A worm-inspired algorithm for the simulation of Abelian gauge theories

Tomasz Korzec and Ulli Wolff

- Worm algorithms
- U(1) gauge theory
- Updates for strong coupling graphs
- Comparison with a local algorithm

The XXVIII International Symposium on Lattice Field Theory

Worm Algorithms

- Worm algorithms have been formulated for various models
  - Ising model [N.Prokof'ev, B.Svistunov (2001)], [U.Wolff (2008)], [W.Janke, T.Neuhaus, A.Schakel (2010)]
  - Nonlinear σ-models [U.Wolff (2009)]
  - ϕ^4 theory [U.Wolff (2009)]
  - CP(N-1) models [U.Wolff (2010)]
  - Fermions (Gross-Neveu model) [U.Wolff (2009)], [U.Wenger (2009)], [V.Maillart's talk]
  - Supersymetric quantum mechanics [talks by U.Wenger and D.Baumgartner]
- Nearly no critical slowing down
- More general than e.g. cluster algorithms
- Degrees of freedom other than in standard approach
  - Improved estimators
  - Alternative actions

$\Rightarrow$ Plenary talk by Ulli Wolff, Saturday, 09:00 AM

Abelian gauge theories on the lattice

We specialize to the $U(1)$ model with Wilson action, but ideas are easily generalized to e.g. Villain action or Z_2 gauge model.
Action:

$$S = -\beta \sum_{x,\mu<\nu} Re\, U_\Box \,, \qquad U_\Box = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^{-1}(x+\hat{\nu}) U_\nu^{-1}(x)$$

- In 3D
  - Super-renormalizable
  - Duality transformation $\rightarrow$ monopole gas or Z-ferromagnet
  - Mass-gap $a\, m \sim \sqrt{8\pi^2\beta}\ \exp[-\pi^2\beta\ 0.2527]$ [M.Göpfert, G.Mack (1982)]
  - Confinement at every β: $a^2\sigma \geq \sqrt{\frac{1}{2\pi^2\beta}}\ \exp[-\pi^2\beta\ 0.2527]$
  - Continuum limit $m = \mathrm{const}$ exists (Free massive bosons)
- In 4D
  - QED without matter fields (renormalizable)
  - $\beta_c = 1.0111331(21)$ separates a confined phase from a Coulomb-like phase
  - Duality transformation $\rightarrow$ a gauge theory on the dual lattice

Strong coupling expansion

$$Z[j] = \int DU \left(\prod_{\Box} \exp[\beta \, Re \, U_{\Box}] \right) \prod_{x,\mu} U_{\mu}(x)^{j_{\mu}(x)}$$

- $DU = \prod_{x,\mu} dU_{\mu}(x)$, invariant measure on U(1)
- j, integer valued link field
- Correlators $\langle \mathcal{O}[j] \rangle = \frac{Z[j]}{Z[\emptyset]}$
- Non-zero expectation values only if (gauge symmetry)
  - $\partial^{*}_{\mu} j_{\mu}(x) = 0$
  - $\sum_x j_{\mu}(x) = 0$
  - e.g. sets of Wilson loops, pairs of Polyakov loops with opposite orientation

Taylor: $e^{Re\ U} = \sum_{n_1,n_2} \frac{(U/2)^{n_1} (U^*/2)^{n_2}}{n_1! n_2!}$

$$\begin{aligned} Z[j] &= \sum_{\{n_1\},\{n_2\}} \int DU \left[\prod_{\Box} \frac{\left(\frac{\beta}{2}\right)^{n_{1\Box}+n_{2\Box}} U_{\Box}^{n_{1\Box}-n_{2\Box}}}{n_{1\Box}! n_{2\Box}!} \right] \prod_{x,\mu} U_{\mu}(x)^{j_{\mu}(x)} \\ &= \sum_{\{n\}} \left[\prod_{\Box} I_{n_{\Box}}(\beta) \right] \delta(\partial^* n - j) \,, \qquad n_{\Box} = -\infty \ldots \infty \end{aligned}$$

constraint: $\delta(\partial^* n - j) \equiv \prod_{x,\mu} \delta_{\partial^*_{\nu} n_{\nu\mu}(x), j_{\mu}(x)}$

