Renormalization Constants for 1-derivative operators in twisted mass QCD

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OUTLINE

A Non-perturbative renormalization

- Fermion and gluon actions
- Computation of Green's functions

B Renormalization Conditions

- fermion field
- ultralocal and twist-2 operators

C Perturbative renormalization

- $\mathcal{O}(a^2)$ corrections to ultralocal fermion bilinears
- $\mathcal{O}(a^2)$ corrections to twist-2 fermion bilinears

D Results

- $N_F = 2: \quad Z_A, Z_V$
- $N_F = 2: \quad Z_{DA}, Z_{DV}$
- $N_F = 2+1+1: \quad Z_A, Z_V, Z_{DA}$ (Preliminary)

NON-PERTURBATIVE RENORMALIZATION

$N_F = 2$ Twisted mass fermions (twisted basis)

$$S_F = a^4 \sum_x \overline{\chi}(x) \big(\frac{1}{2} \gamma_\mu (\overrightarrow{\nabla}_\mu + \overrightarrow{\nabla}^*_\mu) - \frac{ar}{2} \overrightarrow{\nabla}_\mu \overrightarrow{\nabla}^*_\mu + m_0 + i\mu_0 \gamma_5 \tau^3 \big) \chi(x)$$

physical basis at maximal twist

$$\psi(x) = \exp\left(\frac{i\pi}{4} \gamma_5 \tau^3 \right) \chi(x), \qquad \overline{\psi}(x) = \overline{\chi}(x) \exp\left(\frac{i\pi}{4} \gamma_5 \tau^3 \right)$$

Tree-level Symanzik improved gluons

$$S_g = \frac{\beta}{3} \sum_x \left(\frac{5}{3} \sum_{\substack{\mu,\nu=1 \\ 1\le\mu<\nu}}^{4} \{1 - \mathrm{Re}\,\mathrm{Tr}(U^{1\times 1}_{x,\mu,\nu})\} - \frac{1}{12} \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{4} \{1 - \mathrm{Re}\,\mathrm{Tr}(U^{1\times 2}_{x,\mu,\nu})\} \right)$$

$N_F = 2$: Statistics

β	a (fm)	$a\mu_0$	m_π (GeV)	$L^3 \times T$
3.9	0.089	0.0040	0.3021(14)	$24^3 \times 48$
3.9	0.089	0.0064	0.37553(80)	$24^3 \times 48$
3.9	0.089	0.0085	0.4302(11)	$24^3 \times 48$
4.05	0.070	0.006	0.4082(31)	$24^3 \times 48$
4.05	0.070	0.006	0.404(2)	$32^3 \times 64$
4.05	0.070	0.008	0.465(1)	$32^3 \times 64$
4.20	0.055	0.0065	0.476(2)	$32^3 \times 64$

(n_t,n_x,n_y,n_z)	$\beta = 3.9$ $24^3 \times 48$ $\mu_0 = 0.004$	$\beta = 3.9$ $24^3 \times 48$ $\mu_0 = 0.0064$	$\beta = 3.9$ $24^3 \times 48$ $\mu_0 = 0.0085$	$\beta = 4.05$ $24^3 \times 48$ $\mu_0 = 0.006$	$\beta = 4.05$ $32^3 \times 64$ $\mu_0 = 0.006$	$\beta = 4.05$ $32^3 \times 64$ $\mu_0 = 0.008$	$\beta = 4.20$ $32^3 \times 64$ $\mu_0 = 0.0065$
(4,2,2,2)	100	50	80	—	50	50	15
(5,2,2,2)	100	60	60	—	—	33	15
(6,2,2,2)	100	50	50	—	—	50	15
(3,3,3,2)	—	—	27	—	—	15	15
(7,2,2,2)	—	—	20	—	—	15	15
(2,3,3,3)	—	—	20	—	—	15	15
(8,2,2,2)	—	—	20	—	—	15	15
(3,3,3,3)	100	50	80	15	—	50	15
(4,4,4,4)	—	—	—	—	15	—	—
(4,3,3,3)	100	60	60	—	—	50	15
(5,3,3,3)	100	60	60	—	—	50	15
(6,3,3,3)	—	—	15	—	—	15	15
(10,2,2,2)	—	—	15	—	—	15	15
(9,3,3,3)	—	—	—	—	—	15	15
(10,3,3,3)	—	—	—	—	—	15	15
(13,2,2,2)	—	—	—	—	—	15	15
(11,3,3,3)	—	—	—	—	—	15	15
(14,2,2,2)	—	—	—	—	—	15	15

Non-amputated Green's function (Physical basis)

$$G_{\alpha\delta}(p) = \frac{a^{12}}{V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle u_\alpha(x) \mathcal{O}_{\Gamma}(z) \bar{d}_\delta(y) \rangle$$

