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Reflection Positivity of Free Overlap Fermions

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based on arXiv:1005.3751

Contents

Setup

- Four dimensional hypercubic lattice $\Lambda = [-L+1, L]^4 \subset \mathbb{Z}^4$
- Boundary condition of Λ :

Periodic b.c. in Space directions
Anti-priodic b.c. in Time direction

- Fermionic fields $\{\psi_\alpha(x), \bar{\psi}_\alpha(x)\}_{\alpha, x\in\Lambda}$
- Grasmann algebra of fermionic fields $\mathcal{A}$
- 'Positive (Negative) time sites' Λ_+ (Λ_-)
- $\mathcal{A}_\pm$=Algebra generated by fields on $\Lambda_\pm$
- Fermionic action (D_Λ: Lattice Dirac operator on Λ, D: on $\mathbb{Z}^4$)

$$A(\bar{\psi}, \psi) = \sum_{x\in\Lambda} \bar{\psi}(x) D_\Lambda \psi(x)$$

$$D_\Lambda(x, y) = \sum_{n\in\mathbb{Z}^4} (-1)^{n_0} D(x + 2nL, y)$$

- The expectation functional $\langle\cdot\rangle$

$$\langle F \rangle = \frac{1}{Z} \int D[\psi] D[\bar{\psi}] \mathrm{e}^{A(\bar{\psi}, \psi)} F(\bar{\psi}, \psi)$$

Reflection Positivity (RP)

We define θ reflection as follows :

- For sites : $\theta(t, \boldsymbol{x}) = (-t+1, \boldsymbol{x})$ (Reflection with respect to the $t = 1/2$ plane)
- For fields :

$$\theta\psi(x) = \bar{\psi}(\theta x)\gamma_0, \quad \theta\bar{\psi}(x) = \gamma_0\psi(\theta x)$$

- Extend this θ to $\mathcal{A}$ by:

$$\theta(FG) = \theta(G)\theta(F)$$

$$\theta(\alpha F + \beta G) = \alpha^*\theta(F) + \beta^*\theta(G)$$

Definition. We say a lattice theory is reflection positive (w.r.t. θ) if for any $F \in \mathcal{A}_+$

$$\langle\theta(F)F\rangle \geq 0.$$

This condition ensures that the quantum theory can be reconstructed. i.e. Hilbert space of state vectors, equipped with the positive definite metric, and hermitian Hamiltonian operator acting on it.

[K. Osterwalder, R. Schrader '73]
[K. Osterwalder, E. Seiler '78]

In the case of Wilson Fermion

- Wilson Dirac operator.

$$D_{\mathrm{w}} = \frac{1}{2}\sum_{\mu=0}^{3}\gamma_\mu(\partial_\mu - \partial_\mu^\dagger) + \frac{1}{2}\sum_{\mu=0}^{3}\partial_\mu\partial_\mu^\dagger$$

- Reflection positivity has been rigorously proven :

> **Theorem.** The system of Wilson fermion satisfies the reflection positivity condition (not only for the free case but also for the gauge interacting case).

[M. Lüscher '77]
[R. Osterwalder, E. Seiler '78]
[P. Menotti, A. Pelissetto '87]

- The use of Wilson fermions in numerical applications to Lattice QCD, for example, has a completely sound basis.

In the case of overlap Fermion

- Overlap Dirac operator

$$D = \frac{1}{2}\Big(1 + \frac{X}{\sqrt{X^\dagger X}}\Big), \quad X = D_{\mathrm{w}} - m, \quad 0 < m < 2$$

[H. Neuberger '98]

- The reflection positivity of the overlap fermion is not completely established yet.

[M. Lüscher '01][M. Creutz '04][J. E. Mandula '09]

- Our main result in this talk is :

> **Theorem.** The expectation functional of *free* overlap Dirac fermion $\langle \cdot \rangle$ is reflection positive if $0 < m \leq 1$.

[Y. Kikukawa and KU, arXiv:1005.3751]

- Therefore, when there is no gauge interaction, the overlap fermion system *is* indeed a quantum theory satisfying unitarity!