Surface ensemble

In analogy to worm algorithms we consider the ensemble with partition function

$$\mathcal{Z} = \sum_{\{j\}} \rho^{-1}[j]\, Z[j]$$

- Summation over all currents that satisfy the constraints
- ρ is some non-negative weight.
  - Choice of ρ defines the algorithm, important for τ_{int}
  - Cancels in observables of the original ensemble

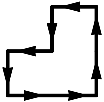
Observables in the strong-coupling ensemble

$$\langle\langle \mathcal{O}[n] \rangle\rangle = \frac{1}{\mathcal{Z}} \sum_{\{n\}} \mathcal{O}[n] \left[\prod_{\square} I_{n_\square}(\beta) \right] \rho^{-1}[\partial^* n]$$

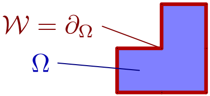
A particularly useful class of observables: 'vacuum expectation values'

$$\langle\langle \mathcal{O}[n] \rangle\rangle_0 = \frac{\langle\langle \mathcal{O}[n]\delta[\partial^* n] \rangle\rangle}{\langle\langle \delta[\partial^* n] \rangle\rangle}$$

Observables

- Wilson loops $\qquad \langle U[\mathcal{W}]\rangle = \Big\langle$  $\Big\rangle$

  Find some surface Ω that has the links of the loop as boundary



$$\mathcal{O}_\mathcal{W}[n] = \prod_{\square\in\Omega} \frac{I_{n_\square+1}(\beta)}{I_{n_\square}(\beta)}$$

$$\langle U[\mathcal{W}]\rangle = \langle\langle \mathcal{O}_\mathcal{W}[n]\rangle\rangle_0$$

- Other correlators, e.g. plaquette-plaquette, accordingly
- Alternative estimators (that collect information from non-vacuum configurations) also exist [under study!]
- Widely used techniques like APE-smearing are not obvious

The algorithm

- Guideline for first attempt: simplicity
- Choose ρ such that only connected, non-intersecting loops contribute to $\mathcal{Z}$
- Let ρ depend on simple characteristics of the loops, like e.g. the perimeter P of the loop, the number of corners etc.
- For instance: $\rho = e^{\theta^P (P-2)}$, (used from here on)
  - The “loop tension” θ_P is a parameter of the algorithm

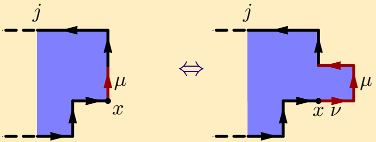
Markov process

- Propose local deformations of the defects $j_\mu(x)$.
- Accept-reject step for detailed balance

Proposed moves

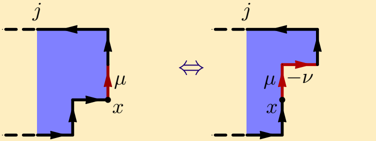
Shift

- choose link (x, μ) with $j_\mu(x) \neq 0$
- choose direction $\nu \perp \mu$
- if possible, propose move depicted on the right



Flip

- choose link (x, μ) with $j_\mu(x) \neq 0$
- choose direction $\nu \perp \mu$
- if possible, propose move depicted on the right



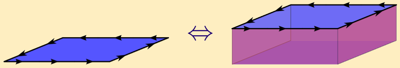
Kick

$P = 2$: self-shielding "trivial loop" $\Rightarrow$ vacuum configuration
If $P = 2$, move the trivial loop to a random link

Updates

Loop-shift

- If $D > 2$ and loop is planar (alternatively: has 4 corners), determine its plane μ, ν
- Choose direction $\rho \perp \mu, \rho \perp \nu$
- Propose to shift the whole loop in direction ρ