- No disconnected diagrams

$$\boxed{\begin{aligned}
\mathcal{O}^{\mu}_{\mathbf{V}}(z) &= \bar{u}(z)\,\gamma^{\mu}\,d(z)\\
\mathcal{O}^{\mu}_{\mathbf{A}}(z) &= \bar{u}(z)\,\gamma^{5}\,\gamma^{\mu}\,d(z)\\
\mathcal{O}^{\{\mu\nu\}}_{\mathbf{DV}}(z) &= \bar{u}(z)\,\gamma^{\{\mu}\,\overleftrightarrow{D}^{\,\nu\}}d(z)\\
\mathcal{O}^{\{\mu\nu\}}_{\mathbf{DA}}(z) &= \bar{u}(z)\,\gamma^{5}\,\gamma^{\{\mu}\,\overleftrightarrow{D}^{\,\nu\}}d(z)
\end{aligned}}$$

$$\mathcal{O}^{\{\mu\nu\}} = \frac{1}{2}\left(\mathcal{O}^{\mu\nu} + \mathcal{O}^{\nu\mu}\right) - \frac{1}{4}\delta_{\mu\nu}\sum_{\rho}\mathcal{O}^{\rho\rho}$$

- No mixing with lower dimension operators

$$\mathcal{O}_{\mathbf{DV,DA}}(z) \equiv \bar{u}(z)\,J_{\mathbf{DV,DA}}(z,z') \qquad z,z' : \text{ nearest neighbors}$$

Upon Wick contraction + Gauge field average

$$G_{\alpha\delta}(p) = \frac{1}{V} \sum_{x,y,z,z'} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle \mathcal{U}_{\alpha\beta}(x,z)\, J^{\Gamma}_{\beta\gamma}(z,z')\, \mathcal{D}_{\gamma\delta}(z',y) \rangle^{G}$$

$\mathcal{U}(x,z)$, $\mathcal{D}(z',y)$: up, down propagators

- Method A: point source

★ Fixed value for z ($z = 0$)

$$G_{\alpha\delta}(p) = a^{8} \sum_{x,y} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle \mathcal{U}_{\alpha\beta}(x,0)\, J^{\Gamma}_{\beta\gamma}(0,z')\, \mathcal{D}_{\gamma\delta}(z',y) \rangle^{G}$$

z' neighbors of $z = 0$

★ Point to all propagators: $\mathcal{S}(z,x)$, $\mathcal{S}(z',x)$

★ Dirac equation solvable with point source at the fixed value of z

★ Translation invariance $\leftrightarrow$ Average over gauge field configurations

★ Less inversions, but larger statistical errors

- Method B: momentum source

$$\sum_z \mathcal{K}^{ac}_{\alpha\gamma}(x,z) \underbrace{\sum_y \mathrm{e}^{\mathrm{i}p\cdot y} \mathcal{D}^{cb}_{\beta\gamma}(z,y)}_{\breve{\mathcal{D}}^{cb}_{\beta\gamma}(z,p)} = \mathrm{e}^{\mathrm{i}p\cdot x}\, \delta_{\alpha\beta}\, \delta_{ab}$$

$\mathcal{K}$: Dirac operator for down quark

★ Perform the summation over z

★ Dirac equation solved with momentum source

★ $\sharp$ of inversion depends on the $\sharp$ of momenta considered

★ Application of any operator

★ High statistical accuracy is achieved

Propagators

Twisted basis:

up quark

$$\mathcal{S}^{ab}_{\alpha\beta}(p) = -\frac{a^8}{4\,V}\sum_z \langle\, \mathrm{e}^{\mathrm{i}p\cdot z}\left[(\hat{1} - i\,\gamma^5)\breve{\mathcal{D}}^\dagger(z,p)(\hat{1} - i\,\gamma^5)\right]^{ab}_{\alpha\beta}\,\rangle$$

down quark

$$\mathcal{S}^{ab}_{\alpha\beta}(p) = \frac{a^8}{4\,V}\sum_z \langle\, \mathrm{e}^{-\mathrm{i}p\cdot z}\left[(\hat{1} - i\,\gamma^5)\breve{\mathcal{D}}(z,p)(\hat{1} - i\,\gamma^5)\right]^{ab}_{\alpha\beta}\,\rangle$$

Amputated Green's Function

Bare $\quad \Gamma(p) = \mathcal{S}^{-1}(p) G(p) \mathcal{S}^{-1}(p)$

Renormalized $\quad \Gamma_R(p) = Z_q^{-1}\, Z_{\mathcal{O}}\, \Gamma(p)$

$Z_q^{-1}\, Z_{\mathcal{O}}$ extracted from $G(p)$, $S(p)$

Renormalization Conditions (RI′-MOM scheme):

$$Z_q = \frac{1}{12}\mathrm{Tr}[(S^L(p))^{-1}\, S^{(0)}(p)]\Big|_{p^2=\mu^2}$$

$$Z_q^{-1}\, Z_{\mathcal{O}}^{\mu\nu}\, \frac{1}{12}\mathrm{Tr}[\Gamma^L_{\mu\nu}(p)\, \Gamma^{(0)-1}{}_{\mu\nu}(p)]\Big|_{p^2=\mu^2} \quad = \quad 1$$