Proof

- Outline of the proof (Following Wilson case given in [R. Osterwalder, E. Seiler '78])
  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
  - Prove lemmas (Only lemma (iii) and (iv) will be shown here)

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  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
  - Prove lemmas (Only lemma (iii) and (iv) will be shown here)
- Preparation to state lemmas :

$$\mathcal{P} := \left\{ \sum_j \theta(F_j^+) F_j^+ \; : \; F_j^+ \in \mathcal{A}_+ \right\}$$

$$\langle F \rangle_0 := \int D[\psi] D[\bar{\psi}] F(\bar{\psi}, \psi)$$

$$A = A_+ + A_- + \Delta A, \quad A_+ \in \mathcal{A}_+, A_- \in \mathcal{A}_-$$

- Four lemmas :

  - Lemma (i) If $F \in \mathcal{P}$, then $\langle F \rangle_0 \geq 0$ [R. Osterwalder, E. Seiler '78]
  - Lemma (ii) $F, G \in \mathcal{P}$ implies $FG \in \mathcal{P}$ [R. Osterwalder, E. Seiler '78]
  - Lemma (iii) $A_- = \theta(A_+)$
  - Lemma (iv) $\Delta A \in \mathcal{P}$

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- Preparation to state lemmas :

$$\mathcal{P} := \overline{\left\{ \sum_j \theta(F_j^+) F_j^+ \ : \ F_j^+ \in \mathcal{A}_+ \right\}}$$

$$\langle F \rangle_0 := \int D[\psi] D[\bar{\psi}] F(\bar{\psi}, \psi)$$

$$A = A_+ + A_- + \Delta A, \quad A_+ \in \mathcal{A}_+, A_- \in \mathcal{A}_-$$

- Four lemmas :
  - Lemma (i) If $F \in \mathcal{P}$, then $\langle F \rangle_0 \geq 0$ [R. Osterwalder, E. Seiler '78]
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- Outline of the proof (Following Wilson case given in [R. Osterwalder, E. Seiler '78])
  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
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Proof

- Outline of the proof (Following Wilson case given in [R. Osterwalder, E. Seiler '78])
  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
  - Prove lemmas (Only lemma (iii) and (iv) will be shown here)
- Remember four lemmas :

  - Lemma (i) If $F \in \mathcal{P}$, then $\langle F \rangle_0 \geq 0$
  - Lemma (ii) $F, G \in \mathcal{P}$ implies $FG \in \mathcal{P}$
  - Lemma (iii) $A_- = \theta(A_+)$
  - Lemma (iv) $\Delta A \in \mathcal{P}$

- Lemmas imply the theorem : [R. Osterwalder, E. Seiler '78]

$$
\begin{aligned}
\langle \theta(F) F \rangle &= \left\langle \mathrm{e}^{A} \right\rangle_0^{-1} \left\langle \mathrm{e}^{A}\, \theta(F) F \right\rangle_0 \\
&= \left\langle \mathrm{e}^{A_+ + A_- + \Delta A} \right\rangle_0^{-1} \left\langle \mathrm{e}^{A_+ + A_- + \Delta A}\, \theta(F) F \right\rangle_0 \\
&= \left\langle \mathrm{e}^{A_+ + \theta(A_+) + \Delta A} \right\rangle_0^{-1} \left\langle \mathrm{e}^{A_+ + \theta(A_+) + \Delta A}\, \theta(F) F \right\rangle_0 \quad \text{by (iii)} \\
&= \left\langle \underbrace{\theta(\mathrm{e}^{A_+}) \mathrm{e}^{A_+}}_{\in \mathcal{P}} \quad \underbrace{\mathrm{e}^{\Delta A}}_{\in \mathcal{P} \text{ (by (ii),(iv))}} \right\rangle_0^{-1} \left\langle \underbrace{\theta(\mathrm{e}^{A_+}) \mathrm{e}^{A_+}\, \mathrm{e}^{\Delta A}}_{\in \mathcal{P} \text{ (by (ii),(iv))}} \underbrace{\theta(F) F}_{\in \mathcal{P}} \right\rangle_0 \geq 0 \quad \text{by (i) and (ii)}
\end{aligned}
$$