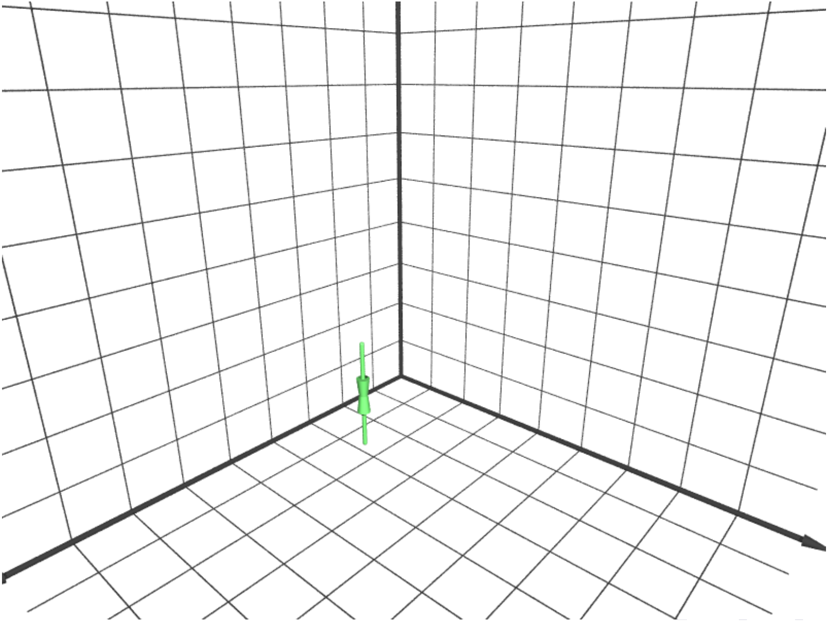


Plane update

- On a torus another update is necessary to ensure ergodicity
- Choose randomly a two dimensional plane Λ
- Choose randomly $s \in \{+1, -1\}$
- Proposal: $\forall \Box \in \Lambda : n_\Box \to n_\Box + s$

- Ergodicity realized already with shifts, flips (and plane updates)
- One iteration $=$ Volume $\times$ (shift, flip, loop-shift, kick), plane-update

Example (thermalization at $\beta = 1.77$)



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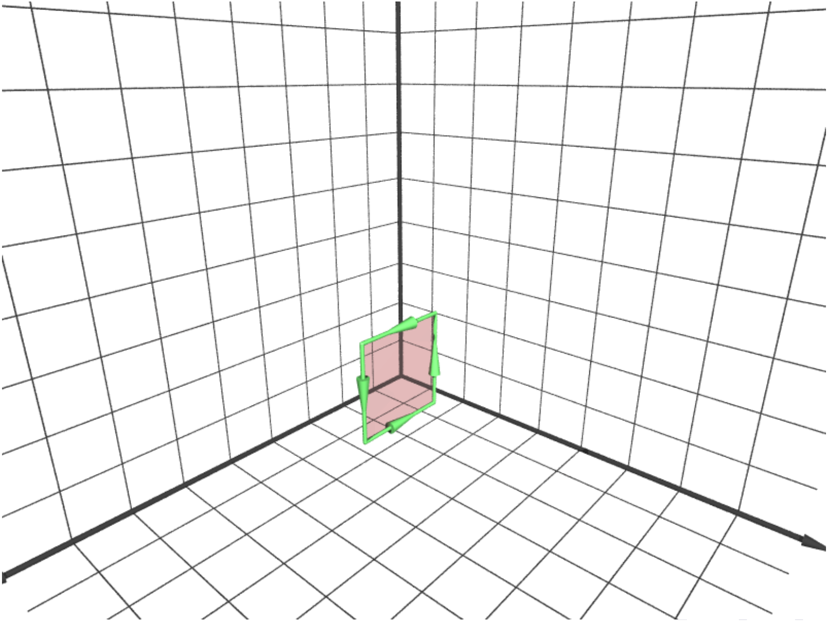

