Staggered chiral perturbation theory in the two-flavor case and SU(2) analysis of the MILC data

Xining Du

Washington University in St.Louis

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1. Introduction to staggered chiral perturbation theory
2. Staggered chiral perturbation theory in the two-flavor case
3. SU(2) chiral analysis of the MILC lattice data

Staggered fermions

- Staggered fermion action in terms of one-component field $\chi(x)$

$$S^{KS} = m_q \sum_x \bar{\chi}(x)\chi(x) + \frac{1}{2a}\sum_{x,\mu} \bar{\chi}(x)\eta_\mu(x)\ [\chi(x+a\hat{\mu}) - \chi(x-a\hat{\mu})],$$

$$\eta_\mu(x) = (-1)^{\sum_{\nu<\mu} x_\nu/a}.$$

- Staggered action in spin-taste basis (free theory)

$$S^{KS} = 16\sum_y \bar{q}(y)\left\{ m(I\otimes I) + \sum_\mu [(\gamma_\mu \otimes I)\bigtriangledown_\mu + a(\gamma_5\otimes\xi_\mu\xi_5)\triangle_\mu] \right\} q(y)$$

$$q(y)_{\alpha i} = \frac{1}{8}\sum_A (\Gamma_A)_{\alpha i}\chi(2y+aA), \quad \bar{q}(y)_{i\alpha} = \frac{1}{8}\sum_A \bar{\chi}(2y+aA)(\Gamma_A)^\dagger_{i\alpha},$$

- In interacting case, the staggered action takes the form

$$S^{KS}_{int} = 16\sum_y \bar{q}(y)\left\{ m(I\otimes I) + \sum_\mu [(\gamma_\mu\otimes I)\bigtriangledown_\mu + aS_5 + a^2 S_6 + \cdots] \right\} q(y) \quad (1)$$

- Lattice discretization effects break taste symmetry
- Taste breaking effects are incorporated in the chiral perturbation theory for staggered fermions → (rooted) staggered chiral perturbation theory (rSχPT)

Staggered chiral perturbation theory (SχPT)

- Constructing the staggered chiral perturbation theory
  - Symanzik effective theory (SET) for staggered fermions
  - Map the terms in SET to operators in the chiral Lagrangian by using spurion analysis
- Power counting scheme depends on the action formalism. For Asqtad staggered action

$$p^2 \sim m_\pi^2 \sim \mu m_q \sim a^2\delta \qquad (p^2 \sim m_q \sim a^2) \tag{2}$$

- Assume the usage of fourth-root procedure is legitimate
  - Violate locality at non-zero lattice spacing.
    C. Bernard, M. Golterman and Y. Shamir [PRD 73:114511 (2006)]
  - Locality and universality are apparently restored in the continuum limit.
    C. Bernard [PRD 73:114503 (2006)], Y. Shamir [PRD 71:034509 (2005)]
- In the chiral theory, fourth-root procedure can be taken by associating factors of 1/4 with sea quark loops in quark flow diagrams, or by using the "replica" method (more systematic)

SU(2) $rS\chi PT$

- m_s is much larger than m_u, m_d. Expansions in m_K^2/Λ_χ^2 do not converge as fast as m_π^2/Λ_χ^2
- Expanding around the two-flavor chiral limit $m_u = m_d = 0, m_s = m_s^{phys}$. $\Rightarrow$ Converges faster
- Generalize SU(2) χPT to SU(2) $rS\chi PT$ in the same way as constructing SU(3) $rS\chi PT$
- Find the most general SU(2) $rS\chi PT$ Lagrangian up to NLO, and calculate pion mass and decay constant in the partially quenched case
- Obtain relations between SU(2) LECs and SU(3) LECs
  - Ordinary LECs: results available in literature
  - Taste-violating parameters: results of this work

