QCD Rotator with Light Quarks up to NNL Order

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- We calculate the low-energy excitations of QCD in finite volume in chiral perturbation theory (ChPT)

- Consider 2 light quark flavours, $m_u = m_d = m$.

- Special environment: δ-regime
  - $L_t \gg L_s$
  - The Compton wavelength of the pion M^{-1} is much larger than the box size L_s

$$ML_s \ll 1\,.$$

- M corresponds to the leading term for the pion mass in infinite volume

$$M^2 = 2mB\,.$$

- The low-energy excitations of this system are described by a $O(4)$ quantum mechanical rotator [Fisher, Privman '83] [Brezin, Zinn-Justin '83] [Leutwyler '87]

$$E_j = \frac{1}{2F^2L_s^3} j(j+2)\,, \qquad j = 0, 1, \ldots\,,$$

where j is the "angular momentum" in the internal $O(4)$ space.

- $\Theta = F^2 L_s^3 (1 + \ldots)$ receives corrections due to ChPT (chiral limit)

- The symmetry breaking terms give corrections to energy E.

- Two dimensionless expansion parameters ($\delta^2 \sim r^4$)

$$\delta^2 = \frac{1}{F^2 L_s^2}\,, \qquad\qquad r^4 = F^8 L_s^{12} M^4$$

- The formula for the energy gap contains low-energy constants (LECs) from ChPT $F, B, \Lambda_1, \Lambda_2$.

- The energy gap will never be measured in experiments, but it can be measured in numerical simulations (lattice).

Motivation

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- Determine the LECs through lattice simulations in the δ-regime.

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1. Chiral perturbation theory in the δ-regime

2. Energy gap E_{L_s} up to NNL order

3. Domain in L_s and M for the rotator

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Chiral perturbation theory

The effective Lagrangian

- QCD at low-energies can be described by an effective field theory (ChPT).

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \dots .$$

- We use the $O(4)$ non-linear σ-model (since $SU(2) \times SU(2) \sim O(4)$).
- The effective Lagrangian is expressed in the fields $\vec{S}$, where $\vec{S}(x)^2 = 1$.

$$\mathcal{L}^{(2)} = \frac{F^2}{2} \partial_\mu \vec{S}(x) \partial_\mu \vec{S}(x) - M^2 F^2 S_0(x) \,,$$

$$\mathcal{L}^{(4)} = -\ell_1 \left(\partial_\mu \vec{S}(x) \partial_\mu \vec{S}(x) \right)^2 - \ell_2 \left(\partial_\mu \vec{S}(x) \partial_\nu \vec{S}(x) \right)^2$$

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Chiral perturbation theory

Separate slow and fast modes

- In δ-regime ($ML_s \ll 1$, $L_t \gg L_s$), collective behaviour sets in
- Global mode performs a slow rotation in internal space (non-perturbative) $\Rightarrow$ slow mode
- Identify the fluctuations around the global mode as the fast modes $\Rightarrow$ fast modes can be integrated out in perturbation theory
- Separate fast modes $\boldsymbol{\pi}(x)$ from slow modes $\vec{e}(t)$ in the partition function $\Rightarrow$ will not discuss this technical issue here

Chiral perturbation theory

Integrate out the fast modes

- Expand the effective action in terms of fast modes
- Integrate out the fast modes in the partition function
- Use dimensional regularisation
- Similar work by Niedermayer & Weiermann (chiral limit, NNL order) but in lattice regularisation

Chiral perturbation theory

Intermediate result after having integrated out the fast modes

After having integrated out the fast modes, we end up with a 1-d problem

$$\mathcal{A}_{\text{eff}} = \int \mathrm{d}t \, \frac{\Theta}{2} \partial_t \vec{e}(t) \partial_t \vec{e}(t) - \eta e_0(t) \, , \qquad \vec{e}(t)^2 = 1 \, ,$$

where

$$\Theta = F^2 L_s^3 \left[1 + \frac{d_2}{F^2 L_s^2} \right.$$ [Hasenfratz, Niedermayer '93]

$$\left. + \frac{1}{F^4 L_s^4} \left[d_4 + d_4' \left(\frac{1}{4} \log(\Lambda_1 L_s)^2 + \log(\Lambda_2 L_s)^2 \right) \right] \right] ,$$ [Hasenfratz '09]

$$\eta = F^2 L_s^3 M^2 \left[1 + \frac{c_2}{F^2 L_s^2} \right] ,$$ [M.W.]