Method 1

$$S^{(0)}(p) = \frac{-i\sum_\rho \gamma_\rho p_\rho}{p^2}$$

$$\Gamma^{(0)}_{\mu\nu}(p) = -i\,\tilde{\Gamma}_{\{\mu}\, p_{\nu\}}$$

Method 2

$$S^{(0)}(p) = \frac{-i\sum_\rho \gamma_\rho \sin(p_\rho)}{\sum_\rho \sin(p_\rho)^2}$$

$$\Gamma^{(0)}_{\mu\nu}(p) = -i\,\tilde{\Gamma}_{\{\mu}\ \sin(p_{\nu\}})$$

$$\frac{1}{L^2} \ll \Lambda^2_{\mathrm{QCD}} \ll \bar{\mu}^2 \ll \frac{1}{a^2}$$

2-3

Reliable perturbation theory

Small $\mathcal{O}(a)$ lattice effects

Motivation for $\mathcal{O}(a^2)$ corrections

★ **Perturbative computation of $\mathcal{O}(a^2)$ terms** $\Rightarrow$ **subtraction of these effects from non-perturbative result** $\Rightarrow$

minimization of their lattice artifacts

Complications with $\mathcal{O}(a^2)$

★ $\mathcal{O}(a^1)$**: No new types of IR divergences**

★ $\mathcal{O}(a^2)$**: Novel IR singularities**

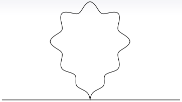
Non-Lorentz invariant contributions, e.g., $a^2\, m \frac{\sum_\mu \gamma_\mu p_\mu^3}{p^2}$

Large number of strong divergent integrals

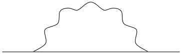
Matrix elements of Green's functions

- **Fermion action: Wilson, Twisted mass, clover**
- **Gluon action: 10 sets of Symanzik improved gluons**
- **Up to 2^{nd} order in the lattice spacing a**
- **General covariant gauge λ**
- **General clover parameter c_{SW}**
- **General values for momentum p and lattice spacing a**
- **General values for the masses m and μ**

Feynman Diagrams



1

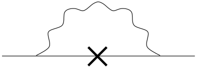


2

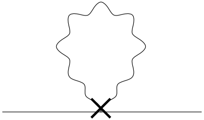
Z_q

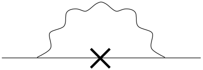
: gluon field

: fermion field

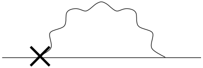


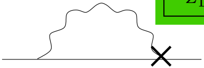
$Z_S\,,\; Z_P\,,\; Z_V\,,\; Z_A\,,\; Z_T\,,\; Z_{T'}$





$Z_{\mathrm{DV}}\,,\; Z_{\mathrm{DA}}\,,\; Z_{\mathrm{DT}}$





Perturbative Procedure

- **Wick contraction of appropriate vertices**
- **Simplification of color dependence, Dirac matrices, tensors**
- **Exploitation of symmetries of the theory and of the diagrams**
- **Isolation of the logarithmic and non-Lorentz invariant terms:**
  - *hundreds of new primitive divergent integrals* *many to $\mathcal{O}(a^3)$*
    - ★ *11 strong IR divergent integrals*

M. Constantinou et al., JHEP 0910:064, 2009 [arXiv:0907.0381]

- **Convergent terms: Taylor expansion in p and a up to $\mathcal{O}(a^3 p^3)$**
- **Numerical integration over the internal momentum k**
- **Extrapolation of results to $L \to \infty$**
  - *only source of systematic errors*

Renormalization Conditions (RI′-MOM scheme):

$$Z_q = \frac{1}{12} \mathrm{Tr}[(S^L(p))^{-1} \, S^{(0)}(p)]\Big|_{p^2=\mu^2}$$

$$Z_q^{-1} \, Z_{\mathcal{O}}^{\mu\nu} \, \frac{1}{12} \mathrm{Tr}[\Gamma^L_{\mu\nu}(p) \, \Gamma^{(0)-1}{}_{\mu\nu}(p)]\Big|_{p^2=\mu^2} \quad = \quad 1$$

Method 1

$$S^{(0)}(p) = \frac{-i \sum_\rho \gamma_\rho p_\rho}{p^2}$$

$$\Gamma^{(0)}_{\mu\nu}(p) = -i \, \tilde{\Gamma}_{\{\mu} \, p_{\nu\}}$$

Method 2

$$S^{(0)}(p) = \frac{-i \sum_\rho \gamma_\rho \sin(p_\rho)}{\sum_\rho \sin(p_\rho)^2}$$

$$\Gamma^{(0)}_{\mu\nu}(p) = -i \, \tilde{\Gamma}_{\{\mu} \, \sin(p_{\nu\}})$$

$$\frac{1}{L^2} \ll \Lambda^2_{\mathrm{QCD}} \ll \bar{\mu}^2 \ll \frac{1}{a^2}$$

2-3

Reliable perturbation theory

Small $\mathcal{O}(a)$ lattice effects

twist-2 renormalization factors

Vector:

$$Z_{\mathrm{DV1}} = Z_{\mathrm{DV}} \text{ with } \mu = \nu$$

$$Z_{\mathrm{DV2}} = Z_{\mathrm{DV}} \text{ with } \mu \neq \nu$$

Axial:

$$Z_{\mathrm{DA1}} = Z_{\mathrm{DA}} \text{ with } \mu = \nu$$

$$Z_{\mathrm{DA2}} = Z_{\mathrm{DA}} \text{ with } \mu \neq \nu$$

Example of Perturbative Results:

- **Tree-level Symanzik gluons ,** $c_{\rm SW} = 0$
- **Landau gauge**
- $m = 0$, $\mu_0 = 0$

$$
\begin{aligned}
\mathrm{Tr}\Big[L^{\rm DV1}(p)\cdot L^{\rm DV1}_{\rm tree}(p)\Big] =\; & -2\,p_\mu^2 - \frac{1}{4}p^2 + a^2\Big(\frac{1}{12}\sum_\rho p_\rho^4 + \frac{2}{3}p_\mu^4\Big)\\
& + \tilde g^2\Big\{\frac{4}{3}\frac{p_\mu^4}{p^2} + p^2\,\Big(3.610062(3) - \frac{2}{3}\ln(a^2\,p^2)\Big) + p_\mu^2\,\Big(27.54716(3) - \frac{16}{3}\ln(a^2\,p^2)\Big)\\
& \quad + a^2\Big[(p^2)^2\,\Big(0.11838(2) + \frac{7}{288}\ln(a^2\,p^2)\Big) + p^2\,p_\mu^2\,\Big(-0.6573(1) - \frac{299}{180}\ln(a^2\,p^2)\Big)\\
& \qquad + \sum_\rho p_\rho^4\,\Big(-1.71886(3) + \frac{397}{720}\ln(a^2\,p^2) - \frac{43}{360}\frac{p_\mu^2}{p^2}\Big)\\
& \qquad + p_\mu^4\,\Big(-16.1049(5) + \frac{94}{15}\ln(a^2\,p^2) + \frac{29}{90}\frac{\sum_\rho p_\rho^4}{(p^2)^2} + \frac{169}{45}\frac{p_\mu^2}{p^2}\Big)\Big]\Big\}\\
& + \quad \mathcal{O}(a^4, g^4)
\end{aligned}
$$

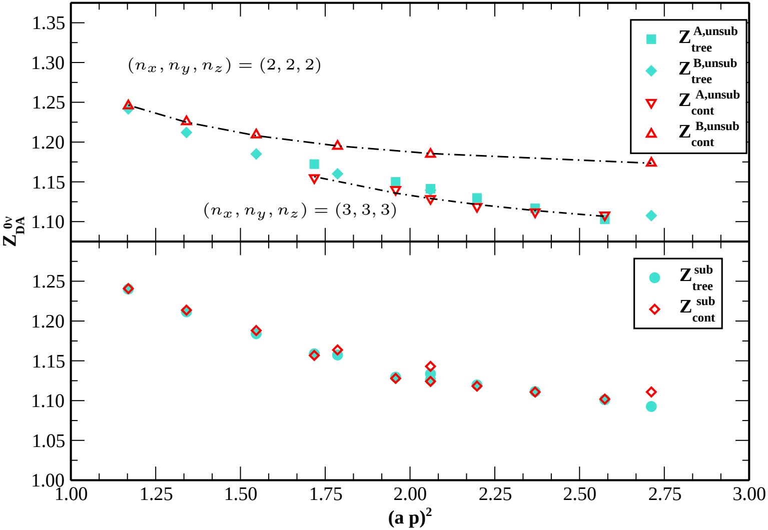
$L^{\rm DV1}(p)$: **matrix element of Green's function up to 1-loop**

$$
L^{\rm DV1}_{\rm tree}(p) = i\gamma_\mu\Big(p_\mu - a^2\,\frac{p_\mu^3}{6}\Big) - \frac{i}{4}\sum_\tau \gamma_\tau\Big(p_\tau - a^2\,\frac{p_\tau^3}{6}\Big) + \mathcal{O}(a^4)
$$

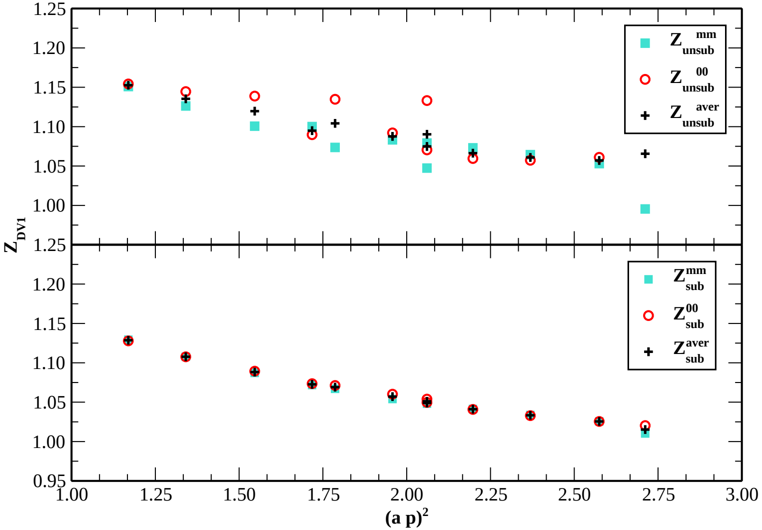
- **Ambiguity on the choice of the momentum direction**

C. Alexandrou et al., [arXiv:1006.1920]

Comparison of renormalization conditions 1 and 2:



Averaging spatial and temporal components (Method 1):

