Wilson fermions at fine lattice spacings: scale setting, pion form factors and $(g-2)_\mu$

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In collaboration with:

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Lattice 2010, Villasimius, 15 June 2010

Motivation

- Continuum limit in simulations with dynamical quarks still poorly understood

  → no continuum extrapolation for many quantities

- Hadronic form factors at fine lattice spacings:

  can study a wide range of momentum transfers

- **C**oordinated **L**attice **S**imulations:

  Berlin – CERN – DESY – Madrid – Mainz – Milan – Rome – Valencia – Wuppertal

- Share configurations and technology

CLS run tables

- $N_\mathrm{f} = 2$ flavours of non-perturbatively $\mathrm{O}(a)$ improved Wilson quarks
- Use deflation accelerated DD-HMC algorithm
- Generated ensembles without serious topology problems:

β	$a[\mathrm{fm}]$	lattice	$L[\mathrm{fm}]$	masses	$m_\pi L$	Labels
5.20	0.08	64×32^3	2.6	4 masses	4.8 – 9.0	A1 — A4
5.30	0.07	48×24^3	1.7	3 masses	4.6 – 7.9	D1 — D3
5.30	0.07	64×32^3	2.2	3 masses	4.7 – 7.9	E3 — E5
5.30	0.07	96×48^3	3.4	2 masses	5.0, 4.2	F6,F7
5.50	0.05	96×48^3	2.5	3 masses	5.3 – 7.7	N3 — N5
5.50	0.05	128×64^3	3.4	1 mass	4.7	O6

Outline:

1. **Setting the scale**
2. **The pion electromagnetic form factor**
3. **Hadronic vacuum polarisation contribution to** $(g-2)_\mu$
4. **Summary & Outlook**

1. Setting the scale

- Procedure by CERN group *[Del Debbio et al., hep-lat/0610059]*

  1. Determine κ_s for each sea quark mass:

     Interpolate $\left(\frac{m_{\mathrm{K}}}{m_{\mathrm{K}^*}}\right)^2$ in $(am_{\mathrm{K}})^2$ to $\left.\frac{m_{\mathrm{K}}}{m_{\mathrm{K}^*}}\right|_{\text{phys}} = 0.554$

  2. Interpolate am_{K} in the sea quark mass to reference point:

     $$\left.\frac{m_\pi}{m_{\mathrm{K}}}\right|_{\text{ref}} = 0.85 \quad \Rightarrow \quad am_{\mathrm{K}}|_{\text{ref}}$$

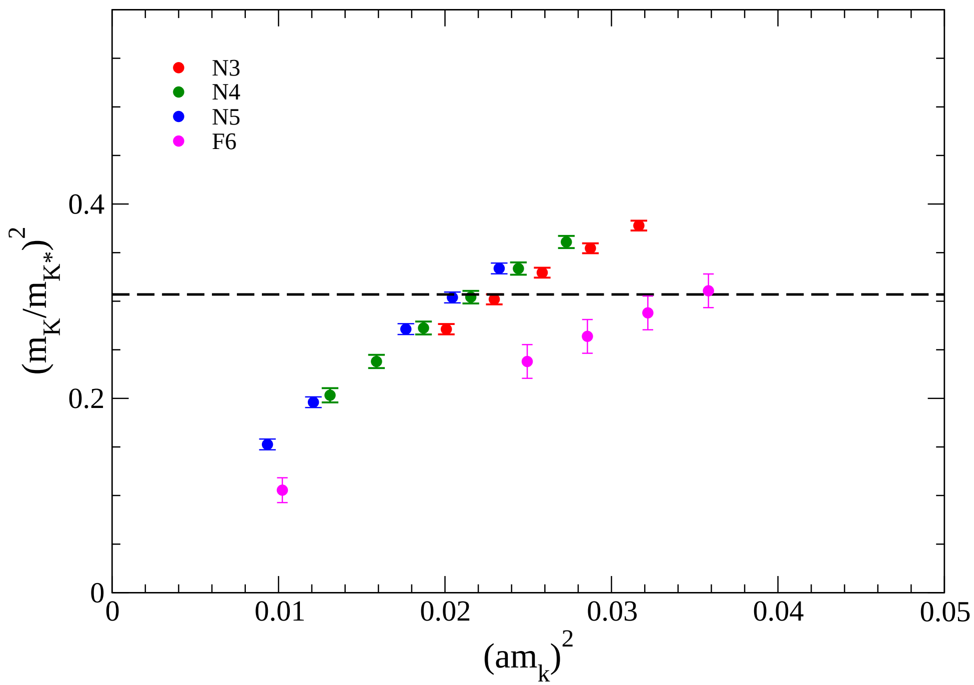
     Using $\quad m_{\mathrm{K}} = 495\,\mathrm{MeV} \quad \Rightarrow \quad a_{\text{ref}}\,[\mathrm{fm}]$

- N.B. Value of $a_{\text{ref}}\,[\mathrm{fm}]$ does not correspond to the physical pion mass.

- Reference point useful to fix the ratio of scales at different bare couplings:

$$a_Q(\beta_1) = \left(\frac{a_{\text{ref}}(\beta_1)}{a_{\text{ref}}(\beta_2)}\right) a_Q(\beta_2), \quad Q = f_{\mathrm{K}},\, r_0,\, m_\Omega, \ldots$$

- Data sets: N3, N4, N5 : 96×48^3 at $\beta = 5.5$; F6 : 96×48^3 at $\beta = 5.3$



- $\beta = 5.5 : \quad am_K|_{\text{ref}} = 0.151(4) \quad \Rightarrow \quad a_{\text{ref}} = 0.0603(15)\,\text{fm}$ (preliminary)

  [Capitani et al., arXiv:0910.5578]

- $\beta = 5.3 : \quad am_K|_{\text{ref}} = 0.197(3) \quad \Rightarrow \quad a_{\text{ref}} = 0.0784(10)\,\text{fm}$

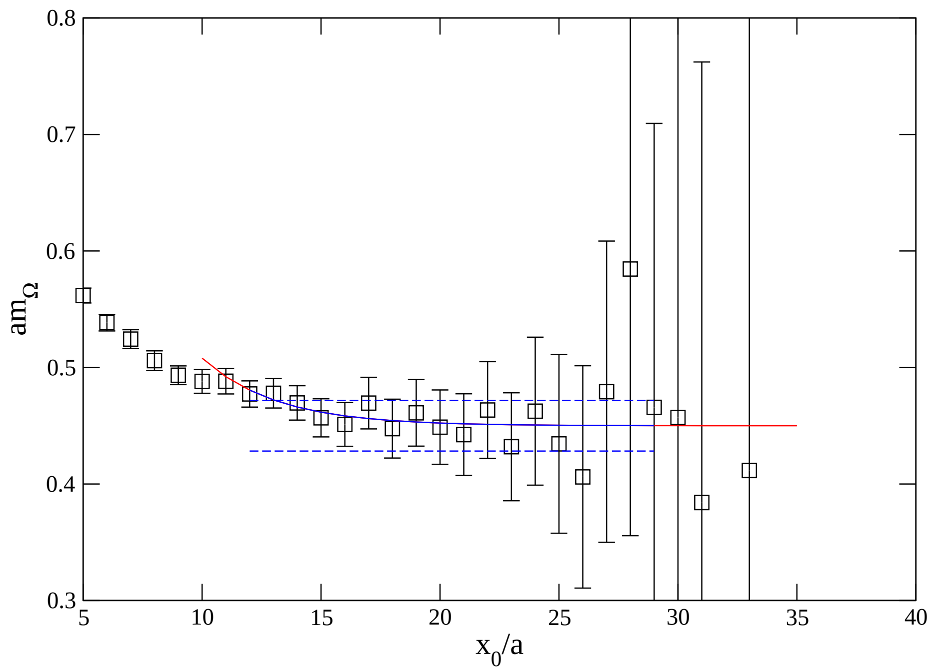
  [Del Debbio et al., hep-lat/0610059]

Setting the scale using m_Ω

- Ω : stable in QCD; weak dependence on sea quark mass

- Effective masses from smeared-local correlator (Jacobi smearing)

  N3 : $\kappa_l = 0.1364, \quad \kappa_{\text{val}} = 0.1365, \quad \kappa_s = 0.13648(2)$



- Fit to ground plus 1^{st} excited state: $m_1 \equiv m_\Omega, \; m_2 \equiv m_\Omega + 2m_\pi$

