_\rho \phi^a, D_\sigma \phi^a])}$$

From $\mathrm{Tr}(\phi^a D_\mu \phi^a) = \frac{1}{2}\partial_\mu \mathrm{Tr}((\phi^a)^2) = 0$ it follows that, in the unitary gauge $\phi^a = \phi^a_0 \equiv \frac{1}{2}\sigma^3$, the covariant derivatives are linear combinations of σ^+ and σ^-.
The trace of three $\sigma^\pm$ is zero and therefore $\boxed{R^a_\nu = 0}$

Conclusions

We have shown that

- monopoles are related to NABI violations
- to detect monopoles not all Abelian Projections are equivalent and the Maximal Abelian Gauge is the correct choice
- monopole condensation is gauge invariant