Proof

- Outline of the proof (Following Wilson case given in [R. Osterwalder, E. Seiler '78])
  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
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Proof

- Outline of the proof (Following Wilson case given in [R. Osterwalder, E. Seiler '78])
  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
  - Prove lemmas (Only lemma (iii) and (iv) will be shown here)
- Proof of the lemma (iii) $A_- = \theta(A_+)$
  - Charge conjugation property of the overlap Dirac operator :

$$\gamma_0 D(x,y)^\dagger \gamma_0 = D(\theta y, \theta x)$$

  - We can explicitly calculate $\theta(A_+)$:

$$\begin{aligned}
\theta(A_+) &= \sum_{x\in\Lambda_+} \sum_{y\in\Lambda_+} \theta\Big(\bar{\psi}(x) D_\Lambda(x,y)\psi(y)\Big) \\
&= \sum_{x\in\Lambda_+} \sum_{y\in\Lambda_+} \bar{\psi}(\theta y) \underbrace{\gamma_0 D_\Lambda(x,y)^\dagger \gamma_0}_{=D(\theta y,\theta x)} \psi(\theta x) \\
&= \sum_{x'\in\Lambda_-} \sum_{y'\in\Lambda_-} \bar{\psi}(x') D_\Lambda(x',y')\psi(y') \quad x' := \theta y,\; y' := \theta x \\
&= A_-
\end{aligned}$$

Proof

- Outline of the proof (Following Wilson case given in [R. Osterwalder, E. Seiler '78])
  - Prepare four small lemmas from (i) to (iv)
  - Show that the theorem immediately follows from these four lemmas
  - Prove lemmas (Only lemma (iii) and (iv) will be shown here)
- Proof of the lemma (iv) $\Delta A \in \mathcal{P}$

$$\Delta A = \sum_{x\in\Lambda_+}\sum_{y\in\Lambda_-} \bar{\psi}(x) D_\Lambda(x,y)\psi(y) + \sum_{x\in\Lambda_-}\sum_{y\in\Lambda_+} \bar{\psi}(x) D_\Lambda(x,y)\psi(y) \in \mathcal{P}$$

- Strategy :
  1. Find the spectral representation of D
  2. Find the spectral representation of D_Λ through

$$D_\Lambda(x,y) = \sum_{n\in\mathbb{Z}^4} (-1)^{n_0} D(x+2nL, y)$$

  3. Write down ΔA in the form of

$$\Delta A = \int_{E_1}^{\infty} \frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}} \left[\theta(C_{E,\boldsymbol{p}})C_{E,\boldsymbol{p}} + \theta(D_{E,\boldsymbol{p}})D_{E,\boldsymbol{p}}\right] \in \mathcal{P}$$

- Spectral representation of D :

$$
\begin{aligned}
D(x,y)\Big|_{x_0>y_0} &= \int_{-\pi}^{\pi}\frac{dp_0}{2\pi}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{ip_0(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\frac{X(p_0,\boldsymbol{p})}{2\sqrt{X^\dagger X(p_0,\boldsymbol{p})}} \\
&= \int_{E_1}^{\infty}\frac{dE}{2\pi}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{-E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\frac{X(iE,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}
\end{aligned}
$$

