Mixed action computations on fine dynamical lattices

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14/06/09

QCD Low Energy Constants (LECs)

QCD low energy dynamics is described by ChPT:

- assume chiral symmetries are spontaneously broken $\Rightarrow \; \pi, (K, \eta)$ are the PseudoGoldstone Bosons (PGBs)
- the form of interactions is constrained by symmetry
- the couplings (LECs) are unconstrained by symmetry: contain the UV information and depend on N_l, m_b, m_c, α_s, (m_s)
- at a given order in M_{PGB}/Λ_χ, p_{PGB}/Λ_χ the number of LECs is finite. At LO the LECs are the pion decay constant F and the quark condensate Σ in the chiral limit:

$$\mathcal{L}_\chi = \frac{F^2}{4}\mathrm{Tr}[\partial_\mu U \partial_\mu U^\dagger] + \frac{\Sigma}{2}\mathrm{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger] \quad U = \exp\left(\frac{2i\xi}{F}\right)$$

In this work we explore the determination of some LECs by matching the distribution of the lowest eigenvalues of the Dirac operator as predicted by a universality class of Random Matrix Theories (RMT) with mixed action Lattice calculations.

Outline

- ChPT in finite volume
    - ϵ-regime
    - mixed-regime
- RMT
    - validity of the Zero modes Chiral Theory
    - scaling with m_{sea} in mixed regime
- Results
    - test of RMT (ratios of eigenvalues)
    - determination of L_6 and Σ
- Conclusions

ChPT in finite volume

No change in the Lagrangian or in LECs, provided $L\Lambda_\chi \gg 1$

p-regime power counting

$$m_q \sim M_\pi^2 \sim p^2 \sim 1/L^2 \qquad \text{as in } \infty \text{ volume}$$

ϵ-regime power counting

$$m_q \sim M_\pi^2 \sim p^4 \sim 1/L^4$$

The dependence on the mass is suppressed ⇒ less LECs appear for a given observable at a given order, with respect to p-regime

mixed-regime power counting

$$m_l \sim p^4 \sim 1/L^4 \quad m_h^2 \sim p^2 \sim 1/L^2$$

- wrt p-regime: less LECs appear for a given observable at a given order
- wrt ϵ-regime: more LECs appear for a given observable at a given order

ϵ-regime

- parametrization for pion field [Gasser Leutwyler 87]:

$$U = U_0 e^{\frac{2i\xi}{F}} \qquad \int dx^4 \; \mathrm{Tr}[T^a \xi(x)] = 0 \qquad U_0 \in SU(N_f)$$

- non trivial dependence on topological sector ν [Leutwyler Smilga 92]

$$\langle O \rangle_\nu = \int_0^{2\pi} d\theta \, e^{-i\nu\theta} \langle O(\theta) \rangle \qquad \theta = \text{vacuum angle}$$

- at NLO there is factorization of zero U_0 and non-zero modes $\xi \Rightarrow$ define a zero modes partition functional ($\mu_i \equiv \Sigma V m_i$):

$$\mathcal{Z}_\nu^{(N_f)}[\{\mu\}] = \int_{U(N_f)} dU_0 \, (\det U_0)^\nu \exp\left(\frac{\Sigma V}{2} \mathrm{Tr}\left[\mathcal{M} U_0 + U_0^\dagger \mathcal{M} \right] \right)$$

The solution is known in terms of Bessel functions. Other necessary integrals are obtained by deriving with respect to the quark masses. [Brower et al. 1982, Leutwyler & Smilga 1992, Jackson et al. 1996]

Mixed regime

$$\mathcal{M} = \left(\begin{array}{cc} \mathcal{M}_l & 0 \\ 0 & \mathcal{M}_h \end{array} \right) \quad m_h \sim L^{-2} \ (\epsilon\text{-regime}) \ \ m_l \sim L^{-4} \ (\text{p-regime})$$

$$\underbrace{\phantom{\mathcal{M}_l}}_{N_l} \quad \underbrace{\phantom{\mathcal{M}_h}}_{N_h}$$