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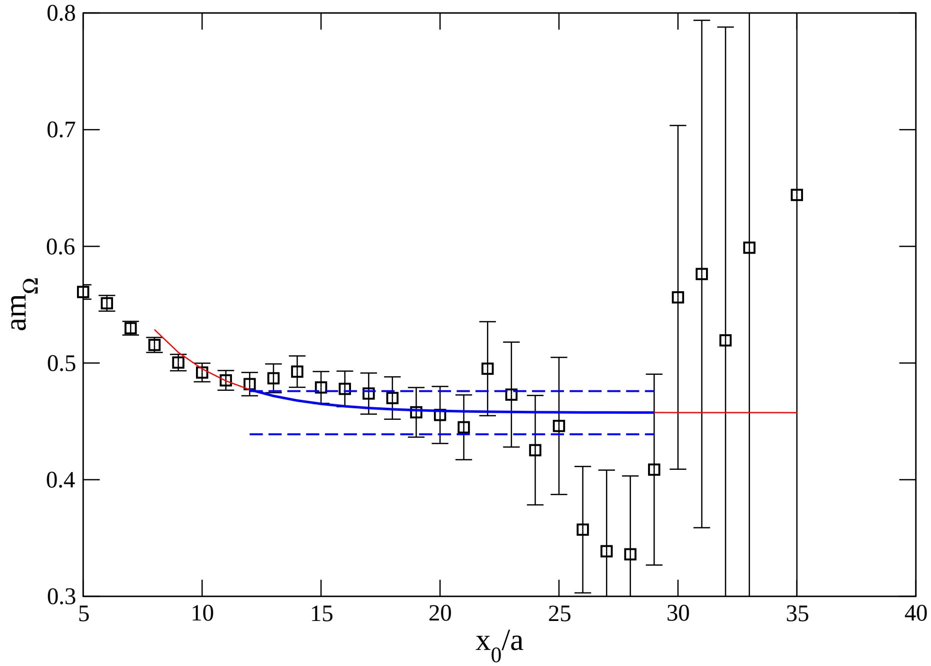
  N4 : $\kappa_l = 0.1365, \quad \kappa_{\text{val}} = 0.1364, \quad \kappa_s = 0.13639(2)$



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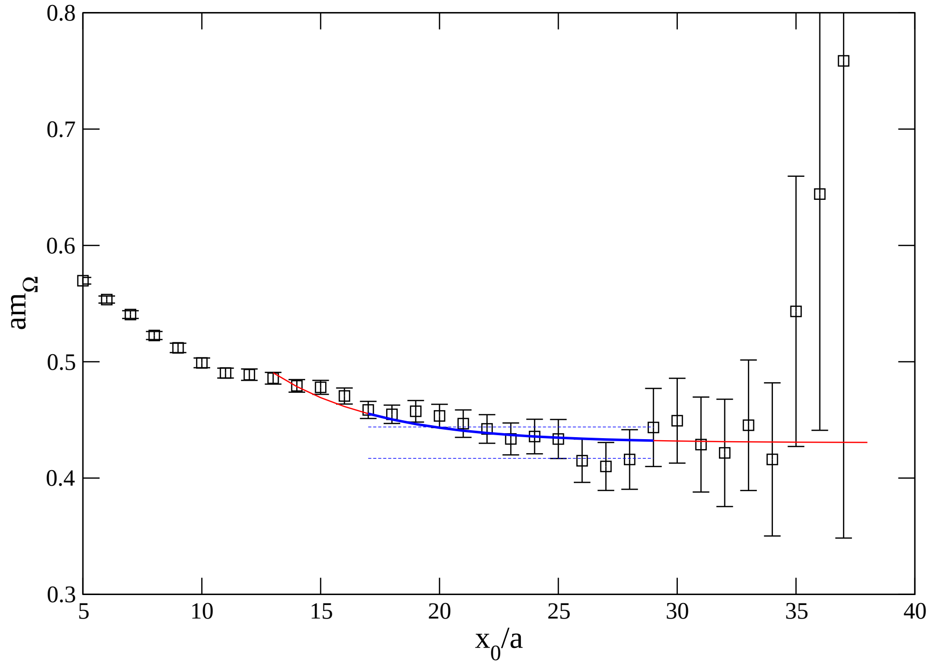
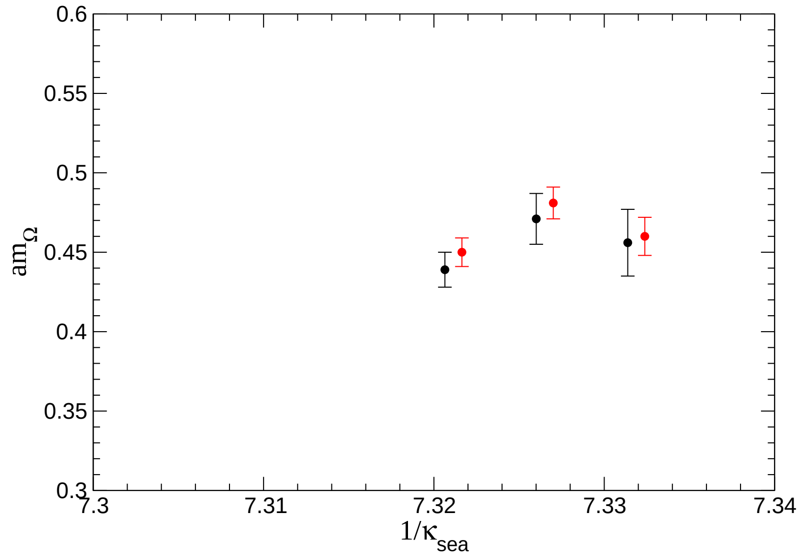
  N5 : $\kappa_l = 0.1366, \quad \kappa_{\text{val}} = 0.1363, \quad \kappa_s = 0.13629(2)$



- Fit to ground plus 1^{st} excited state: $m_1 \equiv m_\Omega, \; m_2 \equiv m_\Omega + 2m_\pi$

- Sea quark mass dependence of am_Ω at $\beta = 5.5$:



- Weak dependence on m_{sea} confirmed: quote am_Ω at lightest quark mass

$\rightarrow \quad \beta = 5.5: \quad a_\Omega = 0.053(1)\,\text{fm}$ (preliminary)

- No determination of m_Ω at $\beta = 5.3$ so far

$$\rightarrow \quad \beta = 5.3: \quad a_\Omega = \left(\frac{a_{\text{ref}}(\beta = 5.3)}{a_{\text{ref}}(\beta = 5.5)}\right) 0.053(1)\,\text{fm} = 0.069(2)\,\text{fm} \quad \text{(preliminary)}$$

To-do list:

- Investigate different conditions to fix κ_s

  $\longrightarrow$ requires extension of the range of valence quark masses

- Refinement of procedures to extract baryon masses:

  Matrix and vector correlators

- Comparison with determination of r_0 on CLS ensembles: (B. Leder's talk)

  $\longrightarrow$ determine $r_0 m_\Omega$

2. Pion electromagnetic form factor

- Provides information on pion structure:

$$\left\langle \pi^+(\vec{p}_f) \middle| \tfrac{2}{3}\overline{u}\gamma_\mu u - \tfrac{1}{3}\overline{d}\gamma_\mu d \middle| \pi^+(\vec{p}_i) \right\rangle = (p_f + p_i)_\mu \, f_\pi(q^2)$$

$$q^2 = (p_f - p_i)^2 : \qquad \text{momentum transfer}$$

- Pion charge radius derived from form factor at zero q^2:

$$f_\pi(q^2) = 1 - \frac{1}{6}\langle r^2\rangle q^2 + \mathrm{O}(q^4) \quad \Rightarrow \quad \langle r^2\rangle = 6\,\frac{\mathrm{d}f_\pi(q^2)}{\mathrm{d}q^2}\bigg|_{q^2=0}$$

- Restriction on minimum accessible momentum transfer:

$$\vec{p}_{i,f} = \vec{n}\frac{2\pi}{L} \quad \Rightarrow \quad |q^2| \geq 2m_\pi\left(m_\pi - \sqrt{m_\pi^2 + (2\pi/L)^2}\right)$$

$\longrightarrow$ Lack of accurate data points near q^2

Twisted boundary conditions

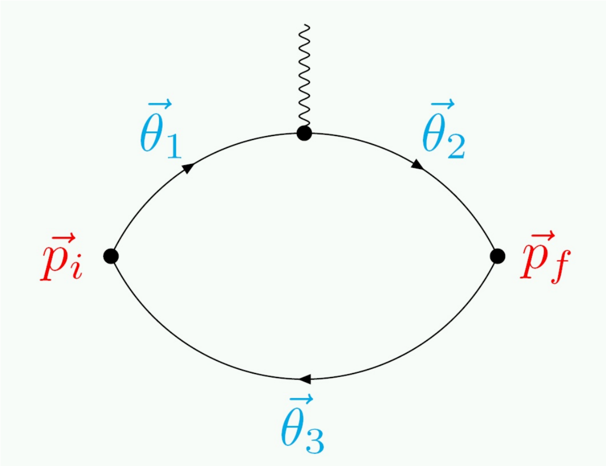
[Bedaque 2004; de Divitiis, Petronzio & Tantalo 2004; Flynn, Jüttner & Sachrajda 2005]

- Apply "twisted" spatial boundary conditions;