Pion mass and decay constant in SU(2) $rS\chi PT$

$$\frac{m^2_{P_5^+}}{(m_x+m_y)} = \mu_{(2)}\Big\{1+\frac{1}{16\pi^2 f^2_{(2)}}\Big[\sum_j R_j^{[2,1]}(\{\mathcal{M}^{[2]}_{XY_I}\})l(m_j^2) - 2a^2\delta_V'^{(2)}\sum_j R_j^{[3,1]}(\{\mathcal{M}^{[3]}_{XY_V}\})l(m_j^2) + (V\leftrightarrow A) + a^2(\tilde{L}''_{(2)}+\tilde{L}'_{(2)})\Big] + \frac{\mu_{(2)}}{f^2_{(2)}}(4l_3+p_1+4p_2)(m_u+m_d) + \frac{\mu_{(2)}}{f^2_{(2)}}(-p_1-4p_2)(m_x+m_y)\Big\}, \quad (3)$$

$$f_{P_5^+} = f_{(2)}\Big\{1+\frac{1}{16\pi^2 f^2_{(2)}}\Big[-\frac{1}{32}\sum_{Q,B} l(m^2_{Q_B}) + \frac{1}{4}\Big(l(m^2_{X_I})+l(m^2_{Y_I})+(m^2_{U_I}-m^2_{X_I})\tilde{l}(m^2_{X_I})+(m^2_{U_I}-m^2_{Y_I})\tilde{l}(m^2_{Y_I})\Big) - \frac{1}{2}\Big(R^{[2,1]}_{X_I}(\{\mathcal{M}^{[2]}_{XY_I}\})l(m^2_{X_I}) + R^{[2,1]}_{Y_I}(\{\mathcal{M}^{[2]}_{XY_I}\})l(m^2_{Y_I})\Big) + \frac{a^2\delta_V'^{(2)}}{2}\Big(R^{[2,1]}_{X_V}(\{\mathcal{M}^{[2]}_{X_V}\})\tilde{l}(m^2_{X_V}) + \sum_j D^{[2,1]}_{j,X_V}(\{\mathcal{M}^{[2]}_{X_V}\})l(m_j^2) + (X\leftrightarrow Y) + 2\sum_j R_j^{[3,1]}(\{\mathcal{M}^{[3]}_{XY_V}\})l(m_j^2)\Big) + (V\leftrightarrow A) + a^2(\tilde{L}''_{(2)}-\tilde{L}'_{(2)})\Big] + \frac{\mu_{(2)}}{2f^2_{(2)}}(4l_4-p_1)(m_u+m_d) + \frac{\mu_{(2)}}{2f^2_{(2)}}(p_1)(m_x+m_y)\Big\}. \quad (4)$$

(All LECs are one-loop renormalized)

Relation of SU(2) and SU(3) $S\chi PT$

- Expect the SU(2) theory to be generated from the SU(3) theory when

$$\frac{m_x}{m_s}, \frac{m_y}{m_s}, \frac{m_l}{m_s}, \frac{a^2\Delta_B}{\mu m_s}, \frac{a^2\delta'_{V(A)}}{\mu m_s} \sim \epsilon \ll 1$$

- Expand NLO SU(3) formulae for m_π^2 and f_π in powers of ϵ
- Expansion has the same pattern as SU(2) formulae
- Compare $\Delta_B^{(2)}$ and $\delta'^{(2)}_{V(A)}$ with Δ_B and $\delta'_{V(A)}$, only need their relations at LO

$$m_{U_B}^2 = 2\mu m_l + a^2\Delta_B^{(2)}$$
$$m_{\eta'_V}^2 = m_{U_V}^2 + \frac{1}{2}a^2\delta'^{(2)}_V$$

$$m_{U_B}^2 = 2\mu m_l + a^2\Delta_B$$
$$m_{\eta_V}^2 = \frac{1}{2}\left(m_{U_V}^2 + m_{S_V}^2 + \frac{3}{4}a^2\delta'_V - Z\right)$$
$$Z = \sqrt{(m_{S_V}^2 - m_{U_V}^2)^2 - \frac{a^2\delta'_V}{2}(m_{S_V}^2 - m_{U_V}^2) + \frac{9(a^2\delta'_V)^2}{16}}$$

$\Rightarrow$

$$a^2\Delta_B^{(2)} = a^2\Delta_B, \tag{5}$$
$$a^2\delta'^{(2)}_V = a^2\delta'_V, \tag{6}$$
$$a^2\delta'^{(2)}_A = a^2\delta'_A \tag{7}$$

