# Loop gas formulation of low dimensional SUSY models

## On the relevance of the sign problem

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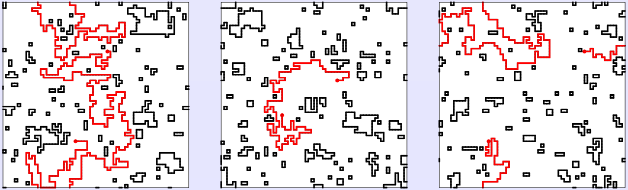


# Introduction

- Supersymmetry is thought to be a crucial ingredient in
  - the unification of the SM interactions,
  - the solution of the hierarchy problem.
- Low energy physics is, however, not supersymmetric:
  - SUSY must be broken spontaneously,
  - this can not be described in perturbation theory.
- Lattice provides a non-perturbative regularisation:
  - discretisation breaks Poincaré symmetry explicitely,
  - Leibniz' rule is absent,
  - fermion doubling,

  $\rightarrow$ recovered in the continuum limit?
- Fermionic sign problem hampers MC simulations.

## Accidental symmetries and fine tuning

- Accidental sym. may emerge from non-sym. lattice action.
- For SUSY theories involving scalar fields this is not easily possible:
  - scalar mass term $m^2|\phi|^2$ breaks SUSY,
  - no other symmetry available to forbid that term.
- Some symmetries can be fine tuned with counterterms:
  - chiral symmetry for Wilson fermions,
  - might be feasible in lower dimensions if theories are superrenormalisable.
- Look for subalgebras of the SUSY algebra [Catterall; Kaplan; Ünsal; etc ’01-’09]:
  - combine Poincaré and flavour group (twisted SUSY),
  - leads to Dirac-Kähler (staggered) fermions.

## Spontaneous SUSY breaking (SSB) and the Witten index

- Witten index provides a necessary but not sufficient condition for SSB:

$$W \equiv \lim_{\beta \to \infty} \mathrm{Tr}(-1)^F \exp(-\beta H) \quad \Rightarrow \begin{cases} = 0 & \text{SSB may occur} \\ \neq 0 & \text{no SSB} \end{cases}$$

det $[\partial\!\!\!/ +$

- Index counts the difference between the number of bosonic and fermionic zero energy states:

$$W \equiv \lim_{\beta \to \infty} \left[ \mathrm{Tr}_B \exp(-\beta H) - \mathrm{Tr}_F \exp(-\beta H) \right] = n_B - n_F$$

- Index is equivalent to partition function with periodic b.c.:

$$W = \int_{-\infty}^{\infty} \mathcal{D}\phi \;\det [D\!\!\!\!/\,(\phi)]\; e^{-S_B[\phi]} = Z_{\mathrm{per}}$$

$\Rightarrow$ Determinant must be indefinite for SSB to occur.

## Example: SUSY QM

- Consider the Lagrangian for supersymmetric quantum mechanics

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}P'(\phi)^2 + \overline{\psi}\left(\frac{\partial}{\partial t} + P''(\phi)\right)\psi\,,$$

- The (regulated) fermion determinant can be calculated exactly:

$$\det\left[\frac{\partial_t + P''(\phi)}{\partial_t + m}\right] = \sinh\int_0^T \frac{P''(\phi)}{2}dt \quad \Longrightarrow \quad Z_0 - Z_1$$

- Under some symmetry $\phi \to \phi'$ we have

$$\int_0^T \frac{P''(\phi')}{2}dt \quad = \quad \begin{cases} +\int_0^T \frac{P''(\phi)}{2}dt & \text{no SSB} \quad \Rightarrow \quad Z_0 \neq Z_1 \\ -\int_0^T \frac{P''(\phi)}{2}dt & \text{SSB} \quad \Rightarrow \quad Z_0 = Z_1 \end{cases}$$

## Example: SUSY QM on the lattice

- On the lattice we find with Wilson type fermions

$$\det\left[\nabla^* + P''(\phi)\right] = \prod_t \left[1 + P''(\phi_t)\right] - 1\,.$$

- For even potentials, e.g. $P(\phi) = \frac{1}{2} m\,\phi^2 + \frac{1}{4} g\phi^4$ we have

$$\det\left[\nabla^* + P''\right] = \prod_t \left[1 + m + 3g\phi_t^2\right] - 1 \geq 0$$

for $m \geq 0, g \geq 0$.

- As a side remark, note that

$$\lim_{a\to 0} \det\left[\nabla^* + P''\right] \sim \exp \int_0^T \frac{P''(\phi)}{2} dt \;\; \det\left[\partial_t + P''(\phi)\right]\,,$$

so this term needs 'fine tuning'.