  Impose periodicity up to a phase $\vec{\theta}$:

$$\psi(x + L\hat{e}_k) = \mathrm{e}^{i\theta_k}\psi(x) \quad \Rightarrow \quad p_k = n_k \frac{2\pi}{L} + \frac{\theta_k}{L}, \quad k = 1, 2, 3$$

- Can tune $|q^2|$ to any desired value *[Boyle, Flynn, Jüttner, Sachrajda, Zanotti, hep-lat/0703005]*



$$\vec{\theta}_i = \vec{\theta}_1 - \vec{\theta}_3,$$
$$\vec{\theta}_f = \vec{\theta}_2 - \vec{\theta}_3$$

$$\Rightarrow \; q^2 = (p_i - p_f)^2 = \Big(E_\pi(\vec{p}_i) - E_\pi(\vec{p}_f)\Big)^2 - \left[\left(\vec{p}_i + \frac{\vec{\theta}_i}{L}\right) - \left(\vec{p}_f + \frac{\vec{\theta}_f}{L}\right)\right]^2$$

Preliminary results

β	$L^3 \cdot T$	a[fm]	L[fm]	m_π [MeV]	Lm_π
5.50	$48^3 \cdot 96$	0.053	2.5	600	7.7
5.50	$48^3 \cdot 96$	0.053	2.5	510	6.5
5.50	$48^3 \cdot 96$	0.053	2.5	410	5.3
5.30	$48^3 \cdot 96$	0.069	3.3	290	5.0

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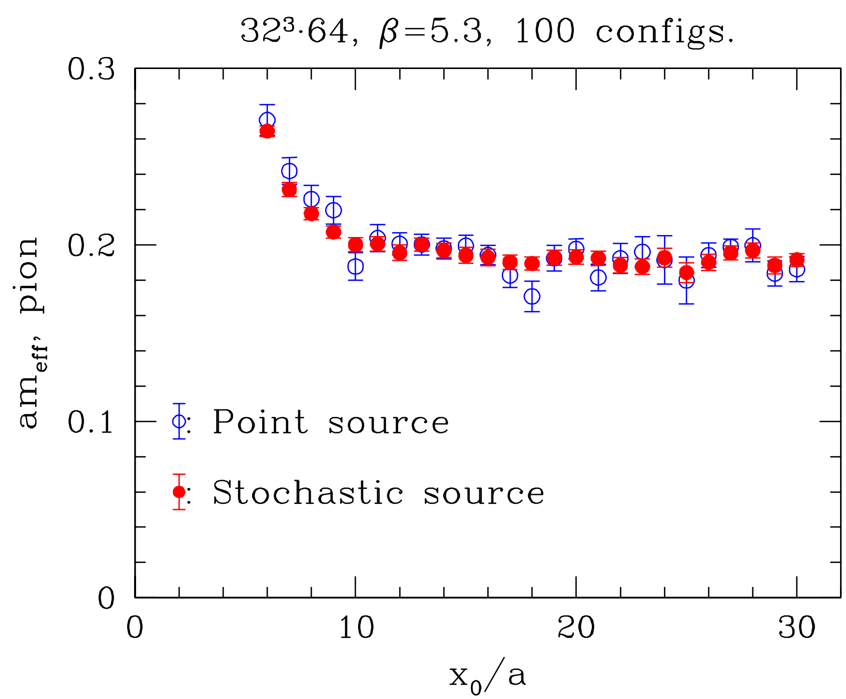
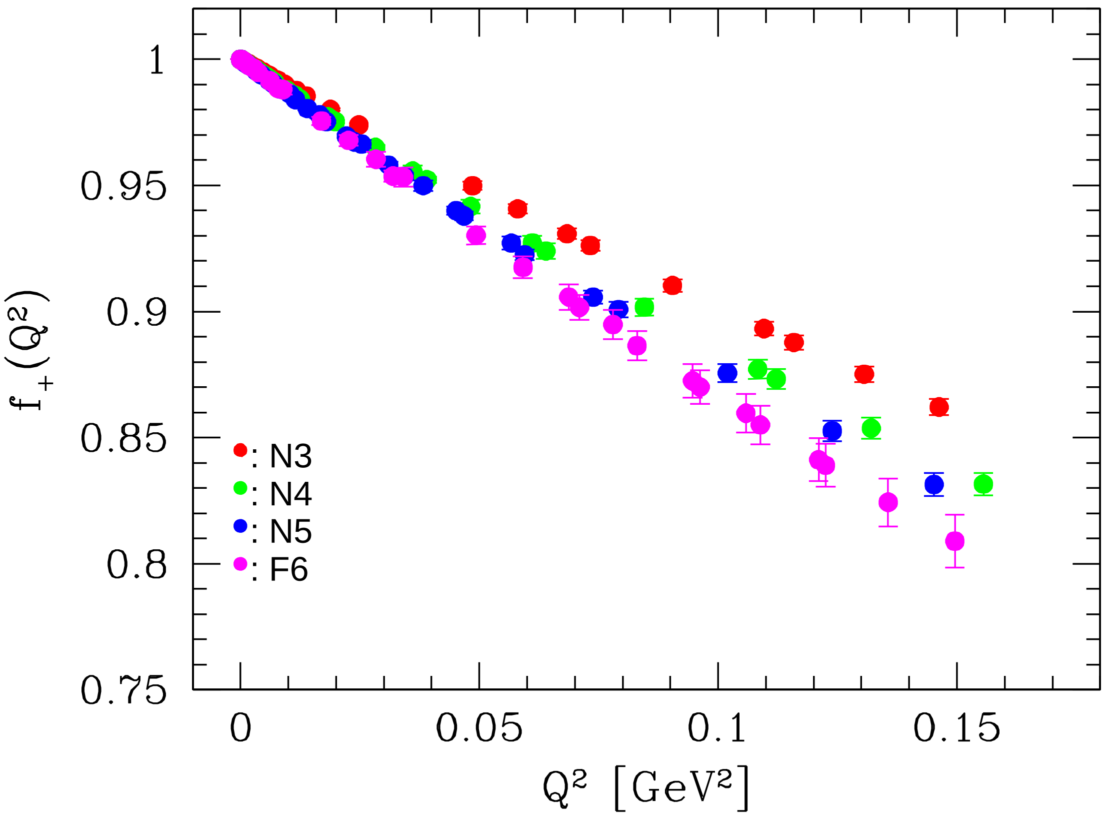
- Use stochastic noise source ("one-end trick") *[E. Endreß, Diploma thesis, 2009]*