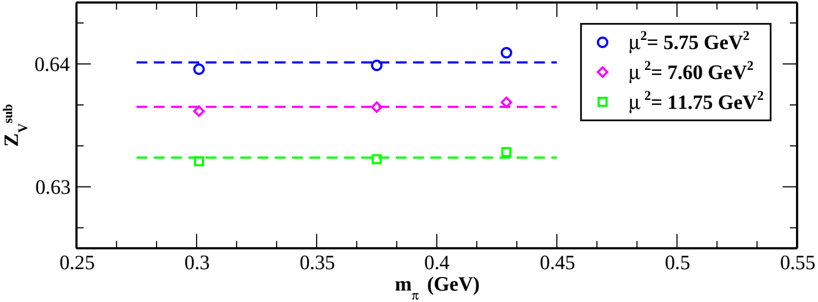


RESULTS

A. Quark mass dependence

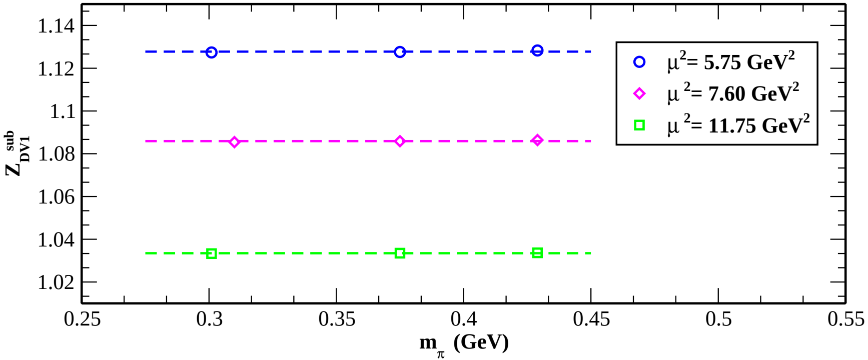
β=3.9, a=0.089 **fm**

$24^3 \times 48$



β=3.9, a=0.089 **fm**

$24^3 \times 48$



★ **Same behavior for** Z_{DV2}, Z_{DA1}, Z_{DA2}

β=4.05, a=0.07 fm

m_π=0.403 GeV

$\mu^2 \sim 16$ GeV^2

B. Volume effects

local operators $Z_V,\ Z_A$ (unsubtracted):

$L^3 \times T$	Z_V	Z_A
$24^3 \times 48$	$0.706833(7)$	$0.793087(8)$
$32^3 \times 64$	$0.706886(5)$	$0.793455(6)$

twist-2 $Z_{DV},\ Z_{DA}$ (unsubtracted):

$L^3 \times T$	Z_{DV1}	Z_{DV2}	Z_{DA1}	Z_{DA2}
$24^3 \times 48$	$1.0700(2)$	$1.0923(2)$	$1.1190(2)$	$1.1117(2)$
$32^3 \times 64$	$1.07123(6)$	$1.0928(2)$	$1.12037(7)$	$1.1122(2)$

- **errors in parenthesis: statistical**

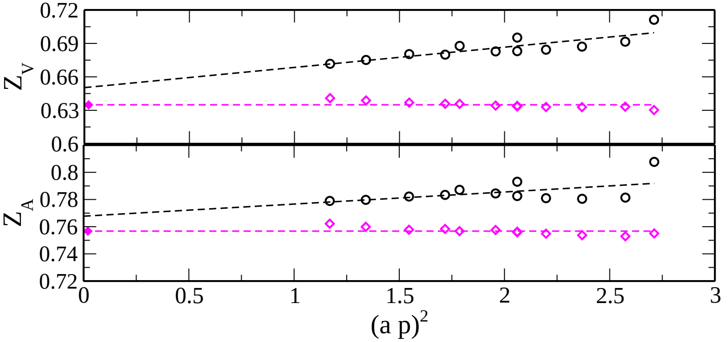
C. Renormalization scale dependence

local operators Z_V, Z_A:

β=3.9, a=0.089 **fm**

m_π=0.429 **GeV**

$24^3 \times 48$



β=4.05 a=0.07 **fm**

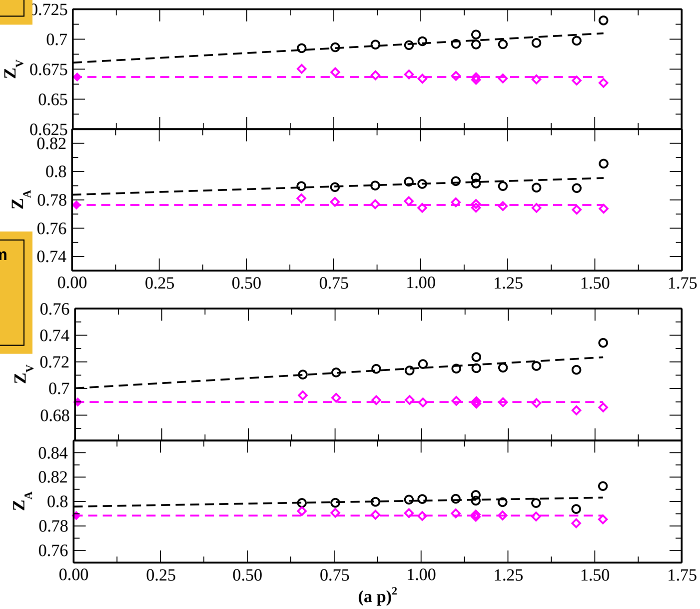
m_π=0.465 **GeV**

$32^3 \times 64$

β=4.20, a=0.055 **fm**

m_π=0.460 **GeV**

$32^3 \times 64$

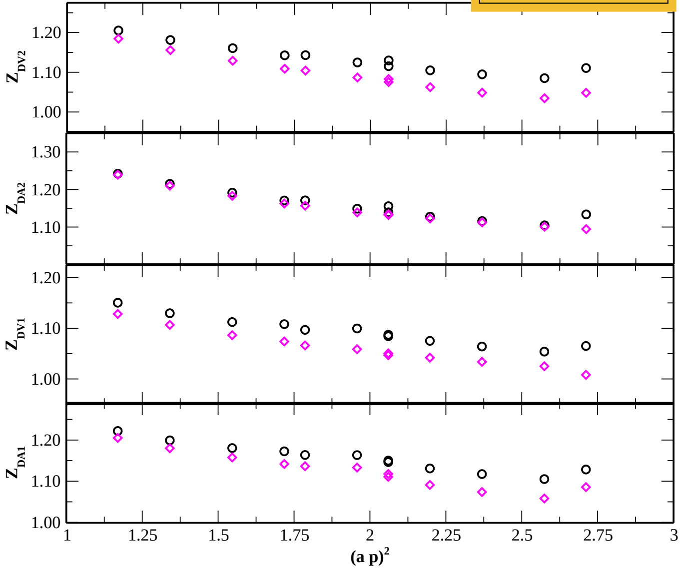


twist-2 Z_{DV}, Z_{DA} in RI′-MOM:

β=3.9, a=0.089 fm

m_π=0.429 GeV

$24^3 \times 48$



D. Conversion to $\overline{\mathrm{MS}}$

$$
\begin{aligned}
C_{\mathrm{DV1}} =\ & 1 + \alpha \left[-\frac{136}{27} + \frac{64}{9} \frac{\mu_\mu^2 - \frac{\mu_\mu^4}{\mu^2}}{\mu^2 + 8\mu_\mu^2} \right] \\
& + \alpha^2 \left[-\frac{128096}{729} + N_F \left(\frac{3208}{243} - \frac{320}{9} \frac{\mu_\mu^2 - \frac{\mu_\mu^4}{\mu^2}}{\mu^2 + 8\mu_\mu^2} \right) + \frac{248}{9} \zeta(3) + \frac{\mu_\mu^2 - \frac{\mu_\mu^4}{\mu^2}}{\mu^2 + 8\mu_\mu^2} \left(\frac{17792}{27} + \frac{320}{9} \zeta(3) \right) \right] \\
& + \alpha^3 \left[-\frac{627867571}{78732} - \frac{64\,\pi^4}{729} + \frac{5588641}{2187} \zeta(3) + N_F^2 \left(-\frac{149552}{6561} + \frac{77440}{729} \frac{\mu_\mu^2 - \frac{\mu_\mu^4}{\mu^2}}{\mu^2 + 8\mu_\mu^2} - \frac{256}{243} \zeta(3) \right) \right. \\
& \quad + N_F \left(\frac{19947676}{19683} + \frac{64\,\pi^4}{243} - \frac{1600}{27} \zeta(3) + \frac{\mu_\mu^2 - \frac{\mu_\mu^4}{\mu^2}}{\mu^2 + 8\mu_\mu^2} \left(-\frac{121024}{27} + \frac{9856}{81} \zeta(3) \right) \right) \\
& \quad \left. - \frac{19420}{27} \zeta(5) + \frac{\mu_\mu^2 - \frac{\mu_\mu^4}{\mu^2}}{\mu^2 + 8\mu_\mu^2} \left(\frac{270701210}{6561} - \frac{2993992}{243} \zeta(3) + \frac{349600}{81} \zeta(5) \right) \right] + \mathcal{O}(\alpha^4)
\end{aligned}
$$

$\alpha = g^2/(16\pi^2)$
$N_c = 3$
$\lambda = 0$

$$
\begin{aligned}
C_{\mathrm{DV2}} = 1 + \alpha \Bigg[& -\frac{124}{27} - \frac{16}{9}\,\frac{\mu_\mu^2 \mu_\nu^2}{\mu^2(\mu_\mu^2+\mu_\nu^2)} \Bigg] \\
& + \alpha^2 \Bigg[-\frac{98072}{729} + N_F \left(\frac{2668}{243} + \frac{80}{9}\,\frac{\mu_\mu^2 \mu_\nu^2}{\mu^2(\mu_\mu^2+\mu_\nu^2)} \right) + \frac{268}{9}\,\zeta(3) + \frac{\mu_\mu^2 \mu_\nu^2}{\mu^2(\mu_\mu^2+\mu_\nu^2)} \left(-\frac{4448}{27} - \frac{80}{9}\,\zeta(3) \right) \\
& + \alpha^3 \Bigg[-\frac{849683327}{157464} - \frac{64\,\pi^4}{729} + \frac{7809041}{4374}\,\zeta(3) + N_F^2 \left(-\frac{105992}{6561} - \frac{19360}{729}\,\frac{\mu_\mu^2 \mu_\nu^2}{\mu^2(\mu_\mu^2+\mu_\nu^2)} - \frac{256}{243}\,\zeta(3) \right) \\
& + N_F \left(\frac{14433520}{19683} + \frac{64\,\pi^4}{243} - \frac{4184}{81}\,\zeta(3) + \frac{\mu_\mu^2 \mu_\nu^2}{\mu^2(\mu_\mu^2+\mu_\nu^2)} \left(\frac{30256}{27} - \frac{2464}{81}\,\zeta(3) \right) \right) \\
& - \frac{36410}{81}\,\zeta(5) + \frac{\mu_\mu^2 \mu_\nu^2}{\mu^2(\mu_\mu^2+\mu_\nu^2)} \left(-\frac{135350605}{13122} + \frac{748498}{243}\,\zeta(3) - \frac{87400}{81}\,\zeta(5) \right) \Bigg] + \mathcal{O}(\alpha^4)
\end{aligned}
$$