Relation of SU(2) and SU(3) $S\chi PT$

- Compare coefficients of terms m_x, m_y, m_l, a^2 separately, obtain

$$f_{(2)} = f(1 - \frac{1}{16\pi^2 f^2}\mu m_s \log\frac{\mu m_s}{\Lambda^2} + \frac{16L_4}{f^2}\mu m_s), \quad (8)$$

$$\mu_{(2)} = \mu(1 - \frac{1}{48\pi^2 f^2}\frac{4\mu m_s}{3}\log\frac{\frac{4\mu m_s}{3}}{\Lambda^2} + \frac{32(2L_6 - L_4)}{f^2}\mu m_s), \quad (9)$$

$$p_1 = 16L_5 - \frac{1}{16\pi^2}(1 + \log\frac{\mu m_s}{\Lambda^2}), \quad (10)$$

$$p_2 = -8L_8 + \frac{1}{16\pi^2}\frac{1}{6}(\log\frac{\frac{4}{3}\mu m_s}{\Lambda^2}) + \frac{1}{16\pi^2}\frac{1}{4}(1 + \log\frac{\mu m_s}{\Lambda^2}), \quad (11)$$

$$l_3 = 8(2L_6 - L_4) + 4(2L_8 - L_5) - \frac{1}{16\pi^2}\frac{1}{36}(1 + \log\frac{\frac{4}{3}\mu m_s}{\Lambda^2}), \quad (12)$$

$$l_4 = 8L_4 + 4L_5 - \frac{1}{16\pi^2}\frac{1}{4}(1 + \log\frac{\mu m_s}{\Lambda^2}), \quad (13)$$

$$\tilde{L}''_{(2)} = \tilde{L}'' - \frac{1}{6}\Delta_I(1 + \log\frac{\frac{4}{3}\mu m_s}{\Lambda^2}) - \frac{1}{2}\Delta_{av}(1 + \log\frac{\mu m_s}{\Lambda^2}), \quad (14)$$

$$\tilde{L}'_{(2)} = \tilde{L}' - \frac{1}{6}\Delta_I(1 + \log\frac{\frac{4}{3}\mu m_s}{\Lambda^2}) + \frac{1}{2}\Delta_{av}(1 + \log\frac{\mu m_s}{\Lambda^2}). \quad (15)$$

Overview of the MILC data and chiral fitting

- Dynamical simulations with $2+1$ asqtad staggered fermions
- Lattice spacings range from $0.15\mathrm{fm}$ to $0.045\mathrm{fm}$
- Various combinations of sea and valence quark masses
- Pseudoscalar meson masses and decay constants are measured on each ensemble
- Data need to be analyzed by using $rS\chi PT$
  - Taste-breaking effects are compatible to effects from quark masses
- Results from SU(3) analysis are very successful
- SU(2) analysis is of interest
  - SU(2) χPT probably converges faster than SU(3) χPT
  - Extract SU(2) LECs: l_3 and l_4
  - Study systematic errors resulting from the truncations of each version of χPT

SU(2) chiral fitting

- Fitting formulae for m_π^2 and f_π up to NNLO
  - NLO formulae from SU(2) $rS\chi PT$ are available
  - Include NNLO analytic terms. NNNLO analytic terms are added to estimate systematic errors.
  - Use continuum NNLO chiral logarithms from Bijnens. RMS pion mass is used.

$$m_{RMS}^2 = m_{xy}^2 + a^2\overline{\Delta} \tag{16}$$

- Select datasets suitable for SU(2) analysis
  - Valence and sea quark masses significantly smaller than the strange quark mass

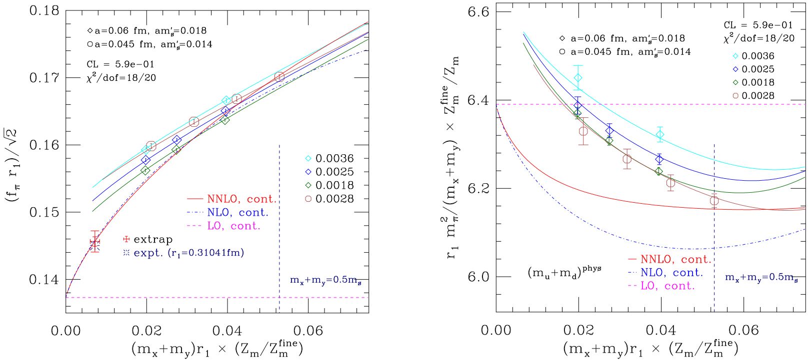
$$m_l \le 0.2 m_s, \qquad m_x + m_y \le 0.5 m_s, \qquad \max(m_x, m_y) < 0.3 m_s \tag{17}$$