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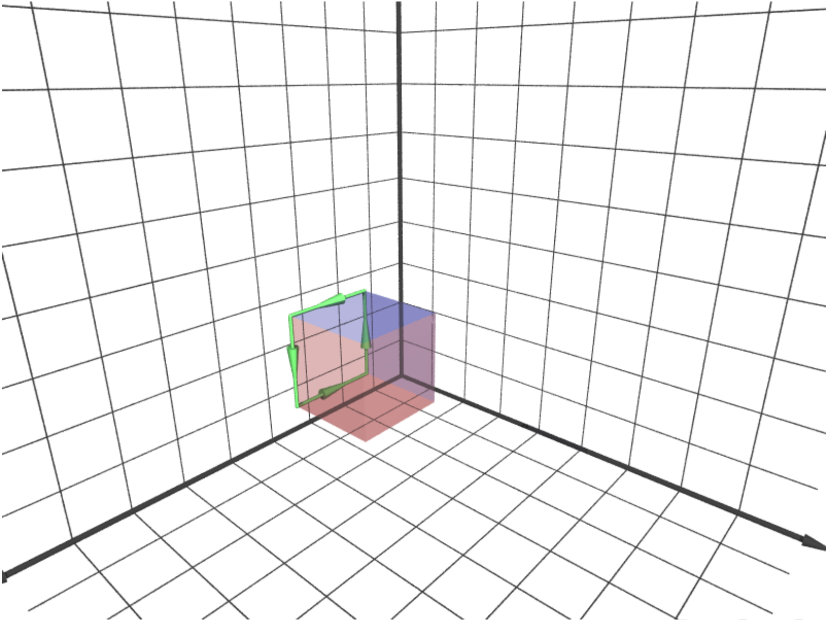

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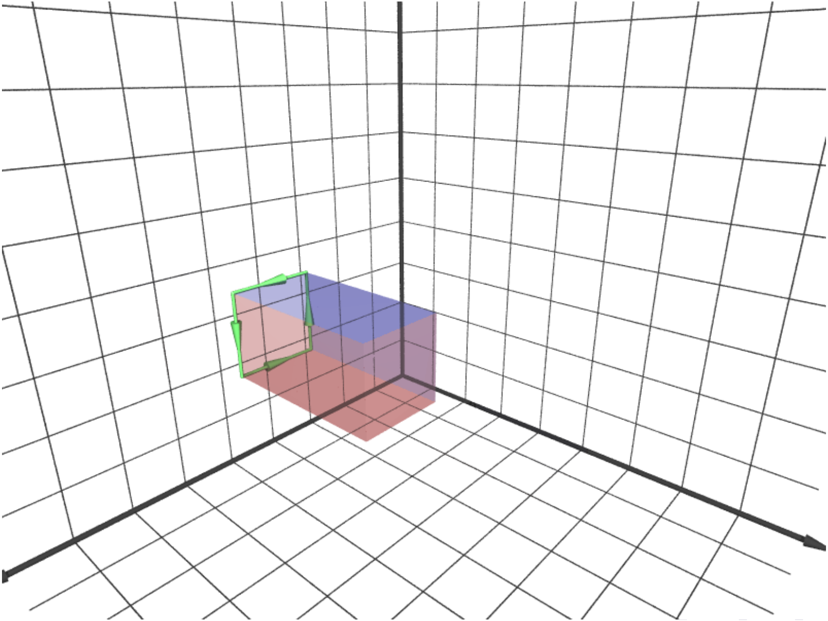