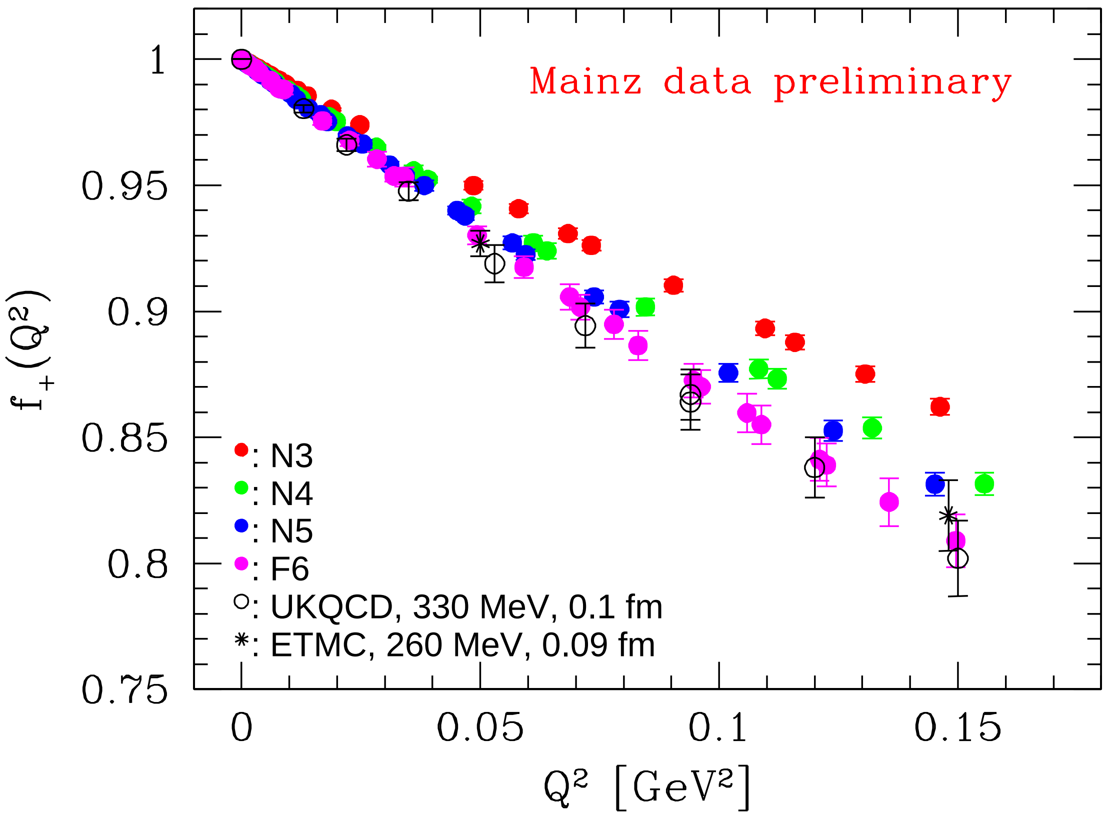
Pion form factor

- Mainz data



Pion form factor

- Comparison with other lattice calculations



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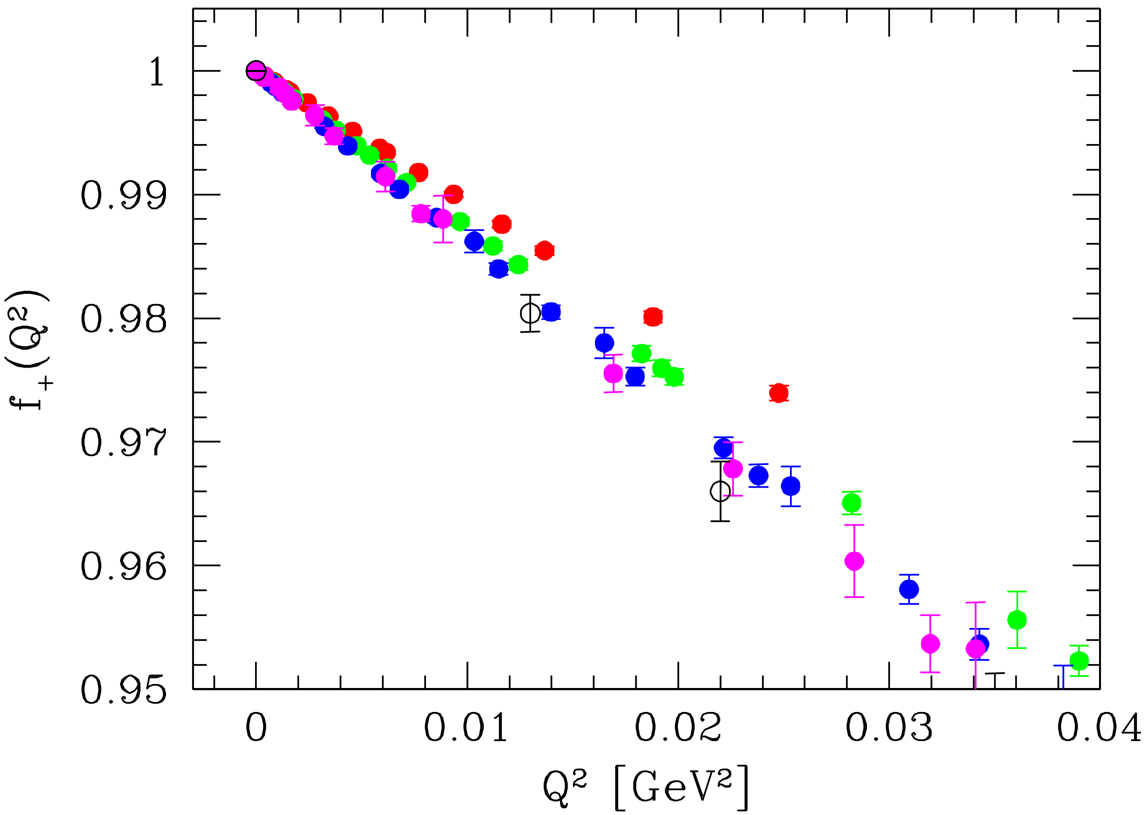

Pion charge radius

- Twisted boundary conditions: accurate data near $Q^2 = 0$

  $\longrightarrow$ extract charge radius from linear slope

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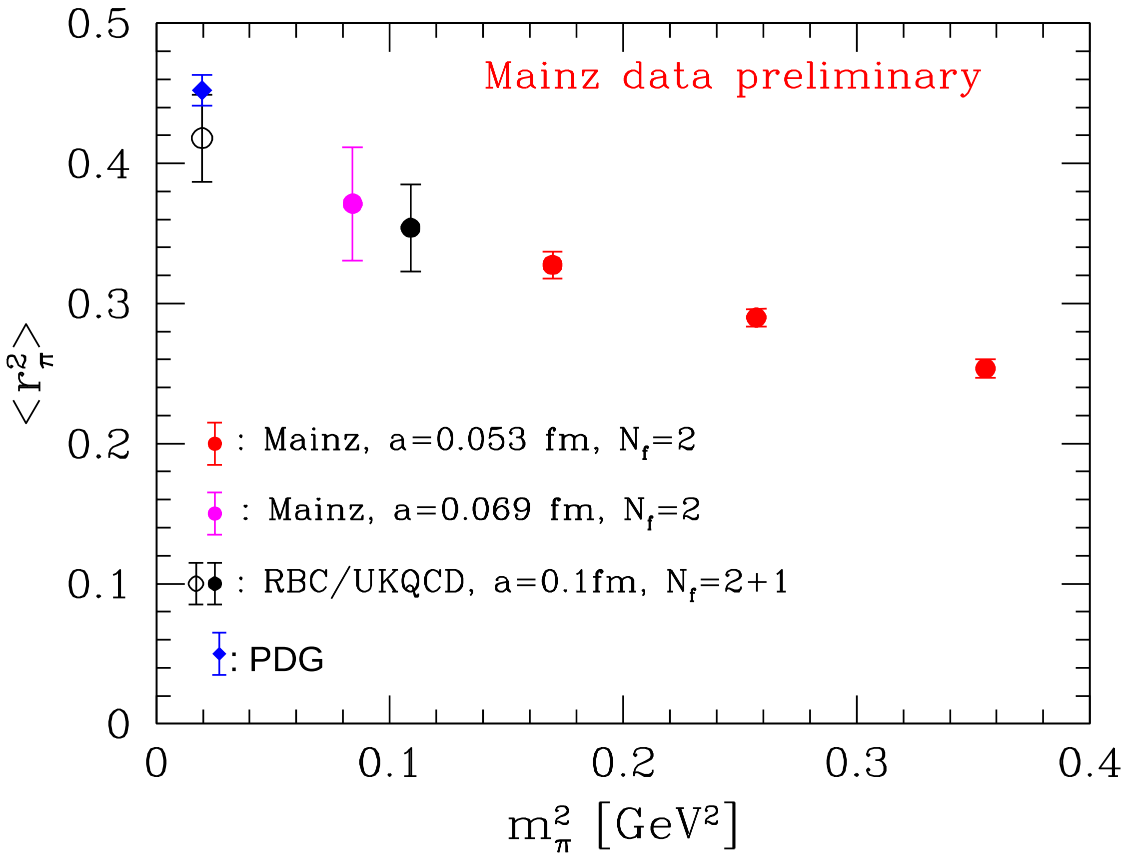

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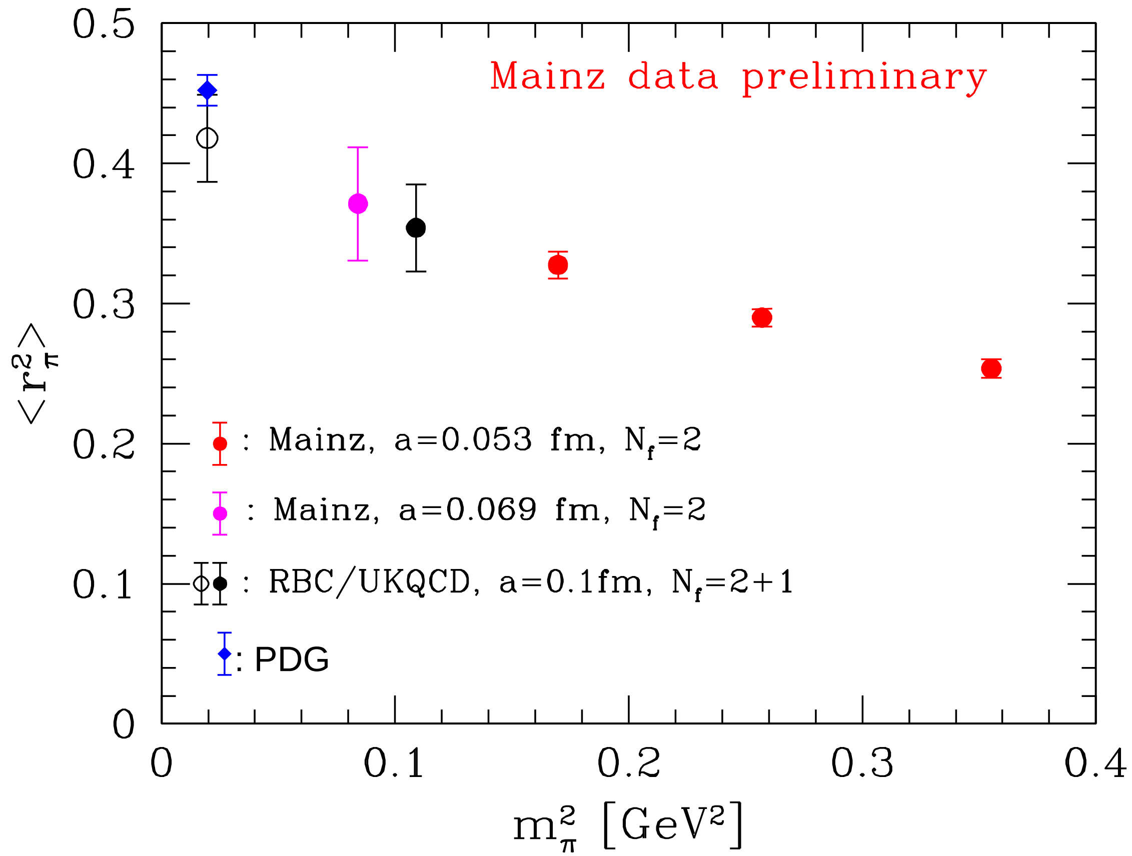