  - Taste-splittings are much smaller than kaon and pion masses. Superfine ($a \approx 0.06\mathrm{fm}$) and ultrafine ($a \approx 0.045\mathrm{fm}$) lattices are used for the central value fit.
- Correlated least chi square fit with Bayesian analysis
- Modify quark masses in NNLO analytic terms to make the NNLO LECs scale invariant on the lattice

$$m \to \tilde{m} = m + \frac{a^2}{2\mu} \tag{18}$$

Central value fit

- Use ensembles $(am_l, am_s) = \{(0.0018, 0.018), (0.0025, 0.018), (0.0036, 0.018), (0.0028, 0.014)\}$, 50 data points
- 30 fitting parameters with appropriate constraints
- $\chi^2/\mathrm{DOF} = 18/20$. CL ≈ 0.6



(a) (b)

Fitting results

- Physical results are obtained by setting taste-violating parameters to zero, valence and light sea quark masses equal to physical values of average u, d quarks masses and using the scale $r_1 = 0.3133(23)\mathrm{fm}$ [C. T. H. Davies *et al.*, PRD 81:034506 (2010)]

$$f_\pi = 130.2 \pm 1.4 \left({}^{+2.0}_{-1.6}\right)\ \mathrm{MeV} \tag{19}$$

  Agrees with PDG 2008: $f_\pi = 130.4 \pm 0.2\,\mathrm{MeV}$

- Using the scale $r_1 = 0.31041\mathrm{fm}$ from the SU(3) f_π analysis, we obtain the following results:

$$\begin{aligned} f_2 &= 123.8 \pm 1.4 \left({}^{+1.0}_{-3.7}\right)\ \mathrm{MeV} & B_2 &= 2.91(5)(5)(14)\,\mathrm{MeV} \\ \bar{l}_3 &= 2.85 \pm 0.81 \left({}^{+0.37}_{-0.92}\right) & \bar{l}_4 &= 3.98 \pm 0.32 \left({}^{+0.51}_{-0.28}\right) \\ \hat{m} &= 3.19(4)(5)(16)\,\mathrm{MeV} & \langle \bar{u}u \rangle_2 &= -[281.5(3.4) \left({}^{+2.0}_{-5.9}\right)(4.0)\,\mathrm{MeV}]^3 \end{aligned} \tag{20}$$

  Quark masses and chiral condensate are evaluated in the $\overline{\mathrm{MS}}$ scheme at $2\mathrm{GeV}$ by using the 2-loop renormalization factor.

Conclusions and outlook

- Conclusions:
  - Calculate the pion mass and decay constant through NLO in SU(2) $rS\chi PT$
  - Obtain relations between SU(2) and SU(3) LECs
  - Perform a systematic NNLO chiral analysis to the MILC lattice data
  - Results agree with experiments and SU(3) chiral analysis
  - χPT converges significantly faster for SU(2) than for SU(3) as expected
  - Including effects from variations of strange quark masses can improve the fit, but not significantly
- Outlook:
  - Include kaon as a heavy particle in SU(2) $S\chi PT$
  - Chiral analysis of the HISQ data


Thank you
From WallPepper

The replica method in the two-flavor case

- Take n_r' copies of each valence quark and n_r copies of each sea quark. Chiral symmetry group is enlarged to $SU(8(n_r' + n_r))_L \times SU(8(n_r' + n_r))_R$.

$$\Phi = \begin{pmatrix}
X^{11} & \dots & X^{1n_r'} & P_+^{11} & \dots & P_+^{1n_r'} & \dots & \dots & \dots & \dots & \dots & \dots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots \\
X^{n_r'1} & \dots & X^{n_r'n_r'} & P_+^{n_r'1} & \dots & P_+^{n_r'n_r'} & \dots & \dots & \dots & \dots & \dots & \dots \\
P_-^{11} & \dots & P_-^{1n_r'} & Y^{11} & \dots & Y^{1n_r'} & \dots & \dots & \dots & \dots & \dots & \dots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots \\
P_-^{n_r'1} & \dots & P_-^{n_r'n_r'} & Y^{n_r'1} & \dots & Y^{n_r'n_r'} & \dots & \dots & \dots & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & U^{11} & \dots & U^{1n_r} & \pi_+^{11} & \dots & \pi_+^{1n_r} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & U^{n_r1} & \dots & U^{n_rn_r} & \pi_+^{n_r1} & \dots & \pi_+^{n_rn_r} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \pi_-^{11} & \dots & \pi_-^{1n_r} & D^{11} & \dots & D^{1n_r} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \pi_-^{n_r1} & \dots & \pi_-^{n_rn_r} & D^{n_r1} & \dots & D^{n_rn_r}
\end{pmatrix}$$