## Spontaneous SUSY breaking (SSB) and the sign problem

- For odd potentials, e.g. $P(\phi) = \frac{1}{2} m \phi^2 + \frac{1}{3}\lambda\phi^3$ we have

$$\prod_t [1 + m + 2\lambda\phi_t] \text{ indefinite,}$$

and hence $\det[\nabla^* + P'']$ is no longer positive...

$\Rightarrow$ sign problem!

- Every supersymmetric model which allows SSB must have a sign problem:
  - SUSY QM with odd potential,
  - $\mathcal{N} = 16$ Yang-Mills quantum mechanics [Catterall, Wiseman '07],
  - $\mathcal{N} = 1$ Wess-Zumino model in 2D [Catterall '03],
  - $\mathcal{N} = (2, 2)$ Super-Yang-Mills in 2D [Giedt '03].

## Solution of the sign problem

- We propose a novel approach circumventing these problems [Wenger '08]:
  - based on the exact hopping expansion of the fermion action,
  - eliminates critical slowing down,
  - allows simulations directly in the massless limit,

  $\Rightarrow$ solves the fermion sign problem.

- Applicable to the
  - Gross-Neveu model in $d = 2$ dimensions,
  - Schwinger model in the strong coupling limit in $d = 2$ and 3,
  - SUSY QM,
  - $\mathcal{N} = 1$ and 2 supersymmetric Wess-Zumino model.

## $\mathcal{N} = 1$ Wess-Zumino model

- Consider now the $\mathcal{N} = 1$ Wess-Zumino model in 2D:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}P'(\phi)^2 + \frac{1}{2}\overline{\psi}\left(\not{\partial} + P''(\phi)\right)\psi$$

  - with $\psi$ a Majorana field,
  - and $\phi$ real bosonic field,
  - superpotential, e.g. $P(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3$.
- Integrating out Majorana fermions yields indefinite Pfaffian.
- For Majorana fermions use exact reformulation in terms of loops [old idea]:

$\Rightarrow$ sign of Pfaffian under perfect control

- For bosonic fields use exact reformulation in terms of loops [Prokof'ev, Svistunov '01].

## Exact hopping expansion for Majorana Wilson fermions

- Using Wilson lattice discretisation for the fermionic part:

$$\mathcal{L} = \frac{1}{2}\xi^T \mathcal{C}(\gamma_\mu \tilde{\partial}_\mu - \frac{1}{2}\partial^* \partial + P''(\phi))\xi \,,$$

  - $\xi$ is a real, 2-component Grassmann field,
  - $\mathcal{C} = -\mathcal{C}^T$ is the charge conjugation matrix.

- Using the nilpotency of Grassmann elements we expand the Boltzmann factor

$$\int \mathcal{D}\xi \prod_x \left(1 - \frac{1}{2}M(x)\xi^T(x)\mathcal{C}\xi(x)\right) \prod_{x,\mu} \left(1 + \xi^T(x)\mathcal{C}P(\mu)\xi(x+\hat{\mu})\right)$$

  where $M(x) = 2 + P''(\phi)$ and $P(\pm\mu) = \frac{1}{2}(1 \mp \gamma_\mu)$.

# Exact hopping expansion for Majorana Wilson fermions

- At each site, the fields $\xi^T\mathcal{C}$ and $\xi$ must be exactly paired to give a contribution to the path integral:

$$\int \mathcal{D}\xi \prod_x \left(M(x)\xi^T(x)\mathcal{C}\xi(x)\right)^{m(x)} \prod_{x,\mu} \left(\xi^T(x)\mathcal{C}P(\mu)\xi(x+\hat{\mu})\right)^{b_\mu(x)}$$

  with occupation numbers
  - $m(x) = 0, 1$ for monomers,
  - $b_\mu(x) = 0, 1$ for fermion bonds (or dimers),

  satisfying the constraint

$$m(x) + \frac{1}{2}\sum_\mu b_\mu(x) = 1.$$

- Only closed, non-intersecting paths survive the integration.

## Exact hopping expansion for scalar fields

- Analogous treatment for the bosonic field [Prokof'ev, Svistunov '01]:
  - $(\partial_\mu \phi)^2 \quad \rightarrow \quad \phi_x \phi_{x-\hat{\mu}},$
  - expand hopping term to all orders:

$$\int \mathcal{D}\phi \prod_{x,\mu} \sum_{n_\mu(x)} \frac{1}{n_\mu(x)!} \left(\phi_x \phi_{x-\hat{\mu}}\right)^{n_\mu(x)} \exp\left(-V(\phi)\right) \, M[\phi]^{m(x)}$$

with bosonic bond occupation numbers $n_\mu(x) = 0, 1, 2, \ldots$

Integrating out $\phi(x)$ yields bosonic site weights

$$Q(N) = \int d\phi \; \phi^N \exp\left(-V(\phi)\right)$$

where $N$ includes powers from $M[\phi]$.