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3. Domain in L_s and M for the rotator

The $O(4)$ rotator

- Interpret the previous system as a quantum mechanical rotator in an external "magnetic" field

$$\mathbf{H} = \frac{1}{\Theta}\left(\frac{\mathbf{L}^2}{2} - (\Theta\eta)\mathbf{e}_0\right),$$

where $\mathbf{L}$ is the "angular momentum" operator in the internal space

- For $\Theta\eta$ small, we calculate the corrections to the unperturbed spectrum
Indeed: $\Theta\eta = F^4 L_s^6 M^2 (1 + \ldots) = r^2 (1 + \ldots)$

- The energy gap: difference of the first excited state to the ground state
- Energy levels of the 1st excited state split up into a triplet and a singlet
- The energy gap up to $\mathcal{O}((\Theta\eta)^4)$ is

$$E = \frac{3}{2\Theta}\left[1 + \frac{(\Theta\eta)^2}{15} - \frac{193}{120}\frac{(\Theta\eta)^4}{15^2}\right].$$

$$E_{L_s} = \frac{3}{2F^2L_s^3}\left\{1 + \frac{2\bar{G}^*}{F^2L_s^2} + \frac{F^8L_s^{12}M^4}{15} + \frac{4(\bar{G}^*)^2 - D_4}{F^4L_s^4} - \frac{\bar{G}^*}{3}\frac{F^8L_s^{12}M^4}{F^2L_s^2} - \frac{193}{120}\left(\frac{F^8L_s^{12}M^4}{15}\right)^2\right\}$$

$$D_4 = d_4 + d_4'\left(\frac{1}{4}\log(\Lambda_1 L_s)^2 + \log(\Lambda_2 L_s)^2\right)$$

Calculate the energy gap up to NNLO

Energy gap up to NLO

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What are the constraints on L_s and M?

How large (small) should L_s (M) be?

- Requiring $1/F^2 L_s^2$ small puts a lower limit on the spatial extent of the box

$$L_s \gtrsim 2.5\,\mathrm{fm}\,.$$

- This gives an upper bound on the pion mass M

$$M \lesssim 70\,\mathrm{MeV}\,.$$

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- We assumed that the two expansion parameters δ^2 and r^4 are of the same order

- $\Rightarrow$ gives even a smaller upper bound on M for a given box size L_s

L_s	M	ML_s
2.0	137	1.40
2.5	63	0.80
3.0	33	0.50
3.5	19	0.34

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How large are the corrections to E_{L_s}?

- Estimate for the size of the corrections (NL, NNL) for E_{L_s}

$$E_{L_s} = \frac{3}{2F^2 L_s^3} \left[1 + \Delta_{\mathrm{NL}} + \Delta_{\mathrm{NNL}}\right]$$

we use $\Lambda_1 = 120\,\mathrm{MeV}$ and $\Lambda_2 = 1200\,\mathrm{MeV}$ and we assume the upper limit of M at a given box size L_s.

- The corrections Δ_{NL}, respectively Δ_{NNL}, for different values of L_s

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Conclusions

- The final result for the energy gap looks quite simple, although the underlying (ChPT) theory is not

- To observe the rotator spectrum:
  - The spatial volume should be quite large $L_s \gtrsim 2.5\,\mathrm{fm}$
  - $F^8 L_s^{12} M^4$ should be small $\Rightarrow M \lesssim 63\,\mathrm{MeV}$
  - When we increase L_s, M must become smaller

- Simulations with quark masses below the physical quark masses are needed