Chiral Lagrangian with replica

- LO chiral Lagrangian

$$\mathcal{L}^{(4)} = \frac{f_{(2)}^2}{8}\mathrm{Tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f_{(2)}^2}{8}\mathrm{Tr}(\chi\Sigma^\dagger + \chi\Sigma)$$
$$+ \frac{2m_0^2}{3}(U_I^{11} + \ldots + U_I^{n_r n_r} + D_I^{11} + \ldots + D_I^{n_r n_r})^2 + a^2\mathcal{V}$$
$$\chi = 2\mu_{(2)} Diag(\underbrace{m_x I, \ldots, m_x I}_{n_r'}, \underbrace{m_y I, \ldots, m_y I}_{n_r'}, \underbrace{m_u I, \ldots, m_u I}_{n_r}, \underbrace{m_d I, \ldots, m_d I}_{n_r}) \tag{21}$$

SU(2) $S\chi PT$ chiral Lagrangian

- SU(2) $S\chi PT$ at LO

$$\mathcal{L}^{(4)} = \frac{f_{(2)}^2}{8}\mathrm{Tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f_{(2)}^2}{8}\mathrm{Tr}(\chi\Sigma^\dagger + \chi\Sigma) + \frac{2m_0^2}{3}(U_I^{11} + \ldots + U_I^{n_r n_r} + D_I^{11} + \ldots + D_I^{n_r n_r})^2 + a^2\mathcal{V}$$

$$-\mathcal{U} = C_1^{(2)}\mathrm{Tr}(\xi_5^{(R)}\Sigma\xi_5^{(R)}\Sigma^\dagger) + C_3^{(2)}\frac{1}{2}\sum_\nu[\mathrm{Tr}(\xi_\nu^{(R)}\Sigma\xi_\nu^{(R)}\Sigma) + h.c.] + C_4^{(2)}\frac{1}{4}\sum_\nu[\mathrm{Tr}(\xi_{\nu 5}^{(R)}\Sigma\xi_{5\nu}^{(R)}\Sigma) + h.c.] + C_6^{(2)}\sum_{\mu<\nu}\mathrm{Tr}(\xi_{\mu\nu}^{(R)}\Sigma\xi_{\nu\mu}^{(R)}\Sigma^\dagger)$$

$$-\mathcal{U}' = C_{2V}^{(2)}\frac{1}{4}\sum_\nu[\mathrm{Tr}(\xi_\nu^{(R)}\Sigma)\mathrm{Tr}(\xi_\nu^{(R)}\Sigma) + h.c.] + C_{2A}^{(2)}\frac{1}{4}\sum_\nu[\mathrm{Tr}(\xi_{\nu 5}^{(R)}\Sigma)\mathrm{Tr}(\xi_{5\nu}^{(R)}\Sigma) + h.c.] + C_{5V}^{(2)}\frac{1}{2}\sum_\nu[\mathrm{Tr}(\xi_\nu^{(R)}\Sigma)\mathrm{Tr}(\xi_\nu^{(R)}\Sigma^\dagger)] + C_{5A}^{(2)}\frac{1}{2}\sum_\nu[\mathrm{Tr}(\xi_{\nu 5}^{(R)}\Sigma)\mathrm{Tr}(\xi_{5\nu}^{(R)}\Sigma^\dagger)]$$

$\Sigma = \exp(i\Phi/f)$, ξ^R is the product of ξ with the identity in flavor and replica space

- LECs in the SU(2) chiral Lagrangian are different from LECs in the SU(3) chiral Lagrangian