- Still to come: Fits to ChPT including vector degrees of freedom

3. Hadronic vacuum polarisation contribution to $(g-2)_\mu$

- Muon anomalous magnetic moment: $a_\mu = \frac{1}{2}(g-2)_\mu$

$$a_\mu = \begin{cases} 11\,659\,208(6.3)\cdot 10^{-10} & \text{Experiment} \\ 11\,659\,179(6.5)\cdot 10^{-10} & \text{SM prediction,} \end{cases} \qquad (3.2\sigma \text{ tension})$$

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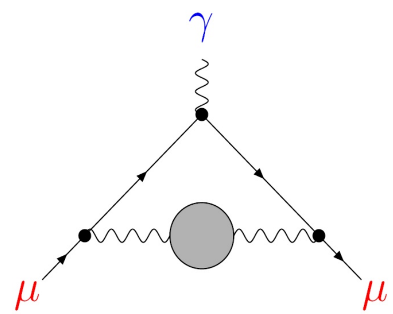
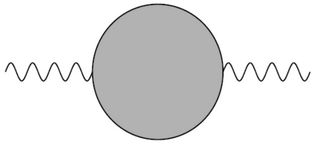
- Hadronic vacuum polarisation; leading contribution:



- Vacuum polarisation tensor:



$$\Pi_{\mu\nu}(q^2) = \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot(x-y)}\, \langle J_\mu(x) J_\nu(y)\rangle \equiv (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)$$

- a_μ^{had} determined from convolution integral: *[Blum; Blum & Aubin]*

$$a_\mu^{\mathsf{had}} = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2)\{\Pi(Q^2) - \Pi(0)\}$$

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Problems for lattice calculations:

- Convolution integral dominated by momenta near m_μ :

  maximum of $f(Q^2)$ located at: $(\sqrt{5}-2)m_\mu^2 \approx 0.003\,\mathrm{GeV}^2$

  lowest momentum transfer: $\left(\frac{2\pi}{T}\right)^2 \approx 0.06\,\mathrm{GeV}^2$

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- Contributions from quark disconnected diagrams

  – Large noise-to-signal ratio

  – Twisted boundary conditions useless:
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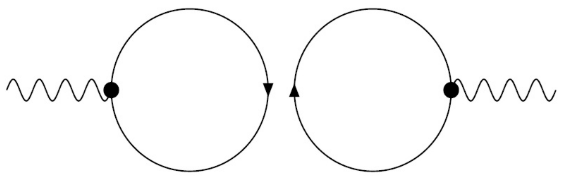

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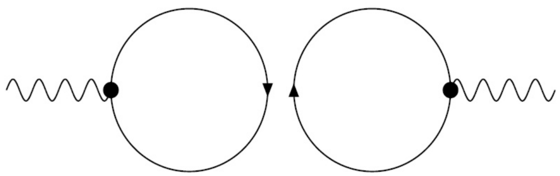

- Resonance effects: $\rho \to \pi\pi$

New strategy for two-flavour QCD

[Della Morte & Jüttner, arXiv:0910.3755]

- QCD with $N_\mathrm{f} = 2$ flavours: $J_\mu(x) = \left(\frac{2}{3} j_\mu^{uu} - \frac{1}{3} j_\mu^{dd}\right)(x)$

$$\langle J_\mu(x) J_\nu(y) \rangle = \frac{4}{9} \left\langle j_\mu^{uu} j_\nu^{uu} \right\rangle - \frac{2}{9} \left\langle j_\mu^{uu} j_\nu^{dd} \right\rangle - \frac{2}{9} \left\langle j_\mu^{dd} j_\nu^{uu} \right\rangle + \frac{1}{9} \left\langle j_\mu^{dd} j_\nu^{dd} \right\rangle$$

- Impose isospin symmetry, $m_u = m_d$, set $y \equiv 0$; Correlation function:

$$C_{\mu\nu}(q) = \frac{5}{9} C_{\mu\nu}^{(\mathsf{con})}(q) + \frac{1}{9} C_{\mu\nu}^{(\mathsf{disc})}(q)$$

$$C_{\mu\nu}^{(\mathsf{con})}(q) = \sum_x \mathrm{e}^{iq\cdot x} \left\langle \mathsf{Tr}\left[\overline{\psi}(x)\gamma_\mu\psi(x)\,\overline{\psi}(0)\gamma_\mu\psi(0)\right] \right\rangle$$

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- $C_{\mu\nu}^{(\mathsf{con})}(q)$ and $C_{\mu\nu}^{(\mathsf{disc})}(q)$ have individual continuum and finite volume limits

- $C_{\mu\nu}^{(\mathsf{con})}(q)$ can be evaluated using twisted boundary conditions

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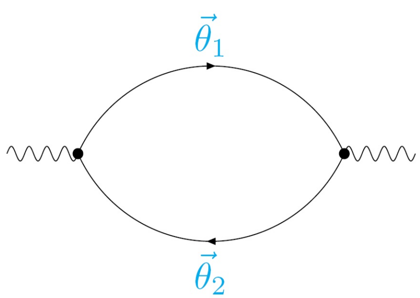

Relative size of the disconnected contribution

[Della Morte & Jüttner, arXiv:0910.3755]

- Compute polarisation tensor in SU(2) ChPT @ NLO
- Determine disconnected and connected contributions to $\Pi(q^2)-\Pi(0)$ (enters convolution integral)

$$\Rightarrow \quad \frac{\Pi^{(\mathrm{disc})}(q^2)-\Pi^{(\mathrm{disc})}(0)}{\Pi^{(\mathrm{con})}(q^2)-\Pi^{(\mathrm{con})}(0)} = -\frac{1}{10}$$

$\longrightarrow$ Effect of disconnected contribution estimated to be a 10% downward shift

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Strategy to compute a_μ^{had} in two-flavour QCD

- Compute connected contribution using twisted boundary conditions

- Compute disconnected contribution for Fourier modes only:

  $\longrightarrow$ validate its relative suppression predicted by ChPT

Preliminary results

- Test runs at $\beta = 5.3, \quad a = 0.069(2)\,\mathrm{fm}, \quad 32{\times}24^3$, $64{\times}32^3$ and $96{\times}48^3$