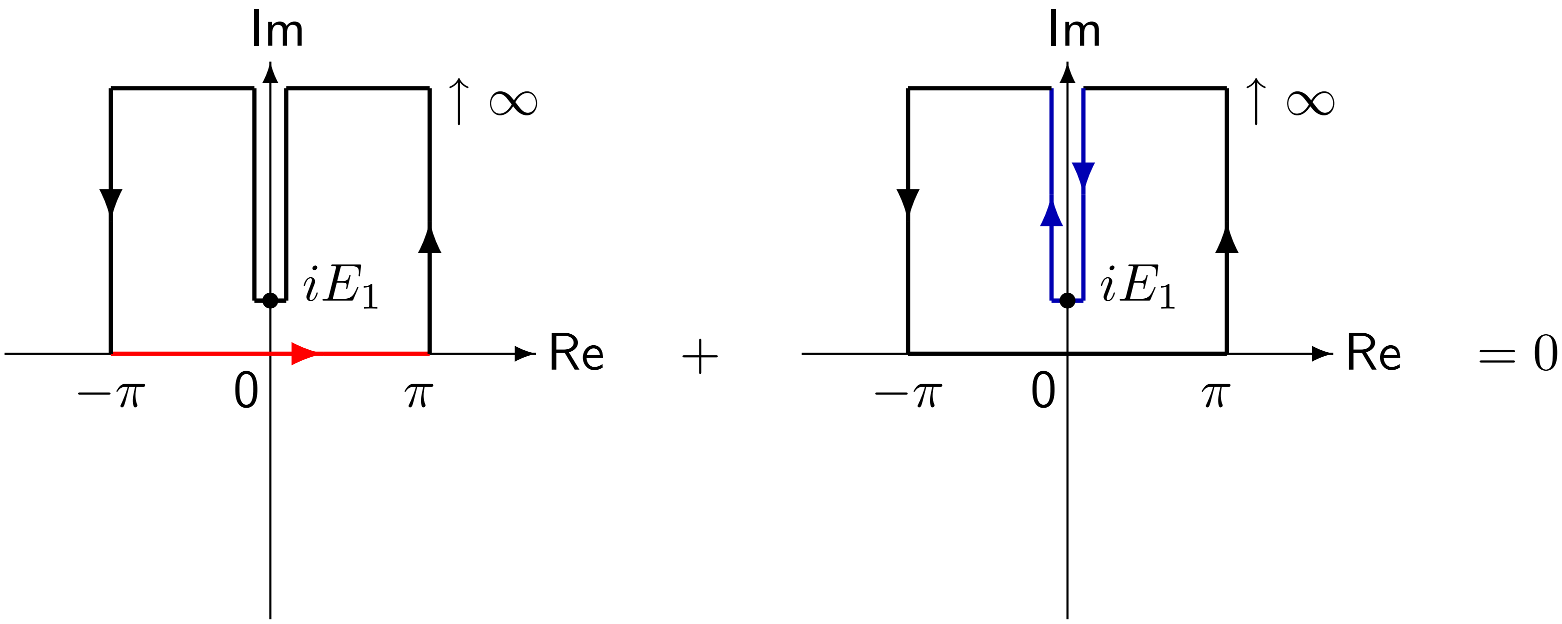

図1　Complex integration contours

- Spectral representation of D_Λ :

$$\begin{cases} D(x,y)\Big|_{x_0>y_0} = \int_{E_1}^{\infty} \frac{dE}{2\pi} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{-E(x_0-y_0)} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{X(iE,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}} \\ D_\Lambda(x,y) = \sum_{n\in\mathbb{Z}^4} (-1)^{n_0} D(x+2nL,y) \end{cases}$$

$$\Longrightarrow \quad D_\Lambda(x,y)\Big|_{x_0>y_0} = \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \sum_{\boldsymbol{p}} \frac{1}{1+\mathrm{e}^{-2EL}} \mathrm{e}^{-E(x_0-y_0)} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{X(iE,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$
$$+ \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \sum_{\boldsymbol{p}} \frac{\mathrm{e}^{-2EL}}{1+\mathrm{e}^{-2EL}} \mathrm{e}^{+E(x_0-y_0)} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{-X(-iE,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$

- Our goal :

$$\sum_{x\in\Lambda_+} \sum_{y\in\Lambda_-} \bar\psi(x) D_\Lambda(x,y) \psi(y) = \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \sum_{\boldsymbol{p}} \Big[\theta(C_{E,\boldsymbol{p}}) C_{E,\boldsymbol{p}} + \theta(D_{E,\boldsymbol{p}}) D_{E,\boldsymbol{p}} \Big]$$

- Crucial Property of X :

There exist spinor matrices $Y_\pm(E,\boldsymbol{p})$ such that

$$X(\pm iE,\boldsymbol{p}) = \mp\gamma_0 Y_\pm^\dagger Y_\pm(E,\boldsymbol{p}) \quad (E \geq E_1).$$