SU(2) $PQ - S\chi PT$ at NLO

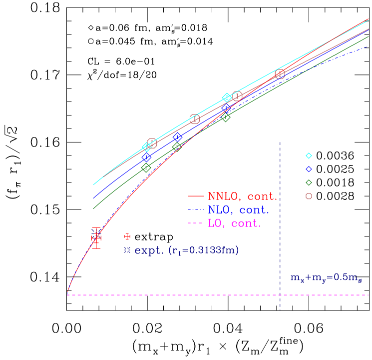
- SU(2) $PQ - S\chi PT$ chiral Lagrangian at NLO

$$
\begin{aligned}
\mathcal{L}^{(6)}_{cont} = & -\frac{l_1^0}{4}[\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)]^2 - \frac{l_2^0}{4}\mathrm{Tr}(D_\mu\Sigma^\dagger D_\nu\Sigma)\mathrm{Tr}(D_\mu\Sigma^\dagger D_\nu\Sigma) \\
& + p_3^0\left(\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma D_\nu\Sigma^\dagger D_\nu\Sigma) - \frac{1}{2}[\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)]^2\right) \\
& + p_4^0\left(\mathrm{Tr}(D_\mu\Sigma^\dagger D_\nu\Sigma D_\mu\Sigma^\dagger D_\nu\Sigma) + 2\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma D_\nu\Sigma^\dagger D_\nu\Sigma)\right. \\
& \left. -\frac{1}{2}[\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)]^2 - \mathrm{Tr}(D_\mu\Sigma^\dagger D_\nu\Sigma)\mathrm{Tr}(D_\mu\Sigma^\dagger D_\nu\Sigma)\right) \\
& - \frac{l_3^0 + l_4^0}{16}[\mathrm{Tr}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)]^2 + \frac{l_4^0}{8}\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{Tr}(\chi\Sigma^\dagger + \Sigma\chi^\dagger) \\
& + \frac{p_1^0}{16}\left(\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma(\chi\Sigma^\dagger + \Sigma\chi^\dagger)) - \frac{1}{2}\mathrm{Tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{Tr}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)\right) \\
& + \frac{p_2^0}{16}\left(2\mathrm{Tr}(\Sigma^\dagger\chi\Sigma^\dagger\chi + \Sigma\chi^\dagger\Sigma\chi^\dagger) - \mathrm{Tr}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)^2 - \mathrm{Tr}(\chi\Sigma^\dagger - \Sigma\chi^\dagger)^2\right) \\
& + \frac{l_7^0}{16}[\mathrm{Tr}(\chi\Sigma^\dagger - \Sigma\chi^\dagger)]^2 \\
& - l_5^0\mathrm{Tr}(\Sigma^\dagger F_{R\mu\nu}\Sigma F_{L\mu\nu}) - \frac{il_6^0}{2}\mathrm{Tr}(F_{L\mu\nu}D_\mu\Sigma^\dagger D_\nu\Sigma + F_{R\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger),
\end{aligned}
\tag{22}
$$

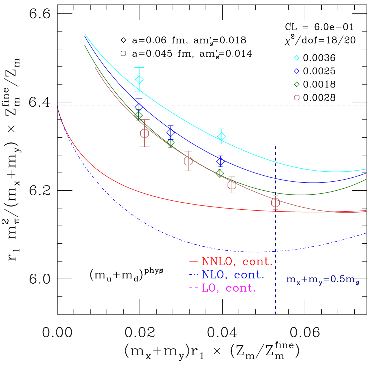
- $l_1^0 - -l_7^0$: (bare) LECs in ordinary SU(2) χPT
- $p_1^0 - -p_4^0$: New (bare) LECs in SU(2) $PQ\chi PT$

red: unphysical operators

Plots with scale $r_1 = 0.3133\text{fm}$



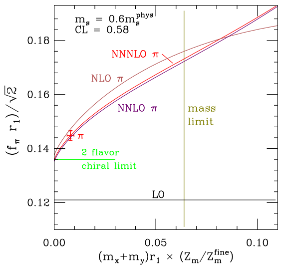
(c)



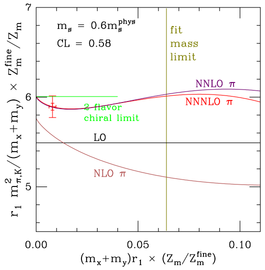
(d)

Comparison with SU(3) fit

- Results from SU(2) and SU(3) fits agree within errors
- SU(2) $S\chi PT$ within its applicable region converges much faster than SU(3) $S\chi PT$ [Figure is from arXiv:0910.3618]



(e)



(f)

At $x = 0.05$, ratios of NNLO corrections to results through NLO are

$$\begin{aligned} SU(2): \quad & 1\% \ (f_\pi) \quad & 1\% \ (m_\pi^2/(m_x+m_y)) \\ SU(3): \quad & 2.9\% \ (f_\pi) \quad & 15.6\% \ (m_\pi^2/(m_x+m_y)) \end{aligned} \tag{23}$$