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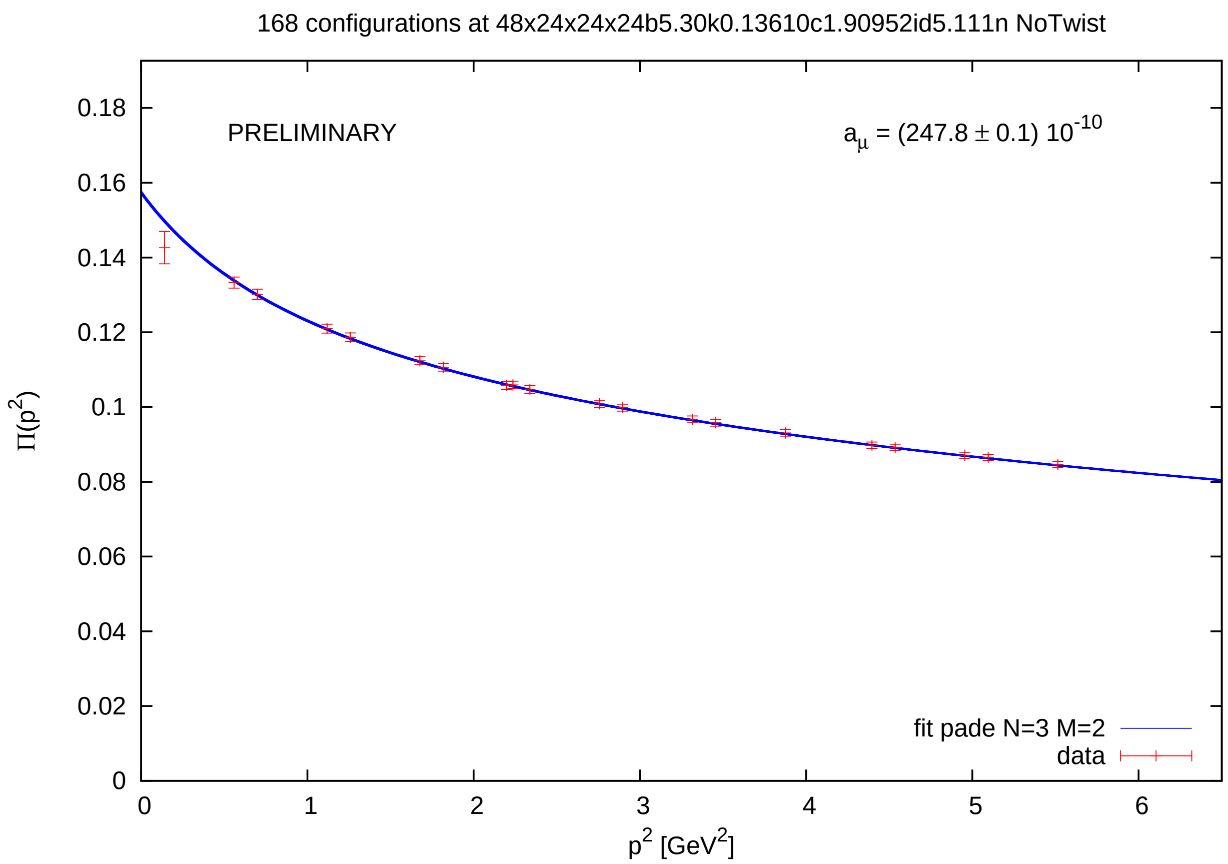

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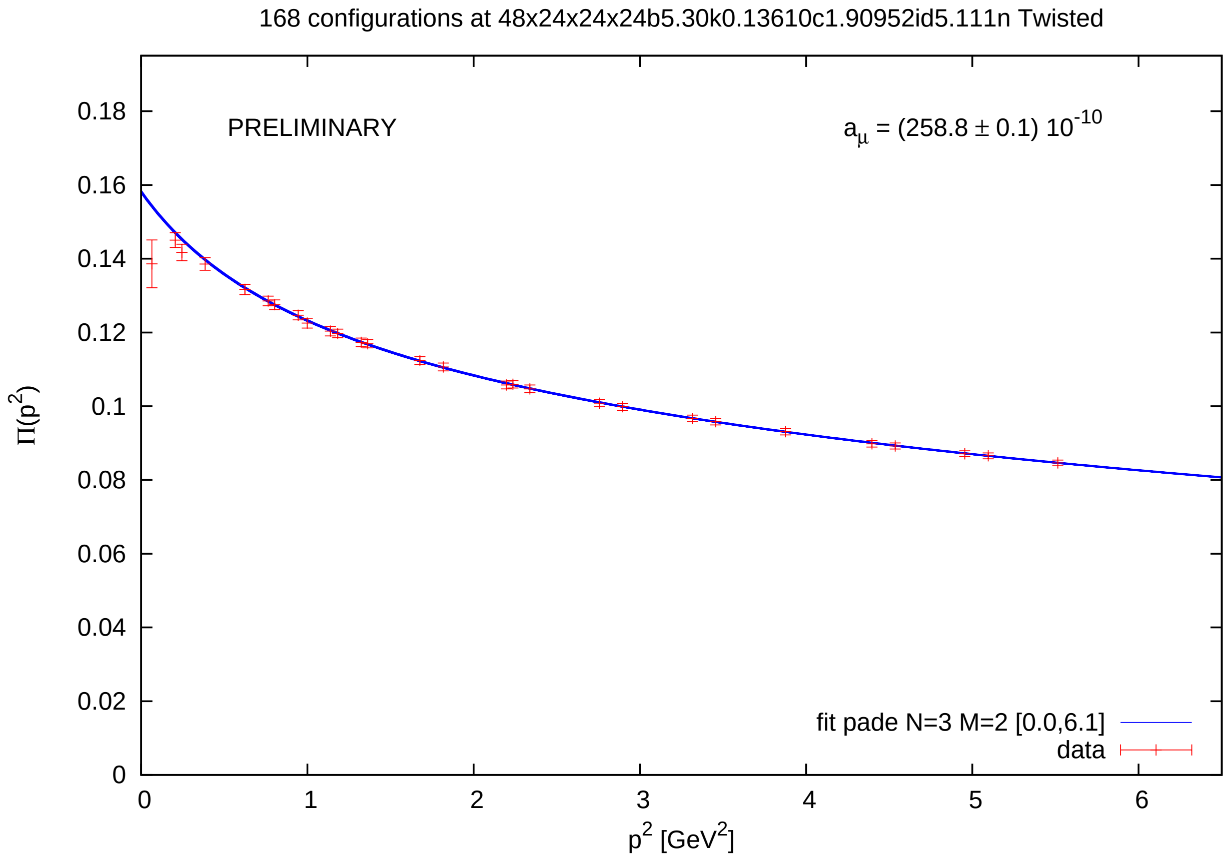

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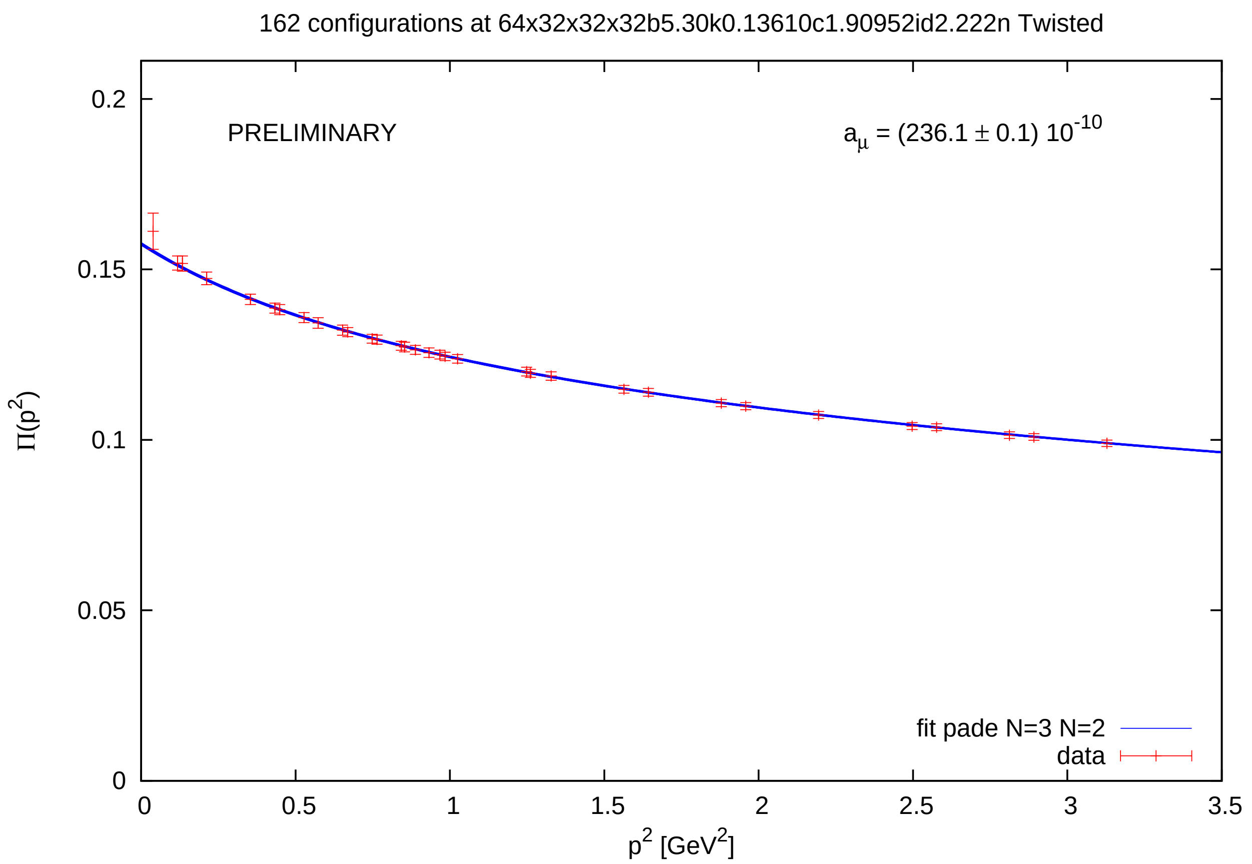
  $m_\pi = 550\,\text{MeV}, \quad L \simeq 2.2\,\text{fm}$