- Spectral representation of D_Λ :

$$\begin{cases} D(x,y)\Big|_{x_0>y_0} = \int_{E_1}^{\infty} \frac{dE}{2\pi} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{-E(x_0-y_0)} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{X(iE,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}} \\ D_\Lambda(x,y) = \sum_{n\in\mathbb{Z}^4} (-1)^{n_0} D(x+2nL, y) \end{cases}$$

$$\Longrightarrow \; D_\Lambda(x,y)\Big|_{x_0>y_0} = \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \sum_{\boldsymbol{p}} \frac{1}{1+\mathrm{e}^{-2EL}} \mathrm{e}^{-E(x_0-y_0)} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{-\gamma_0 Y_+^\dagger Y_+(E,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$

$$+ \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \sum_{\boldsymbol{p}} \frac{\mathrm{e}^{-2EL}}{1+\mathrm{e}^{-2EL}} \mathrm{e}^{+E(x_0-y_0)} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{-\gamma_0 Y_-^\dagger Y_-(E,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$

- Our goal :

$$\sum_{x\in\Lambda_+} \sum_{y\in\Lambda_-} \bar{\psi}(x) D_\Lambda(x,y) \psi(y) = \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \sum_{\boldsymbol{p}} \left[\theta(C_{E,\boldsymbol{p}}) C_{E,\boldsymbol{p}} + \theta(D_{E,\boldsymbol{p}}) D_{E,\boldsymbol{p}} \right]$$

- Crucial Property of X :

There exist spinor matrices $Y_\pm(E,\boldsymbol{p})$ such that

$$X(\pm iE,\boldsymbol{p}) = \mp\gamma_0 Y_\pm^\dagger Y_\pm(E,\boldsymbol{p}) \quad (E \geq E_1).$$

- Remember our goal again :

$$\sum_{x\in\Lambda_+}\sum_{y\in\Lambda_-}\bar{\psi}(x)D_\Lambda(x,y)\psi(y)=\int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\Big[\theta(C_{E,\boldsymbol{p}})C_{E,\boldsymbol{p}}+\theta(D_{E,\boldsymbol{p}})D_{E,\boldsymbol{p}}\Big]$$

- We have gotten the spectral representation of D_Λ :

$$D_\Lambda(x,y)\Big|_{x_0-y_0>0}=\int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\frac{1}{1+\mathrm{e}^{-2EL}}\mathrm{e}^{-E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\frac{-\gamma_0Y_+^\dagger Y_+(E,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$
$$+\int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\frac{\mathrm{e}^{-2EL}}{1+\mathrm{e}^{-2EL}}\mathrm{e}^{E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\frac{-\gamma_0Y_-^\dagger Y_-(E,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$

- Define $C_{E,\boldsymbol{p}}$ and $D_{E,\boldsymbol{p}}$ as :

$$C_{E,\boldsymbol{p}}=\sqrt{\frac{1}{1+\mathrm{e}^{-2EL}}}\sum_{x\in\Lambda_+}\bar{\psi}(x)\gamma_0\frac{Y_+^\dagger(E,\boldsymbol{p})}{(-X^\dagger X(iE,\boldsymbol{p}))^{1/4}}\mathrm{e}^{-Ex_0}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

$$D_{E,\boldsymbol{p}}=\sqrt{\frac{\mathrm{e}^{-2EL}}{1+\mathrm{e}^{-2EL}}}\sum_{x\in\Lambda_+}\bar{\psi}(x)\gamma_0\frac{Y_-^\dagger(E,\boldsymbol{p})}{(-X^\dagger X(iE,\boldsymbol{p}))^{1/4}}\mathrm{e}^{Ex_0}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$

$$D_\Lambda(x,y)\Big|_{x_0-y_0>0} = \int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\frac{1}{1+\mathrm{e}^{-2EL}}\mathrm{e}^{-E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\frac{-\gamma_0 Y_+^\dagger Y_+(E,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$