- for pions corresponding to $SU(N_l)$ $M^2_{ab} \sim L^{-4}$
- for other pions $M^2_{ab} \sim L^{-2}$ in QCD

$$M^2_{ab} = \frac{\Sigma}{F^2}(m_a + m_b) \qquad M^2_{ll} \sim L^{-4} \quad M^2_{hl} \sim L^{-2} \quad M^2_{hh} \sim L^{-2}$$

- $T_\eta \equiv \sqrt{\frac{N_l N_h}{2(N_l+N_h)}}\mathrm{diag}\{\underbrace{\frac{1}{N_l}, \ldots, \frac{1}{N_l}}_{N_l}, \underbrace{-\frac{1}{N_h}, \ldots, -\frac{1}{N_h}}_{N_h}\}$ has an N_l

  dependent mass:

$$M^2_\eta = \frac{N_l M^2_{hh} + N_h M^2_{ll}}{N_h + N_l} \qquad \mathcal{M}_{h,l} = m_{h,l}\mathbf{1}_{h,l}$$

- in a PQ theory with zero sea quarks in the ϵ-regime, $N_l = 0$ and $M^2_\eta \sim L^{-4}$ (like the η' does not decouple in a quenched theory)

[F. B. Hernandez 07, Damgaard Fukaya 07, F.B. et al. 08]

Factorization for mixed regime

Mimicking ϵ-regime, one convenient parametrization is:

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & \mathbf{1}_h \end{pmatrix} e^{\frac{2i\xi}{F}} e^{i\eta T_\eta} \qquad U_0 \in SU(N_l)$$

$$\int dx^4 \ \mathrm{Tr}[T^a \xi(x)] = \int dx^4 \ \mathrm{Tr}[T^\eta \xi(x)] = 0 \qquad T^a \in SU(N_l)$$

- at fixed topology the η mode is coupled to θ and becomes perturbative
- non perturbative and perturbative modes factorize at LO

This formalism can be applied to PQ cases using the Replica Method

Random Matrix Theory (RMT) and QCD in ϵ-regime

Consider the partition functional of a RMT:

$$\hat{\mathcal{Z}}_\nu[\{\hat m_i\}] = \int dW \prod_{l=1}^{N_l} \det(i\hat D + \hat m_l) \exp\left(-\frac{N}{2}\mathrm{Tr}\, V(\hat D^2)\right) \quad \hat D = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix}$$

- the W matrices have rectangular size $N \times (N + \nu)$
- $V(\hat D^2)$ is an arbitrary potential such that the spectral density $\hat\rho$ satisfies: $\lim_{\hat\lambda \to 0} \hat\rho(\hat\lambda) \neq 0$ where $\hat\lambda$ are the eigenvalues of $\hat D$

It has been shown that in the limit $N \to \infty$, if $\hat\mu_l \equiv 2N\hat m_l \hat\rho(0)$:

$$\hat{\mathcal{Z}}_\nu[\{\hat m\}]_{\hat\mu_i = m_i\Sigma V} = \mathcal{Z}_\nu^{(N_l)}[\{\mu\}] \equiv \int_{U(N_l)} dU_0 \det(U_0)^\nu \exp\left(\frac{\Sigma V}{2}\mathrm{Tr}\left[\mathcal{M}_l U_0 + U_0^\dagger \mathcal{M}_l^\dagger\right]\right)$$

[Shuryak Verbaarschot 92]

RMT = zero modes partition functional in ϵ-regime at LO ≡ ZMChT

Random Matrix Theory (RMT) and QCD in ϵ-regime

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The replica limit $N_v \to 0$ of this integral is:

$$\hat{\mathcal{Z}}_\nu[\{\hat{m}\}]_{\hat{\mu}_i = m_i \Sigma V} = \mathcal{Z}_\nu^{(N_l)}[\{\mu\}] \equiv \int_{\mathcal{G}l(N_v|N_v+N_{s_l})} dU_0 \det(U_0)^\nu \exp\left(\frac{\Sigma V}{2} S\mathrm{Tr}\left[\mathcal{M}_l U_0 + U_0^\dagger \mathcal{M}_l^\dagger\right]\right)$$