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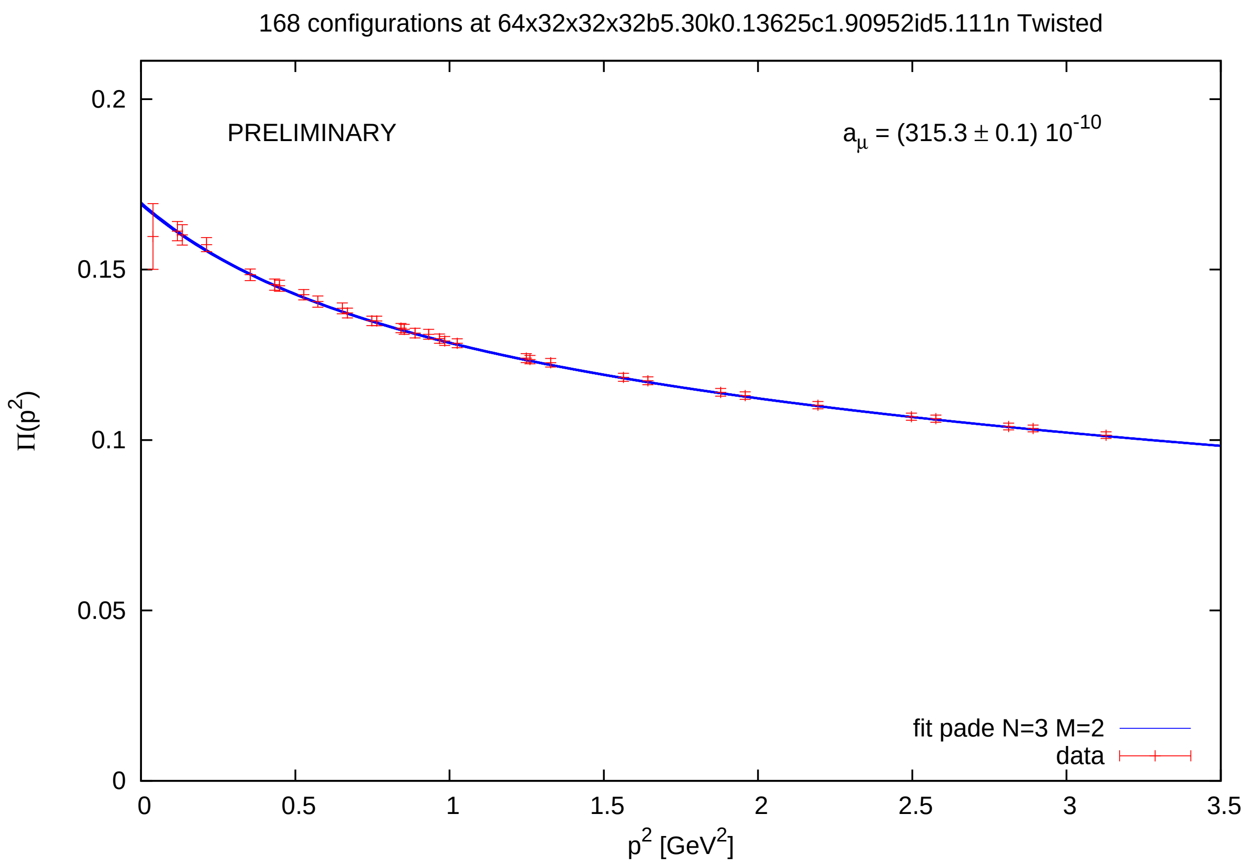
- Test runs at $\beta = 5.3, \quad a = 0.069(2)\,\text{fm}, \quad 32{\times}24^3$, $64{\times}32^3$ and $96{\times}48^3$

  $m_\pi = 420\,\text{MeV}, \quad L \simeq 2.2\,\text{fm}$



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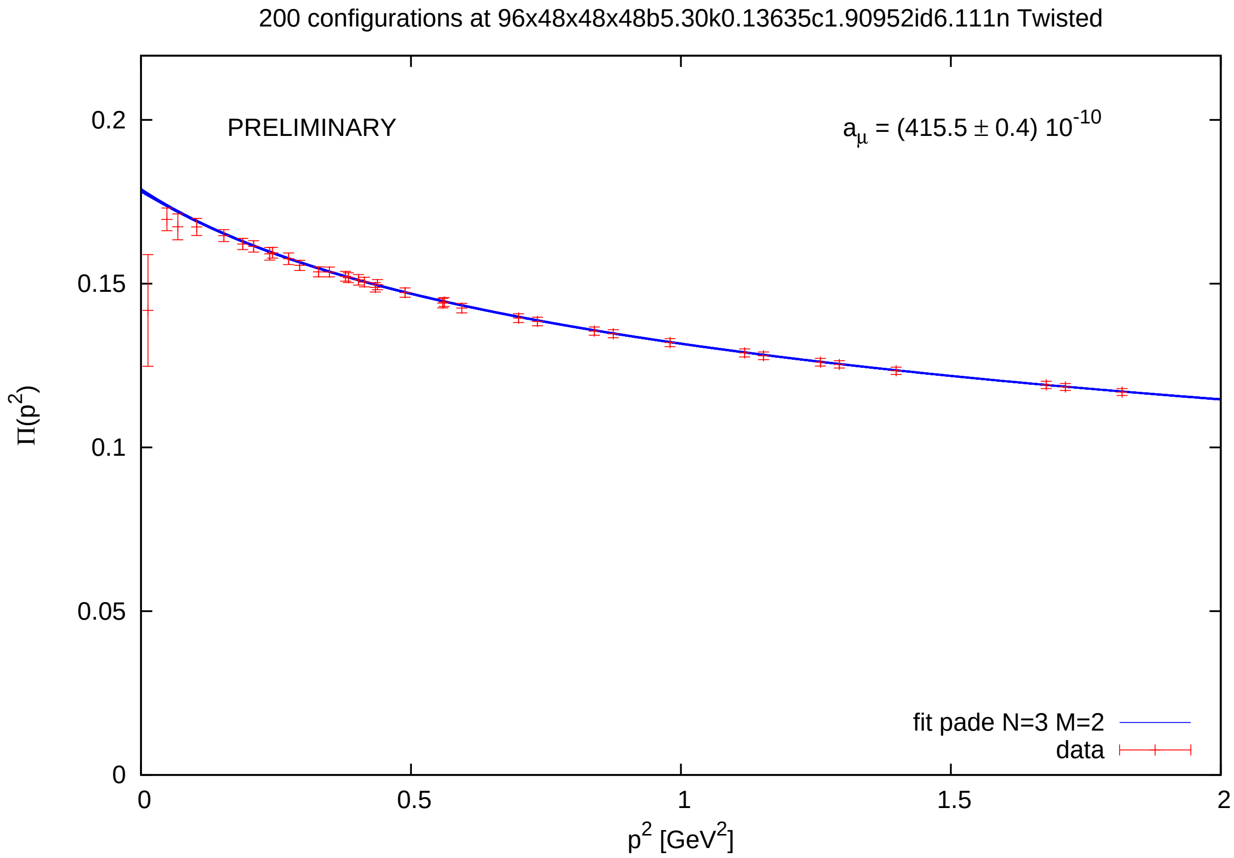
- Test runs at $\beta = 5.3$, $a = 0.069(2)\,\mathrm{fm}$, 32×24^3, 64×32^3 and 96×48^3