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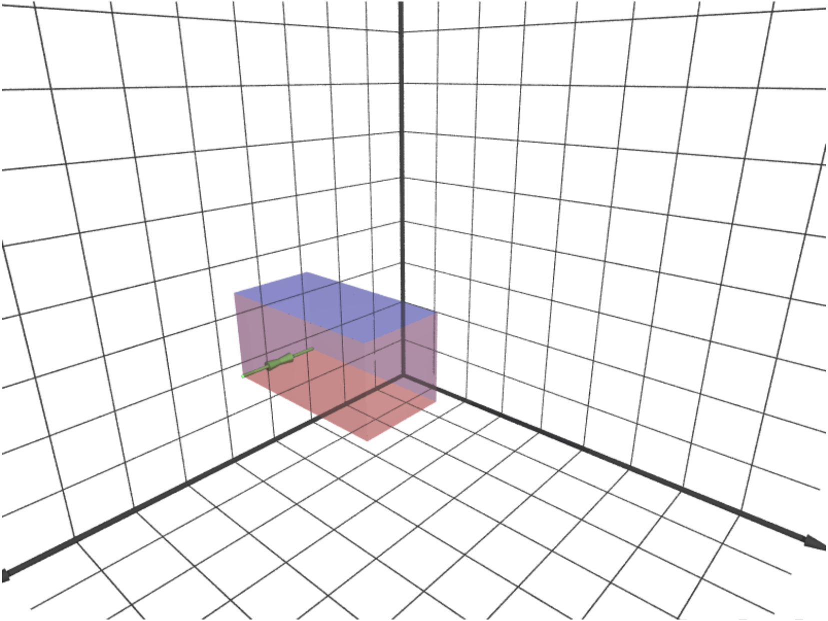

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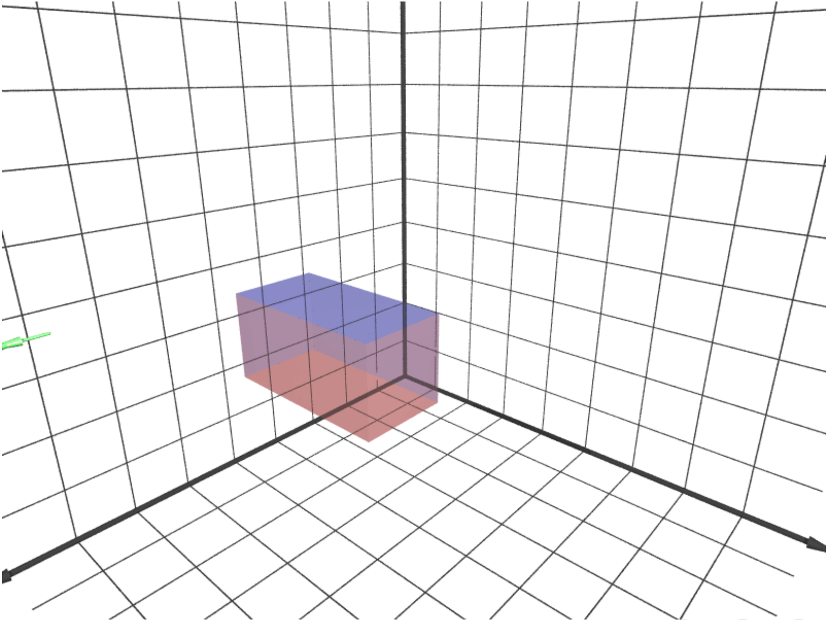