[Damgaard et al 98]

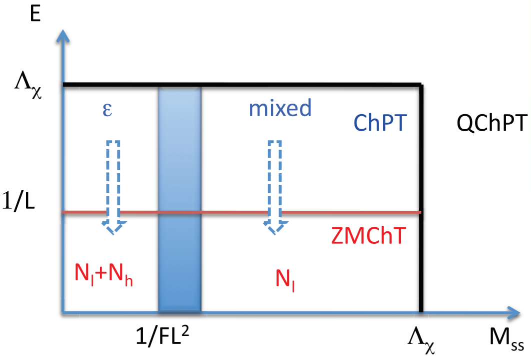
All the conclusions that follow can be extended to the Quenched and PQ theories using this modification.

Matching of QCD in ϵ-regime and ZMChT

In the ϵ-regime at fixed ν the nonzero momentum modes can be integrated out: $M_{ll} < \frac{2\pi}{L}$

$$\mathcal{Z}_\nu^{ChPT(\epsilon)} = \int_{U(N_l)} dU_0\, d\xi\, \det(U_0)^\nu \exp\left(\frac{\Sigma V}{2}\mathrm{Tr}\left[\mathcal{M}_l U_0 + U_0^\dagger \mathcal{M}_l^\dagger\right]\right) e^{-\int d^4x\, \mathrm{Tr}\,[\partial_\mu \xi \partial_\mu \xi]}$$

$$\propto \int_{U(N_l)} dU_0\, \det(U_0)^\nu \exp\left(\frac{\Sigma V}{2}\mathrm{Tr}\left[\mathcal{M}_l U_0 + U_0^\dagger \mathcal{M}_l^\dagger\right]\right) \equiv \mathcal{Z}_\nu^{ZMChT}[\{\mu\}]$$



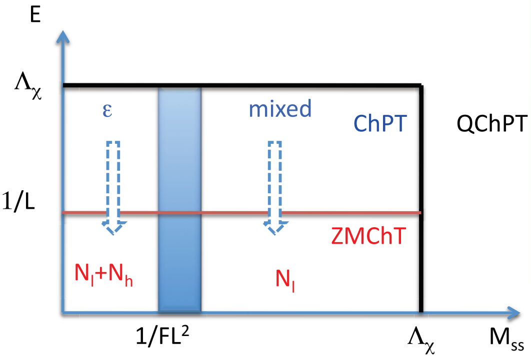
Matching of QCD in ϵ-regime and ZMChT

This matching can be extended to:

- mixed regime: integrating out the zero modes of heavier p-regime PGBs
- NLO: zero and nonzero modes are coupled

$$\mathcal{L}^{NLO}_{ChPT(m)} = \ldots + \frac{\Sigma}{F^2}\mathrm{Tr}\left[\mathcal{M}_l\left(\xi^2 U_0 + U_0^\dagger \xi^2\right)\right] - 16\frac{\Sigma L_6}{F^4}\mathrm{Tr}\left[\mathcal{M}_h\right]\mathrm{Tr}\left[\mathcal{M}_l\left(U_0 + U_0^\dagger\right)\right] + \ldots$$

At NLO the integration of the "heavy modes" induce a renormalization of the Σ: $\mathcal{Z}_\nu^{ZMChT}[\{\mu\}]_{NLO} \propto \mathcal{Z}_\nu^{ZMChT}[\{\tilde{\mu}\}]$



Predictions of RMT

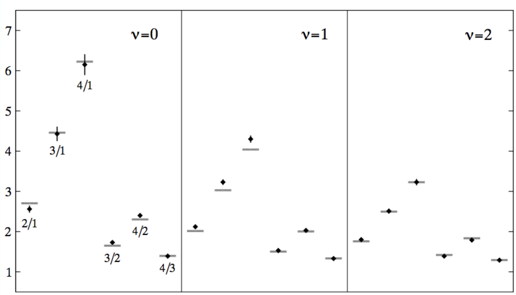
There are analytical calculations of:

- the microscopic spectral density $\rho^{\nu}(\hat{\zeta};\{\mu\})$, where $\hat{\zeta}_i \equiv 2N\hat{\lambda}_i\hat{\rho}(0)$ [Shuryak et al., Damgaard]:
    - ex. the quenched result: $\rho^{\nu}(\hat{\zeta};0) = \frac{\hat{\zeta}}{2}\left[J_{\nu}(\hat{\zeta})^2 - J_{\nu+1}(\hat{\zeta})J_{\nu-1}(\hat{\zeta})\right]$
- the joint probability distributions for N eigenvalues $\rho^{\nu}(\hat{\zeta}_1,\ldots,\hat{\zeta}_N;\{\mu\})$
- probability distribution of the k-th smallest eigenvalue $\hat{\zeta}_k$, $p_k^{\nu}(\hat{\zeta};\{\mu\})$ [Nishigaki et al.]
    - flavor-topology duality (strictly valid at $m_i = 0$): $p_k^{\nu}(\hat{\zeta};\{0\})$ depends on N_f and ν only through $N_f + |\nu|$.
- In the cases where the quantities have been computed in ChPT in the ϵ-regime (eg the microscopic spectral density) the results agree if

$$\hat{\zeta}_i = \tilde{\Sigma}V\lambda_i \qquad \hat{\mu}_i = \tilde{\Sigma}Vm_i$$

where λ_i are the eigenvalues of the Dirac operator. Matching RMT predictions and lattice results allows to extract $\tilde{\Sigma}$

Matching in the Quenched theory



Check of matching RMT-QCD first performed in the Quenched theory [Edwards et al. 99,... Giusti et al. 03]

- diamonds: Lattice data
- horizontal bars: RMT

[Giusti et al. 03]

Since:

$$\langle \hat{\zeta}_k \rangle^{\nu}_{\mathrm{RMT}} = \tilde{\Sigma} V \langle \lambda_k \rangle^{\nu}_{\mathrm{QCD}}$$

- ratios of k-th over l-th eigenvalues $\langle \lambda_k \rangle^{\nu} / \langle \lambda_l \rangle^{\nu'}$ are parameter free predictions
- the Σ (and other LECs) extracted in this framework would be the one of the Quenched theory, $\Sigma_{N_f=0}$

In the dynamical case check performed for $\rho(\lambda)$ [Fukaya et al 07] .

Simulation

Mixed action approach:

- sea: $O(a)$ improved Wilson quarks (CLS configurations: $N_h = 2,\ N_l = 0$) [Del Debbio et al. 2007]
- valence: overlap

<table>
<tr><th colspan="4">$\beta = 5.3,\ c_{\rm sw} = 1.90952,\ V = 48 \times 24^3,\ a = 0.0784(10)$ fm</th></tr>
<tr><th>label</th><th>κ</th><th>aM_{ss}</th><th>$N_{\rm cfg}$</th></tr>
<tr><td>D_4</td><td>0.13620</td><td>0.1695(14)</td><td>86</td></tr>
<tr><td>D_5</td><td>0.13625</td><td>0.1499(15)</td><td>169</td></tr>
<tr><td>D_6</td><td>0.13635</td><td>0.1183(37)</td><td>246 ($D_{6\rm a}$: 159)
($D_{6\rm b}$: 87)</td></tr>
</table>

Table: Simulation parameters.