  $m_\pi = 290\,\mathrm{MeV}$, $L \simeq 3.4\,\mathrm{fm}$



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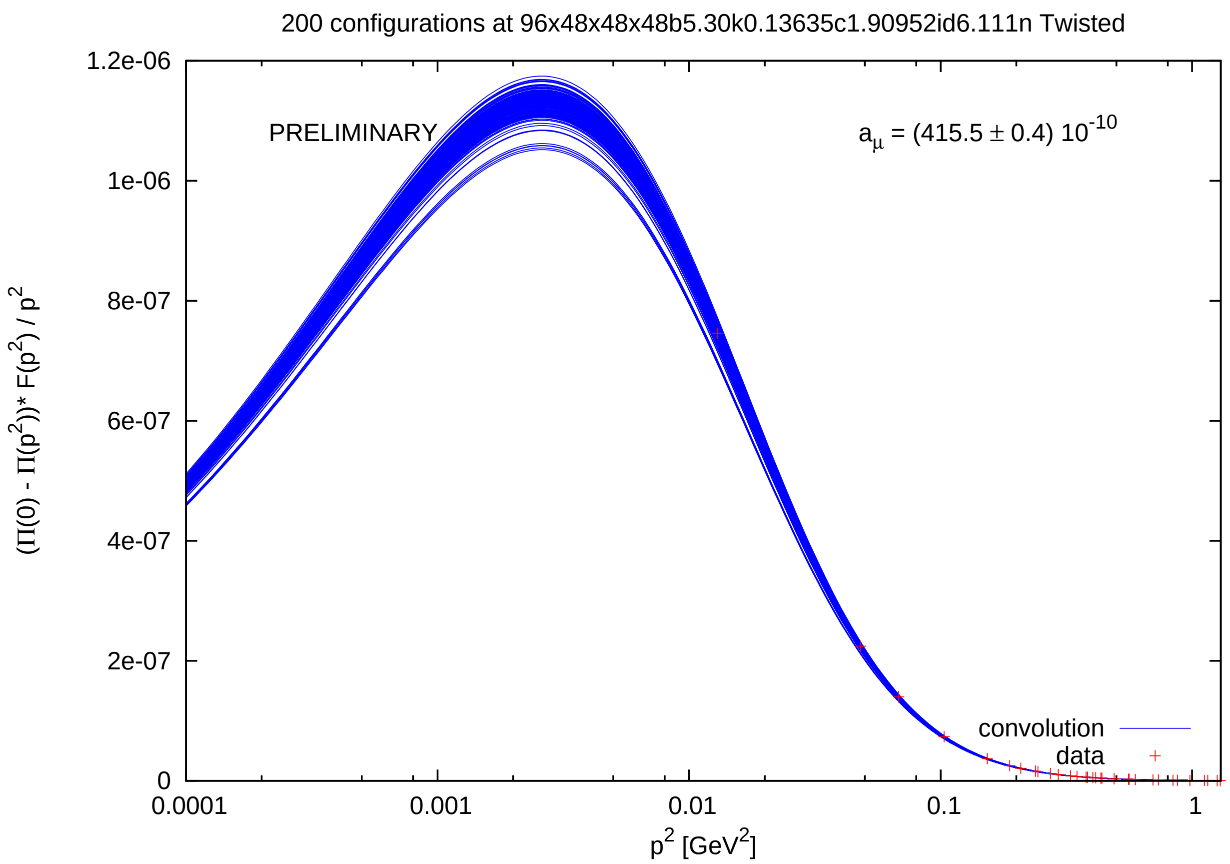
  $m_\pi = 290\,\mathrm{MeV}, \quad L \simeq 3.4\,\mathrm{fm}$



Preliminary results

- Twisted boundary stabilise fits to Q^2-dependence and extrapolation to $\Pi(0)$

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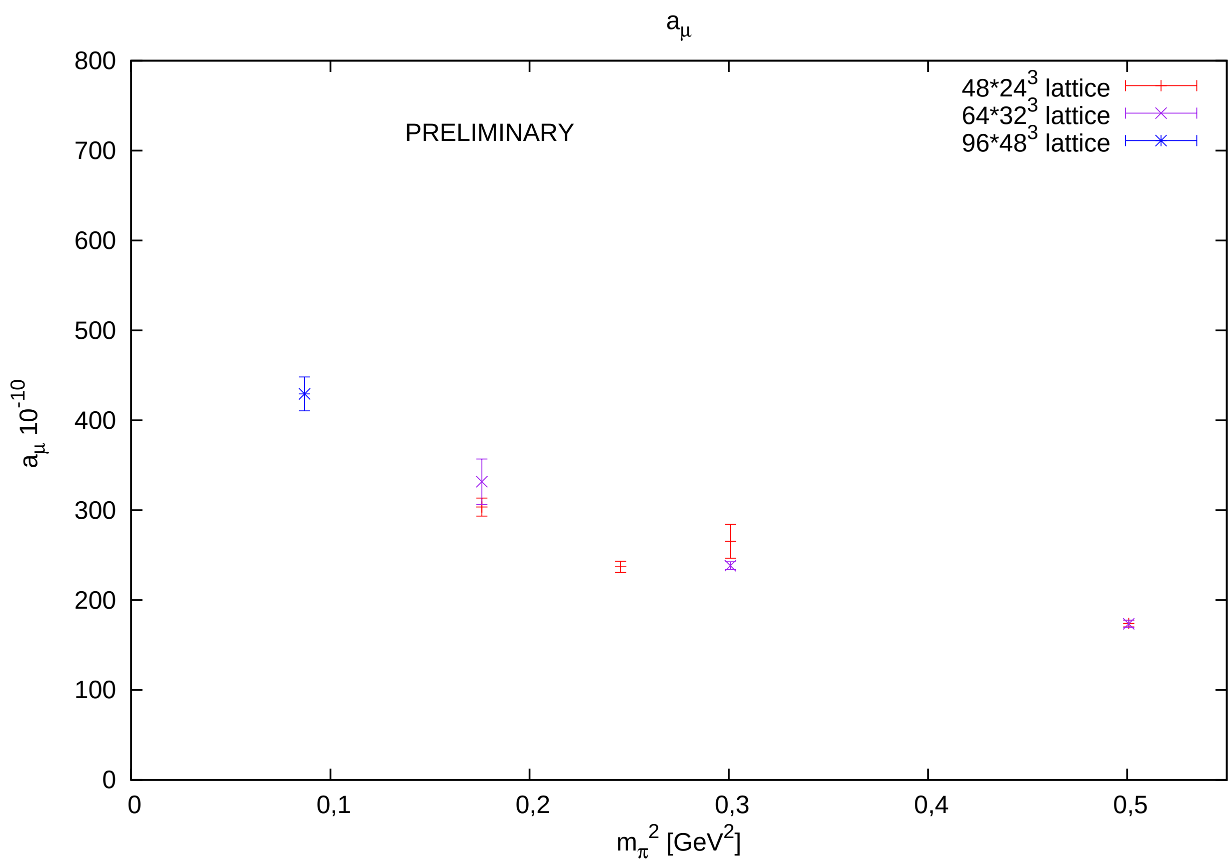

Alternative approach

[Bernecker, Meyer]

- Relate dispersion integral to $\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$ via integral transform

- Split dispersion integral into perturbative and non-perturbative part

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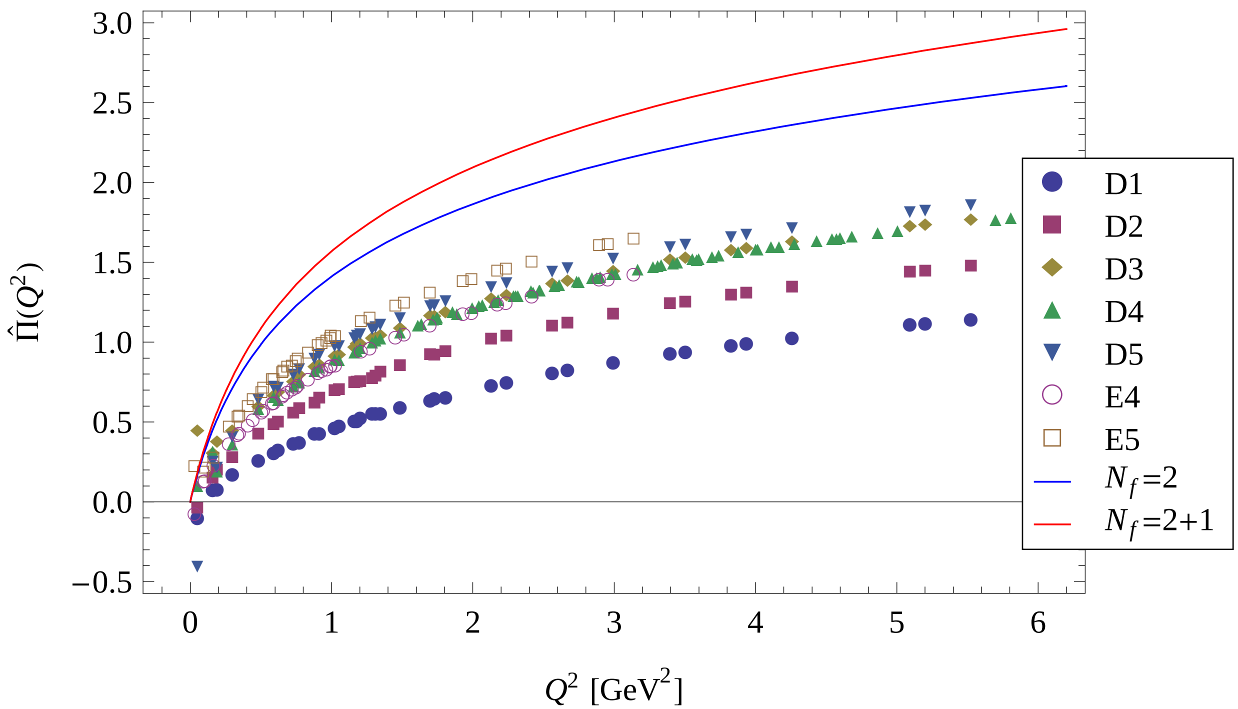

4. Summary & Outlook

- CLS ensembles allow for comprehensive investigation of systematic effects:
  - lattice artefacts
  - finite-volume effects

- Twisted boundary conditions crucial for precision calculations of $f_+(Q^2)$

- Calculations of $(g-2)_\mu$ also profit from the use of twisted boundary conditions

- Pion form factor:
  - Study lattice artfacts
  - Perform chiral fits

- $(g-2)_\mu$:
  - Incorporate quenched strange quark
  - Estimate size of disconnected contribution