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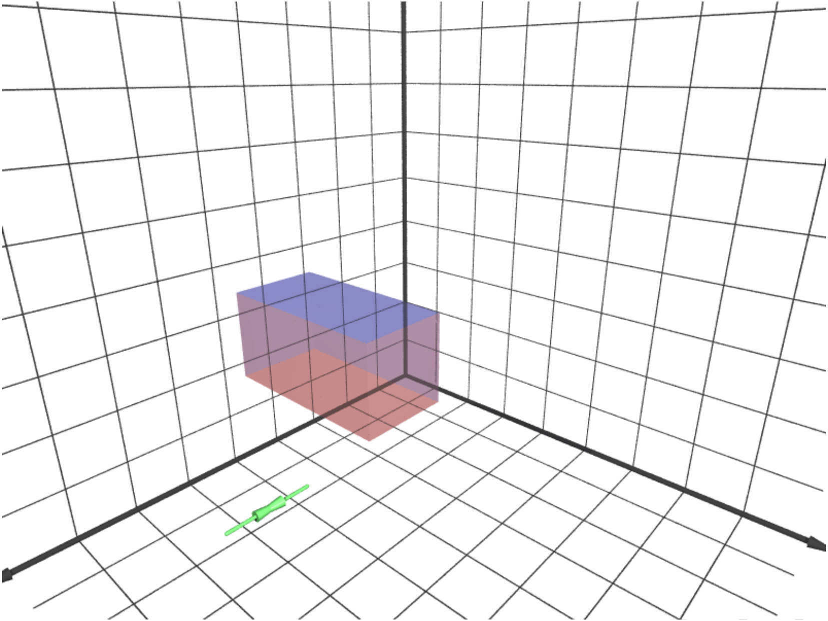

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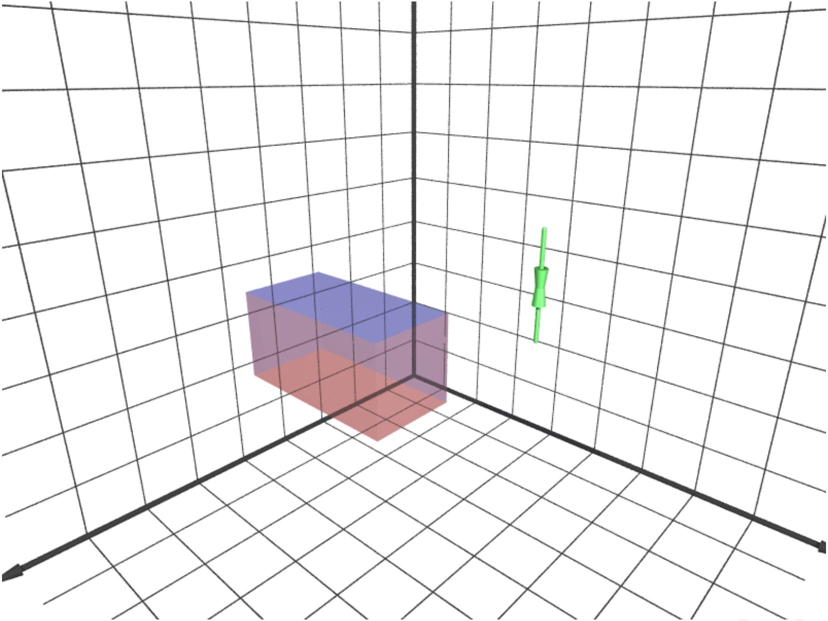

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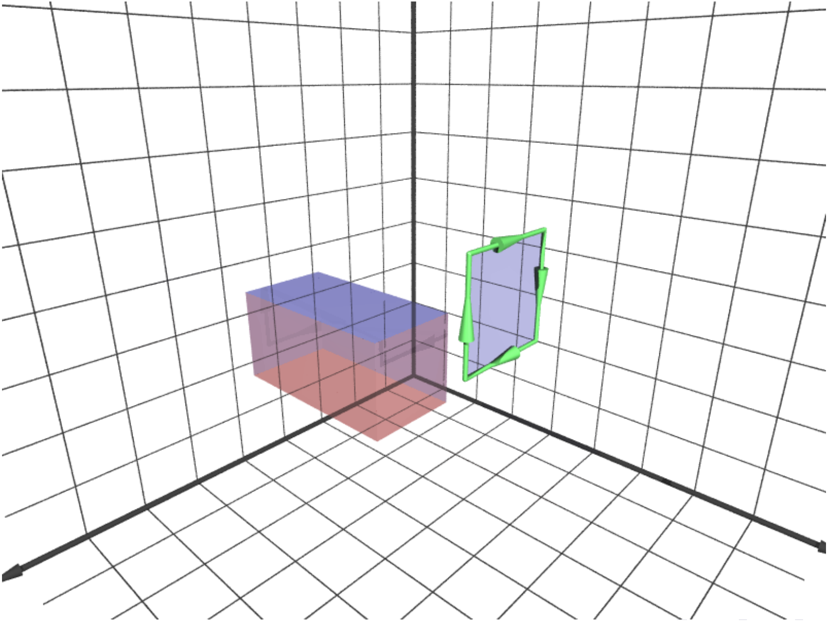