We match with the Quenched RMT prediction: but this time $\tilde{\Sigma} = \tilde{\Sigma}(\Sigma|_{N_f=2},\ L_6|_{N_f=2})$

- overlap allow a determination of ν through the index theorem:

$$\nu = n_R - n_L$$

$n_{R,L}$ is the number of right, left handed zero modes

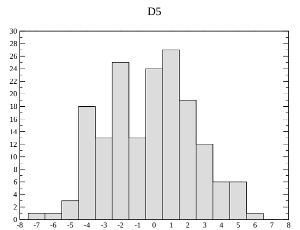


Figure: Distribution of topological charge for D_5

Locality Check

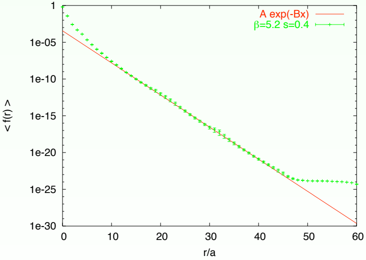
$$\mathcal{D}^{(O)} = \frac{1}{a}(1 - A(A^\dagger A)^{-1/2}), \qquad A = 1 + s - a\mathcal{D}^{(W)}$$

$$\zeta(x) \equiv \text{Sign(A)}\eta(x) \qquad \eta_\alpha(x) = \delta_{xy}\delta_{\alpha 1}$$

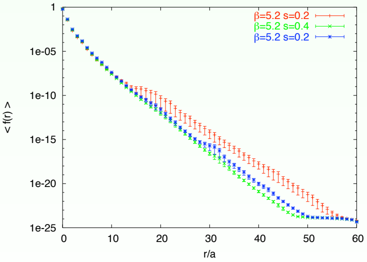
$$f(r) \equiv \max\{||\zeta(x)||^2 \mid ||x-y||_1 = r\}$$

$$g(r) \equiv Ae^{-Br}$$

- $||\dots||_1$ is the "taxi driver" distance
- tune s to maximize B
- OK in quenched case [Hernandez et al.]



- precision of Sign(A) $\sim 10^{-8} \Rightarrow ||\zeta(x)||^2$ calculated reliably until $||\zeta(x)||^2 > 10^{-16} \Rightarrow$ fitrange: 14-28



- locality works better for $s = 0.4$
- B is generally smaller than its quenched analog

Numerical Results for eigenvalues ratios 1

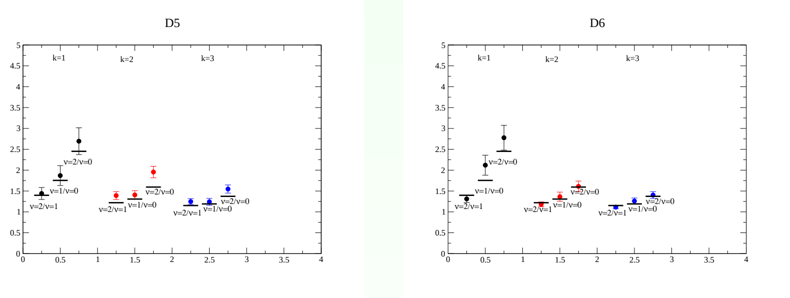


Figure: ratios of k-th eigenvalues at different topological sectors for $M_{ss} = 377$ MeV (left) and $M_{ss} = 257$ (right). The black line indicates the qRMT prediction

Rather good agreement with Quenched RMT predictions

Numerical Results for eigenvalues ratios 2

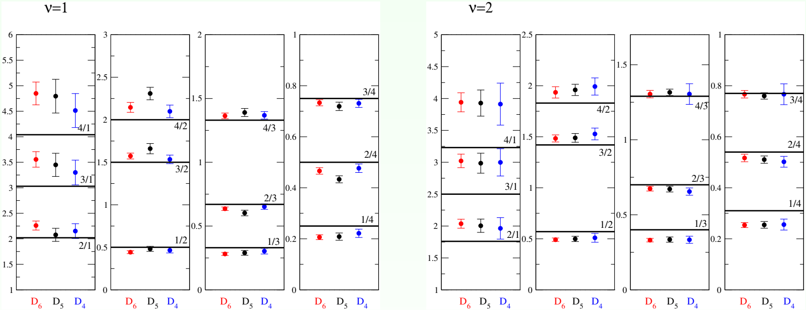


Figure: ratios of Dirac eigenvalues for $\nu = 1$ (left) and $\nu = 2$ (right). k/l is a shorthand for $\langle\lambda_k\rangle/\langle\lambda_l\rangle$ The horizontal bar indicates the qRMT prediction

Rather good agreement with Quenched RMT predictions except for ratios of eigenvalues at fixed ν involving the lowest eigenvalue.