$$+\int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\frac{\mathrm{e}^{-2EL}}{1+\mathrm{e}^{-2EL}}\mathrm{e}^{E(x_0-y_0)}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\frac{-\gamma_0 Y_-^\dagger Y_-(E,\boldsymbol{p})}{\sqrt{-X^\dagger X(iE,\boldsymbol{p})}}$$

$$C_{E,\boldsymbol{p}} = \sqrt{\frac{1}{1+\mathrm{e}^{-2EL}}}\sum_{x\in\Lambda_+}\bar{\psi}(x)\gamma_0\frac{Y_+^\dagger(E,\boldsymbol{p})}{(-X^\dagger X(iE,\boldsymbol{p}))^{1/4}}\mathrm{e}^{-Ex_0}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

$$D_{E,\boldsymbol{p}} = \sqrt{\frac{\mathrm{e}^{-2EL}}{1+\mathrm{e}^{-2EL}}}\sum_{x\in\Lambda_+}\bar{\psi}(x)\gamma_0\frac{Y_-^\dagger(E,\boldsymbol{p})}{(-X^\dagger X(iE,\boldsymbol{p}))^{1/4}}\mathrm{e}^{Ex_0}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$

- We obtain :

$$\sum_{x\in\Lambda_+}\sum_{y\in\Lambda_-}\bar{\psi}(x)D_\Lambda(x,y)\psi(y) = \int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\Big[-C_{E,\boldsymbol{p}}\theta(C_{E,\boldsymbol{p}})-D_{E,\boldsymbol{p}}\theta(D_{E,\boldsymbol{p}})\Big]$$

$$= \int_{E_1}^{\infty}\frac{dE}{2\pi}\frac{1}{V}\sum_{\boldsymbol{p}}\Big[\theta(C_{E,\boldsymbol{p}})C_{E,\boldsymbol{p}}+\theta(D_{E,\boldsymbol{p}})D_{E,\boldsymbol{p}}\Big]\in\mathcal{P}$$

Summery and Discussions

- It is rigorously shown that the system of free overlap Dirac fermion satisfies the reflection positivity condition, which leads to the unitarity property of the corresponding quantum theory.
- The proof given here can be extended to other free fermions — Majorana and Weyl fermions — if there is no gauge interaction.
- Our proof can be easily extended to non-gauge theories with interactions such as Chiral Yukawa theory.
- However, the case of gauge theory — the most important case — does not seem to be easily treated. In fact, it is still an open problem to prove the reflection positivity condition for the overlap gauge theory. We will leave it for future study.
- Or, we should examine to what extent the reflection positivity is violated, if any, in overlap Dirac fermions. Work in these directions is in progress.

Majorana case

- Charge conjugation operator C :

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T, \quad C\gamma_5 C^{-1} = \gamma_5^T, \quad C^\dagger C = 1, \quad C^T = -C$$

- Majorana condition $\bar{\psi} = \psi^T C$ is consistent with the definition of θ reflection :

$$\begin{cases} \theta\psi(x) = \bar{\psi}(\theta x)\gamma_0 \\ \theta\bar{\psi}(x) = \gamma_0\psi(\theta x) \end{cases} \Longrightarrow \theta(\psi(x)) = C\gamma_0\psi(\theta x)$$

- Lemma (i)(ii)(iii)(iv) are still true in this case, which imply that the reflection positivity condition is satisfied in Majorana case.

Weyl case

- Chiral projection

$$\psi_{\pm}(x) = \left(\frac{1 \pm \hat{\gamma}_5}{2}\right) \psi(x), \quad \bar{\psi}_{\pm}(x) = \bar{\psi}(x) \left(\frac{1 \mp \gamma_5}{2}\right)$$

$$(\hat{\gamma}_5 := \gamma_5(1 - 2D))$$

- Gamma matrices

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Chiral basis :

$$\{v^i_{\pm}(x) \,|\, \hat{\gamma}_5 v^i_{\pm}(x) = \pm v^i_{\pm}(x);\ i = 1, \cdots, 2(2L)^4\}$$

$$\psi_{\pm}(x) = \sum_i v^i_{\pm}(x) c^i_{\pm} \quad (c^i_{\pm} : \text{Grasmann coefficients})$$