Evolving to μ=2 GeV: running coupling, anomalous dimension

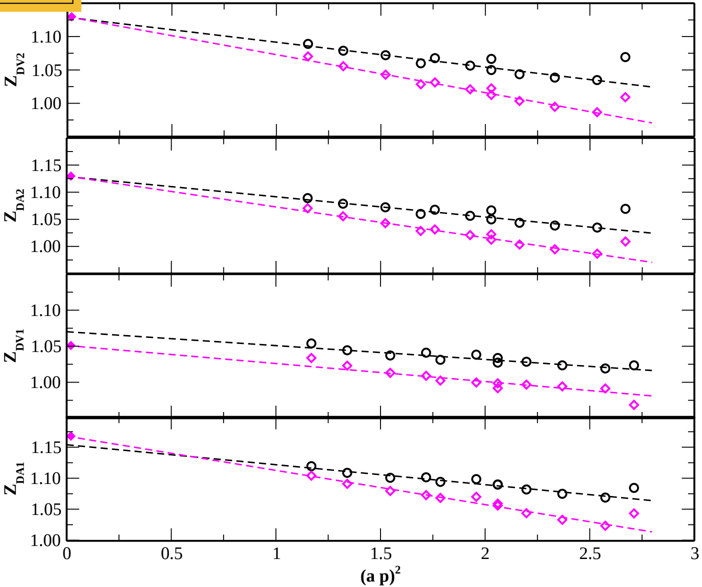
$$
\boxed{Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(2GeV) = R_{\mathcal{O}}(2GeV,\mu)\cdot C_{\mathcal{O}}(\mu)\cdot Z_{\mathcal{O}}^{\mathrm{RI}'}}
$$

$\beta=3.9,\ a=0.089$ **fm**

$m_\pi=0.429$ **GeV**

$24^3 \times 48$

Remaining $(a\,p)^2$ **artifacts** $\Rightarrow$ **fitting:** $Z^{\overline{\mathrm{MS}}}(2GeV, a) = C\,(a\,p)^2 + Z^{\overline{\mathrm{MS}}}(2GeV)$



Results ($\overline{\mathrm{MS}}$ at 2GeV)

β	Z_V	Z_A
3.9	0.635(4)	0.757(3)
4.05	0.669(5)	0.776(3)
4.20	0.690(4)	0.789(3)

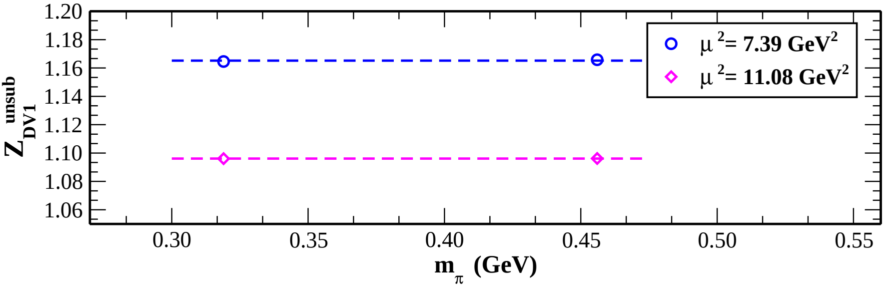
β	Z_{DV1}	Z_{DV2}	Z_{DA1}	Z_{DA2}
3.90	1.038(10)(20)	1.1293(69)(34)	1.174(8)(11)	1.153(6)(16)
4.05	1.0969(48)(42)	1.110(14)(26)	1.147(13)(24)	1.159(7)(16)
4.20	1.114(11)(17)	1.103(21)(42)	1.139(21)(40)	1.159(9)(20)

Systematic errors:

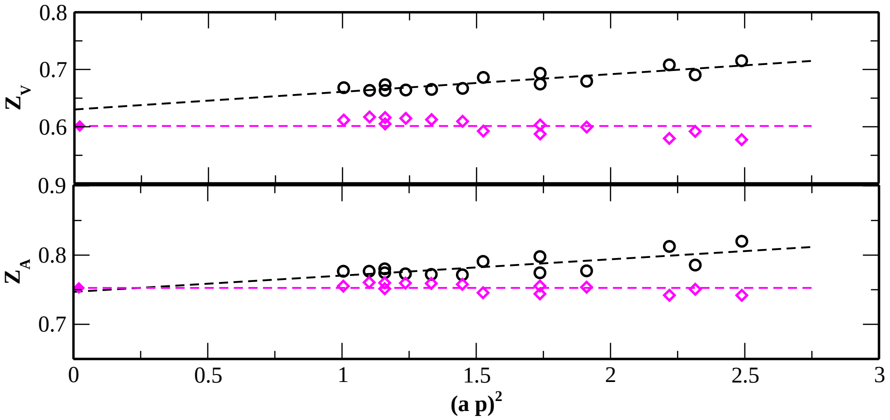
- $(a\,p)^2$ within [1.2, 2.7]
- $(a\,p)^2$ within [1, 2.7]
- $(a\,p)^2$ within [1.2, 2.2]

Preliminary results on $N_F = 2+1+1$, $a = 0.078$ fm

Quark mass dependence



$m_{\text{pion}} = 0.456$ **GeV,** $a = 0.078$ **fm**



Summary

- $\mathcal{O}(a^2)$ **subtraction are crucial**
- **Quark mass dependence is insignificant**
- **Similarly for** $N_F = 2 + 1 + 1$
- **Volume dependence is very small**