## Loop gas formulation

- Loop gas representation in terms of fermionic monomers and dimers and bosonic bonds.
- Partition function summing over all non-oriented, self-avoiding fermion loops

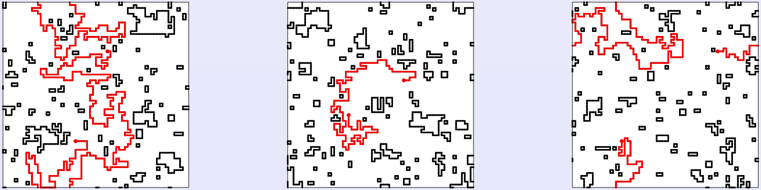
$$Z_{\mathcal{L}} = \sum_{\{\ell\}\in\mathcal{L}} |\omega[\ell, n_\mu(x), m(x)]|, \qquad \mathcal{L} \in \mathcal{L}_{00} \cup \mathcal{L}_{10} \cup \mathcal{L}_{01} \cup \mathcal{L}_{11}$$

represents a system with unspecified fermionic b.c. [Wolff '07].

- Simulate bosons with worm algorithm [Prokof'ev, Svistunov '01].
- Simulate fermions by enlarging the configuration space by one open fermionic string [Wenger '08].

# Reconstructing the fermionic boundary conditions

- The open fermionic string corresponds to the insertion of a Majorana fermion pair $\{\xi^T(x)\mathcal{C}, \xi(y)\}$ at position $x$ and $y$:



- It samples the relative weights between $Z_{\mathcal{L}_{00}}, Z_{\mathcal{L}_{10}}, Z_{\mathcal{L}_{01}}, Z_{\mathcal{L}_{11}}$.
- Reconstruct the Witten index a posteriori

$$Z^{pp} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}},$$

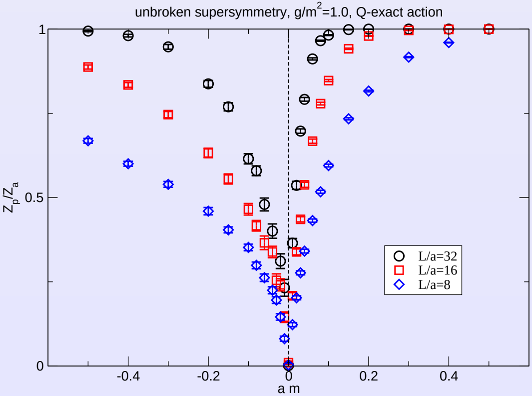
or a system at finite temperature

$$Z^{pa} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} + Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}}.$$

# Example: SUSY QM

- Especially simple for supersymmetric QM:

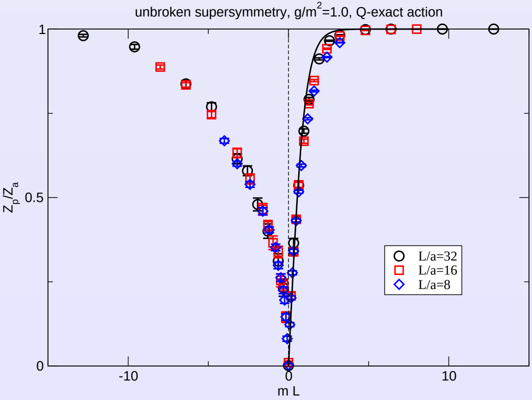
$$Z^p = Z_{\mathcal{L}_0} - Z_{\mathcal{L}_1} \quad \Rightarrow \text{Witten index}$$
$$Z^a = Z_{\mathcal{L}_0} + Z_{\mathcal{L}_1} \quad \Rightarrow \text{finite temperature}$$



# Example: SUSY QM continuum limit

- Especially simple for supersymmetric QM:

$$Z^p = Z_{\mathcal{L}_0} - Z_{\mathcal{L}_1} \quad \Rightarrow \text{Witten index}$$
$$Z^a = Z_{\mathcal{L}_0} + Z_{\mathcal{L}_1} \quad \Rightarrow \text{finite temperature}$$



## Summary and conclusions

- The fermionic sign problem and its relevance to the Witten index.
- Representation of SUSY QM and $\mathcal{N} = 1, 2$ Wess-Zumino model in terms of interacting bosonic and fermionic loops.
- Relevance of topological boundary conditions for the solution of the sign problem.
- Subtleties for taking the continuum limit in the supersymmetry breaking case.