- Path integration measure :

$$\mathcal{D}[\psi_{\pm}]\mathcal{D}[\bar{\psi}_{\pm}] = \prod_i dc^i_{\pm} \prod_{x \in \Lambda; \alpha_{\mp}} d\bar{\psi}_{\alpha_{\mp}}(x)$$

- Left-handed expectation $\langle\cdot\rangle^{(-)}$

$$\langle F\rangle^{(-)} := \frac{1}{Z^{(-)}}\int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\,\mathrm{e}^{A^{(-)}(\bar{\psi}_-,\psi_-)}F(\bar{\psi}_-,\psi_-)$$

- θ reflection : $(\alpha_+ = 1,2,\ \alpha_- = 3,4)$

$$\theta(\psi_{-\alpha_-}(x)) = \{\bar{\psi}_-(\theta x)\gamma_0\}_{\alpha_-} = \bar{\psi}_{-\alpha_+}(\theta x)$$
$$\theta(\bar{\psi}_{-\alpha_+}(x)) = \{\gamma_0\psi_-(\theta x)\}_{\alpha_+} = \psi_{-\alpha_-}(\theta x)$$

- ${\mathcal{A}_\pm}^{(-)}$:= algebra of all the polynomials of the left-handed field components $\psi_{-\alpha_-}(x)$ and $\bar{\psi}_{-\alpha_+}(x)$ on $\Lambda_\pm$
  (NOTE : The field components $\psi_{-\alpha_+}(x)$ are completely excluded from observables)
- Relations between Weyl and Dirac Expectations :

$$\left\langle F(\psi_{-\alpha},\bar{\psi}_{-\alpha_+})\right\rangle^{(-)} = \left\langle F(\psi_{\alpha_-},\bar{\psi}_{\alpha_+})\right\rangle$$

  This follows from the fact that

$$\left\{\left(\frac{1-\hat{\gamma}_5}{2}\right)D^{-1}\ \right\}_{\alpha_-,\alpha_+} = \left\{\left(\frac{1-\gamma_5}{2}\right)D^{-1}\right\}_{\alpha_-,\alpha_+}$$

Another proof via domain wall fermion

- Domain wall fermion with Pauli Villars fields

$$
\begin{aligned}
S_{\mathsf{DW}}(\bar{\Psi}, \Psi, \bar{\Xi}, \Xi, \bar{\Phi}, \Phi, \bar{\chi}, \chi) = & \sum_{x,s;y,t} \bar{\Psi}(x,s)(D_{\mathsf{w5}}^{\mathsf{Dir}} - m)(x,s;y,t)\Psi(y,t) \\
& + \sum_{x,s;y,t} \bar{\Xi}(x,s)(D_{\mathsf{w5}}^{\mathsf{AP}} - m)(x,s;y,t)\Xi(y,t) \\
& + \sum_{x,s;y,t} \Phi(x,s)^\dagger (D_{\mathsf{w5}}^{\mathsf{AP}} - m)^\dagger (D_{\mathsf{w5}}^{\mathsf{AP}} - m)(x,s;y,t)\Phi(y,t) \\
& + \sum_{x} \bar{\chi}(x)\chi(x)
\end{aligned}
$$

  - $D_{\mathsf{w5}}^{\mathsf{xxx}}$: Five dimensional Wilson Dirac operator with the boundary condition xxx
  - Ψ, Ξ : Fermionic fields in five dimensional space-time
  - Φ : Pauli Villars field
  - χ : Auxiliary filed in four dimensional space-time
- Relation between overlap fermion and domain wall fermion :

$$
\left\langle F(\bar{\psi}, \psi) \right\rangle_{\mathsf{overlap}} = \lim_{a_5 \to 0} \lim_{N \to \infty} \langle F(\bar{q} + \bar{\chi}, q + \chi) \rangle_{\mathsf{domain\ wall}}^{N, a_5} .
$$

The positivity of the RHS can be shown in the free case.