LECs from matching in mixed regime

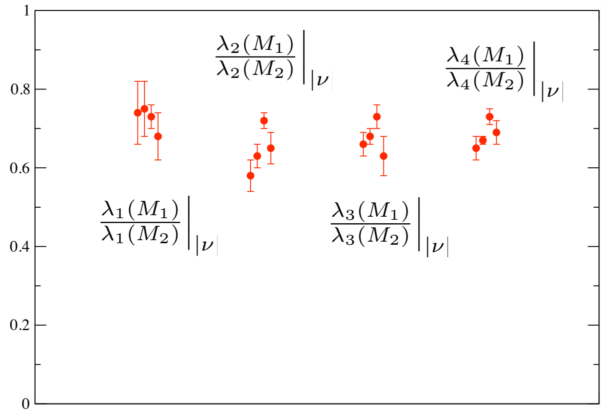


Figure: ratios of k-th eigenvalues at $M_1 = 257$ MeV $M_2 = 377$ MeV

- Ratios of k-th eigenvalues at different masses independent from ν and k:

$$\langle \zeta_k \rangle^{\nu}_{\mathrm{qRMT}} = \tilde{\Sigma}(M_{hh}) V \langle \lambda_k \rangle^{\nu}_{\mathrm{QCD}}(M_{hh}) \Rightarrow R_{\Sigma} \equiv \frac{\langle \lambda_k \rangle^{\nu}(M_1)}{\langle \lambda_k \rangle^{\nu}(M_2)} = \frac{\tilde{\Sigma}(M_2)}{\tilde{\Sigma}(M_1)}$$

LECs from matching in mixed regime

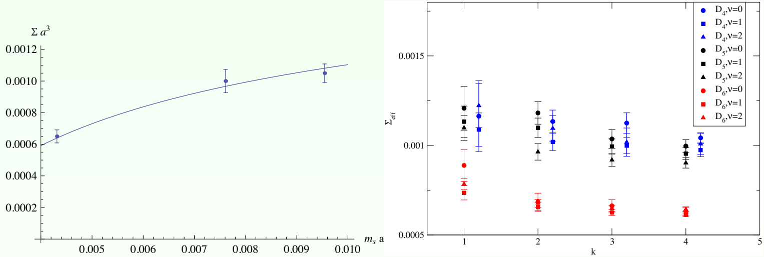


Figure: Left: (PRELIMINARY) $\tilde{\Sigma}(m_h)$. Input: $F = 90$ MeV. $\Sigma a^3 = 0.00103$, $L_6(M_\rho) = -0.000074$. Right: $\tilde{\Sigma}(M_{hh})$ as obtained at different ν. $M_{D_4} = 426$ MeV, $M_{D_5} = 379$ MeV, $M_{D_4} = 257$ MeV

$$\frac{\tilde{\Sigma}}{\Sigma} = 1 + \frac{M_{hh}^2}{F^2}\left(\frac{\beta_2}{N_h} + \frac{\log(\mu L)}{8\pi^2 N_h} + 16 N_h {L_6}^r(\mu) - \frac{N_h}{(4\pi)^2}\log\left(\frac{M_h}{\mu}\right)\right) - \frac{N_h}{F^2} g_1\left(M_h^2, L\right)$$

Conclusions and Outlook

We have tested the matching between Quenched RMT and Partially Quenched QCD using a mixed action approach:

- results for ratios of eigenvalues in agreement with RMT predictions
- the scaling with m_{sea} is as predicted by the mixed regime approach
- from chiral fits of the effective condensate we could extract L_6, Σ (putting $F = 90$ as input)

Outlook:

- calculate renormalization factor Z_S to extract Σ
- analyze 2-point functions results (in mixed regime sensitive to: F, Σ, L_4, L_6)