Nilpotency expansion for QCD at finite chemical potential

S.Caracciolo[1] and F. Palumbo[2]

1: Milano University
2: INFN - Laboratori Nazionali di Frascati

June, 2010

Sign problem

Large cancellations among fermion determinants at nonzero chemical potential make numerical simulations difficult

Working assumptions

1) energetically stable configurations of fermions give contributions of the same sign to the free energy

2) diquarks are stable substructures at low baryon density (atomic nucleons, multiquark mesons) and basic constituents of the color superconducting phases at high baryon density

Formalism for introducing diquarks

Time-dependent Bogoliubov transformations

Operator form of the partition function for Kogut-Susskind fermions

$$\mathcal{Z} = \int [dU] \exp[-S_G(U)] \, \mathrm{Tr}^{\mathrm{F}} \left\{ \prod_{t=0}^{L_0-1} \left(\hat{T}^{\dagger} \hat{V}_t \exp(\mu \, \hat{n}_B) \hat{T}_{t+1} \right) \right\}$$

$$\hat{T}_t = \exp \, (\hat{v} \, N_t \, \hat{u})$$

$$\hat{V}_t = \exp \left(\hat{u}^{\dagger} \ln U_{0,t} \, \hat{u} + \hat{v}^{\dagger} \ln U^{*}_{0,t} \, \hat{v} \right)$$

S_G = gluon action , μ = chemical potential , $\hat{n}_B$ = baryon number

Tr^{F} trace on the fermion Fock space

$\hat{u}, \hat{v}$ = fermion-antifermion canonical annihilation operators

$U_{k,t}$ = spatial link variables , $U_{0,t}$ = temporal link variables

$N_t = N(U_{k,t})$ depends spatial link variables.

The matrix N in the flavor basis

Kogut-Susskind fermions in the flavor basis have Dirac and taste indices on which the matrices γ_μ, t_μ act

$$\begin{aligned} N &= -2\,\gamma_0 \otimes \mathbb{1} \left\{ m + \sum_{j=1}^{3} \gamma_j \otimes \mathbb{1} \left[P_j^{(-)} \nabla_j^{(+)} + P_j^{(+)} \nabla_j^{(-)} \right] \right\} \\ &= \text{Dirac Hamiltonian} \end{aligned}$$

$$P_\mu^{(\pm)} = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \pm \gamma_\mu \gamma_5 \otimes t_5 t_\mu)$$

$$\nabla_j^{(+)} = \frac{1}{2}\left(U_j\, T_j^{(+)} - 1\right), \quad \nabla_j^{(-)} = \frac{1}{2}\left(1 - T_j^{(-)} U_j^{\dagger}\right) = \text{covariant derivatives}$$

$T_\mu^{(\pm)}$ = forward / backward translation operators of one block

Bogoliubov transformations and symmetries

$$\hat{\alpha} = \mathrm{R}^{\frac{1}{2}} \left(\hat{u} - \mathcal{F}^{\dagger}\, \hat{v}^{\dagger}\right) \qquad \hat{\beta} = \left(\hat{v} + \hat{u}^{\dagger}\, \mathcal{F}^{\dagger}\right) \mathring{\mathrm{R}}^{\frac{1}{2}}$$

$$\mathrm{R} = (1 + \mathcal{F}^{\dagger}\mathcal{F})^{-1} \qquad \mathring{\mathrm{R}} = (1 + \mathcal{F}\mathcal{F}^{\dagger})^{-1}$$

Quasiparticle operators $\hat{\alpha}, \hat{\beta}$ do not have definite transformation properties

Gauge transformations

$$\hat{u}_{\mathrm{x}} \rightarrow g_{\mathrm{x}}(t)\, \hat{u}_{\mathrm{x}}\,, \quad \hat{v}^{\dagger}_{\mathrm{x}} \rightarrow g_{\mathrm{x}}(t)\, \hat{v}^{\dagger}_{\mathrm{x}}$$

Quasiparticles will transform as particles if

$$\mathcal{F}_{\mathrm{x,y}} \rightarrow g_{\mathrm{x}}(t)\, \mathcal{F}_{\mathrm{x,y}}\, g_{\mathrm{y}}(t)^{\dagger}$$

This condition is fulfilled if $\mathcal{F}$ is a function of time according to

$$\mathcal{F}_t = \mathcal{F}(U_k(t))$$

Chiral transformations

$$\hat{u}_{\mathbf{x}} \to e^{i\frac{1}{2}\theta_t}\, \hat{u}_{\mathbf{x}}\,, \quad \hat{v}^{\dagger}_{\mathbf{x}} \to e^{-i\frac{1}{2}\theta_t}\, \hat{v}^{\dagger}_{\mathbf{x}}$$