Future Work

- **Complete** $N_F = 2 + 1 + 1$ **computations**
- **Consider** $N_F = 4$ **computation and compare with** $N_F = 2 + 1 + 1$

THANK YOU

Backup Slides

- **Method B: momentum source**

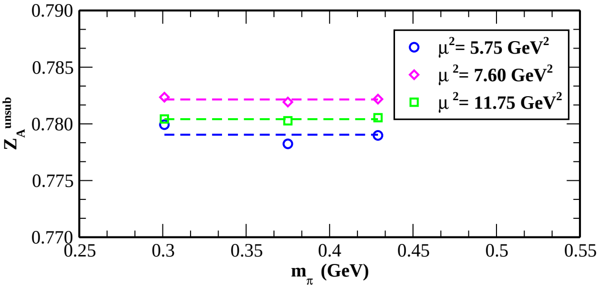
Twisted basis:

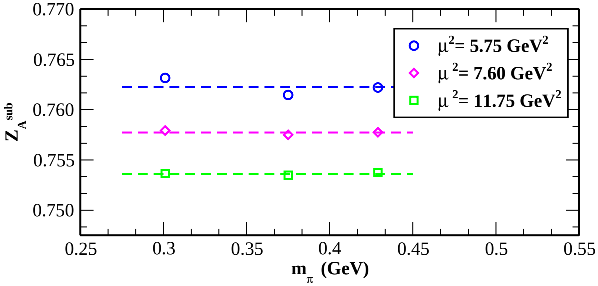
$$G^{ad}_{\alpha\delta}(p) = \frac{1}{4\,V}\sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\,\langle\,\Big[(\hat{1}+i\,\gamma^5)\boxed{\mathcal{U}(x,z)(\hat{1}}+i\,\gamma^5)\Big]^{ab}_{\alpha\beta}\,\tilde{J}^{bc}_{\beta\gamma}(z,z')\,\Big[(\hat{1}-i\,\gamma^5)\mathcal{D}(z',y)(\hat{1}-i\,\gamma^5)\Big]^{cd}_{\gamma\delta}\,\rangle^G$$

exact relation:

$$\boxed{\mathcal{U}(x,z) = \gamma^5\,\mathcal{D}^\dagger(z,x)\,\gamma^5}$$

$$G^{ad}_{\alpha\delta}(p) = -\frac{1}{4\,V}\sum_{z}\,\langle\,\Big[(\hat{1}-i\,\gamma^5)\sum_x \mathcal{D}^\dagger(z,x)\mathrm{e}^{-\mathrm{i}p\cdot x}(\hat{1}-i\,\gamma^5)\Big]^{ab}_{\alpha\beta}\,\tilde{J}^{bc}_{\beta\gamma}(z,z')\,\Big[(\hat{1}-i\,\gamma^5)\sum_y \mathcal{D}(z',y)\mathrm{e}^{\mathrm{i}p\cdot y}(\hat{1}-i\,\gamma^5)\Big]^{cd}_{\gamma\delta}\,\rangle^G$$

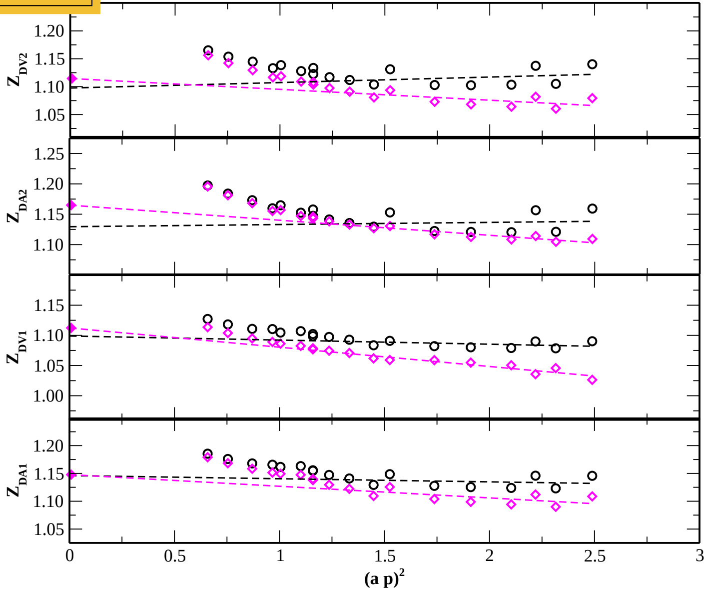

0.790
0.785
0.780
0.775
0.770
0.25
0.3
0.35
0.4
0.45
0.5
0.55
Z_A^{unsub}
m_π (GeV)
μ^2= 5.75 GeV^2
μ^2= 7.60 GeV^2
μ^2= 11.75 GeV^2


0.770
0.765
0.760
0.755
0.750
0.25
0.3
0.35
0.4
0.45
0.5
0.55
Z_A^{sub}
m_π (GeV)
μ^2= 5.75 GeV^2
μ^2= 7.60 GeV^2
μ^2= 11.75 GeV^2

β=4.20, a=0.055 **fm**

m_π=0.460 **GeV**

$32^3 \times 64$



$m_{\text{pion}} = 0.456$ **GeV,** $a = 0.078$ **fm**