Quasiparticles will transform as particles if

$$\mathcal{F}_{\mathbf{x},\mathbf{y}} \to e^{-i\theta_t}\, \mathcal{F}_{\mathbf{x},\mathbf{y}}$$

This can be obtained if

$$(\mathcal{F}_t)_{\mathbf{x}_1,\mathbf{x}_2} = \sum_K \varphi_K(\mathbf{x},t)\left(\Phi_{K,\mathbf{x}}(U_{k,t})\right)^{\dagger}_{\mathbf{x}_1,\mathbf{x}_2}$$

and under chiral transformations

$$\varphi_K(\mathbf{x},t) \to e^{-i\theta_t}\, \varphi_K(\mathbf{x},t)$$

$\varphi_K(\mathbf{x},t)$ are bosonic fields with quantum numbers K, and $\left(\Phi_{K,\mathbf{x}}(U_{k,t})\right)^{\dagger}_{\mathbf{x}_1,\mathbf{x}_2}$ their structure functions which must depend on the spatial link variables

Since nothing depends on the bosonic fields φ_t, we can integrate over them in the partition function with an arbitray measure $d\mu(\varphi^\dagger,\varphi)$.

The trace over the transformed states in the partition function can be performed exactly yielding its functional form

$$\mathcal{Z} = \int [dU] \exp\left[-S_G(U)\right] \int d\mu(\varphi^\dagger,\varphi) \exp\left[S_{eff}\right]$$

$$\begin{aligned} S_{\rm eff} \;&=\; S_{\rm mesons} - \sum_t \alpha_t^* \left(\nabla_t - \mathcal{H}_t\right) \alpha_{t+1} \\ &-\; \beta_{t+1}\left(\mathring{\nabla}_t - \mathring{\mathcal{H}}_t\right)\beta_t^* + \beta_t \mathcal{I}_t^{(2,1)} \alpha_t + \alpha_t^* \mathcal{I}_t^{(1,2)} \beta_t^* \end{aligned}$$

Meson effective action

$$S_{\text{mesons}} = \sum_t \mathbf{tr}_- \left[-\ln \mathsf{R}_t + \ln \mathcal{R}_t \right]$$

$$\mathcal{R}_t = \left[1 + \left(N_t + U^\dagger_{0,t-1} \mathcal{F}_{t-1} U_{0,t-1} \right)^\dagger \times (N_t + \mathcal{F}_t) \right]^{-1} .$$

Index of nilpotency

The quasiparticle vacuum is

$$|\mathcal{F}\rangle = \exp\left(\hat{u}^{\dagger}\mathcal{F}^{\dagger}\hat{v}^{\dagger}\right)|0\rangle$$

definition: Ω_K = index of nilpotency = largest integer such that

$$\left(\hat{u}^{\dagger}\mathcal{F}^{\dagger}\hat{v}^{\dagger}\right)^{\Omega_K} \neq 0$$

Ω_K = maximum number of composites we can put in the state K

Necessary condition for composites to be interpreted as bosons

$$\Omega_K >> 1$$

Ω_K = number of fermionic states in structure function Φ_K, much greater than the number of intrinsic degrees of freedom

Nilpotency expansion

The partition function can be expanded in inverse powers of the index of nilpotency

$$\mathcal{F}_t = \overline{\mathcal{F}} + \delta\mathcal{F}_t$$

$\overline{\mathcal{F}}$, solution of the saddle point equations, determines the vacuum energy

The fluctuations $\delta\mathcal{F}_t$ represent mesonic fields, and reproduce correctly the results of a four-fermion model

We found an exact solution to the saddle point equations which requires stationarity of gauge fields in the sense that

Spatial-temporal plaquettes vanish, spatial plaquettes are constant in time

In the saddle point approximation quasiparticles propagate only in point-like color singlets

Diquarks

In the hamiltonian formalism diquarks are constructed in terms of positive energy states

In the transfer matrix formalism they are constructed by a Bogoliubov transformation as Cooper pairs of quasiparticles

$$\hat{\sigma} = r^{\frac{1}{2}} \left(\hat{\alpha} - \mathcal{D}^{\dagger} \hat{\alpha}^{\dagger} \right) , \quad \hat{\sigma}^{\dagger} = r^{*\frac{1}{2}} \left(\hat{\alpha}^{\dagger} - \tilde{\mathcal{D}}\, \hat{\alpha} \right)$$

where

$$r = \frac{1}{1 + \mathcal{D}^{\dagger} \mathcal{D}} .$$

The matrix $\mathcal{D}$ has the quantum numbers of the diquark field.

The vacuum of the new quasiparticle operators $\hat{\sigma}$ is

$$|\mathcal{D}, \mathcal{F}\rangle = \exp\left(\frac{1}{2} \hat{\alpha}^{\dagger} \mathcal{D}^{\dagger} \hat{\alpha}^{\dagger} \right) |\mathcal{F}\rangle = \exp\left(\frac{1}{2} \hat{\alpha}^{\dagger} \mathcal{D}^{\dagger} \hat{\alpha}^{\dagger} \right) \exp\left(\hat{u}^{\dagger} \mathcal{F}^{\dagger} \hat{v}^{\dagger} \right) |0\rangle ,$$

namely a condensate of Cooper pairs of quasiparticles.

Minima of the diquark action

The saddle point equations for $\mathcal{F}$ are not changed by the presence of diquarks. Using their solution the diquark field action can be written

$$\begin{aligned} \mathcal{S}_{diquark} &= L_0 \,\mathrm{tr} \left\{ \ln\left(1 - 2e^{-2\mu}\,\overline{\mathcal{H}}\right) + \frac{1}{2}\ln\left(1 + \mathcal{D}^\dagger \mathcal{D}\right) \right. \\ &- \left. \frac{1}{2}\ln\left[1 + \mathcal{D}\left(e^{2\mu} - 2\overline{\mathcal{H}}\right)\mathcal{D}^\dagger \left(e^{2\mu} - 2\overline{\mathcal{H}}\right)^T\right]\right\} \quad (1) \end{aligned}$$

This action takes its minima when $\mathcal{D}$ is zero or infinity

We will restrict ourselves to the so called simple pairing structure functions

$$\mathcal{D}_{i,j} = \delta_{j,i_c}\,\epsilon_{i,i_c}\,\mathcal{D}_i\,, \quad \mathcal{D}_i = \mathcal{D}_{i_c}$$

in which any quasiparticle state i is associated to one and only one conjugate state i_c

We will denote by i_p the states for which $|\mathcal{D}_{i_p}| = \infty$.Therefore

$$\mathcal{S}_{diquark} = L_0 \left\{ -\sum_i 2\mu + \sum_{i \neq i_p} \ln\left(e^{2\mu} - s\overline{\mathcal{H}}_i\right) \right\}$$

For given chemical potential this action is minimal if the i_p are all the states for which

$$e^{2\mu} - 2\overline{\mathcal{H}}_{i_p} > 1$$

Since the operator $e^{2\mu} - 2\overline{\mathcal{H}}_{i_p}$ is hermitean and positive definite, the above provides also a practical prescription free of the sign problem for the evaluation of the quark contribution to the free energy.

The eigenvalues $\overline{\mathcal{H}}_i$ are functions of the spatial link variables.

Averaging over these variables will smooth out the distribution of the values of $\mathcal{D}_i = 0, \infty$.

Perturbative expansion in the gauge coupling constant

For sufficiently high values of the chemical potential an expansion with respect to the gauge coupling constant can be justified

$$e^{2\mu} - 2\overline{\mathcal{H}} \approx 1 + A + g\,B + g^2\,C\,.$$

Assuming simple pairing (using the label i for all quasiparticle quantum numbers) we get the standard expression

$$\begin{aligned} S_{diquark} \;\approx\; & -L_0 \sum_i \Big\{ -\ln\Big(1 - s e^{-s\mu}\overline{\mathcal{H}}\Big) + \rho_i\,(A + g\,B + g^2\,C)_{ii} \\ & -\frac{1}{2}\,g^2 \rho_i\,B_{ij}\rho_j B_{ji} + \frac{1}{2}\Big(\psi_i^*\,\triangle_i + \triangle_i^*\,\psi_i\Big)\Big\} \end{aligned} \qquad (2)$$

where

$$\rho_i = \frac{|\mathcal{D}_i|^2}{1 + |\mathcal{D}_i|^2}\,, \quad \psi_i = \frac{1}{1 + |\mathcal{D}_i|^2}\,\mathcal{D}_i$$

$$\triangle_i = \frac{1}{2}\,g^2 \epsilon_{ii_c} \sum_k \epsilon_{kk_c}\,B_{ik}\,B_{i_ck_c}\,\psi_k$$

is the celebrated gap function. .

Summary

We have transformed the QCD action in an effective action containing quasiparticles plus bosonic fields with the quantum numbers of mesons and diquarks

We did it by means of time-dependent Bogoliubov transformations. Time dependence is necessary in order to maintain symmetries in each term of the action, and provides a natural way to generate not only condensates of composite fields, but also dynamical ones

We designed an expansion of the effective action in inverse powers of the index of nilpotency of the composites

We found an exact solution of the saddle point equation

We tested the nilpotency expansion on a four-fermion model at zero and finite chemical potential reproducing the known results

In the application to QCD we found that in the saddle point approximation quasiparticles propagate only in point-like color singlets

We found an expression of the free energy at nonvanishing chemical potential free of the sign problem

whose expansion with respect to the gauge coupling constant gives results compatible with the standard ones