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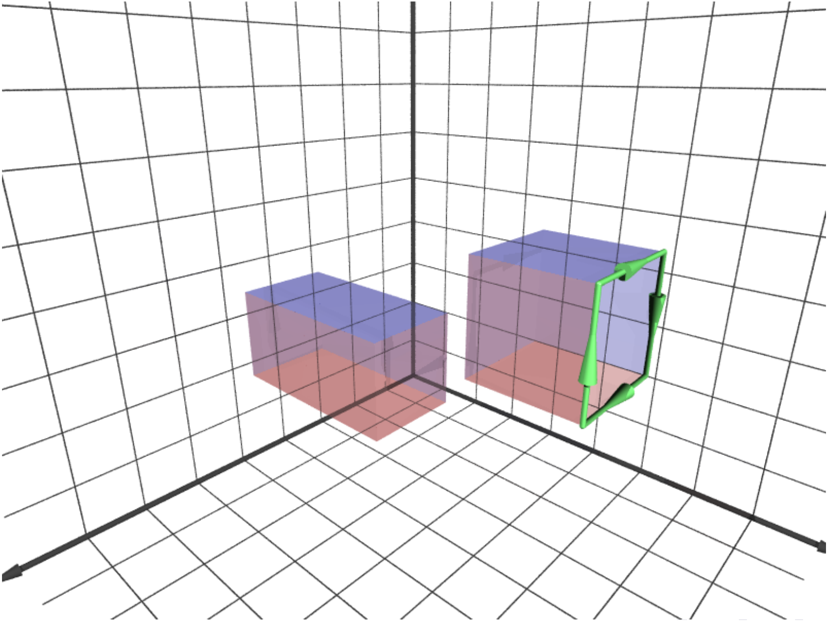

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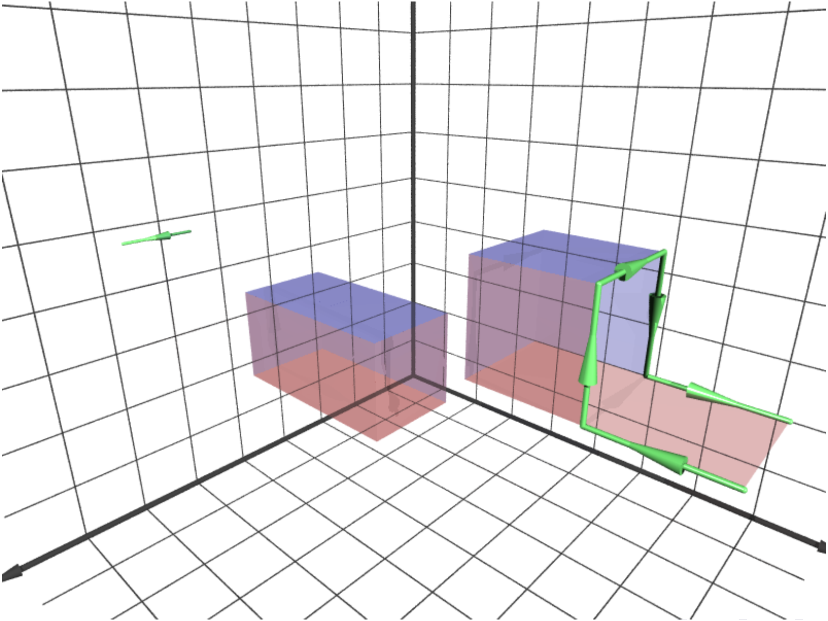

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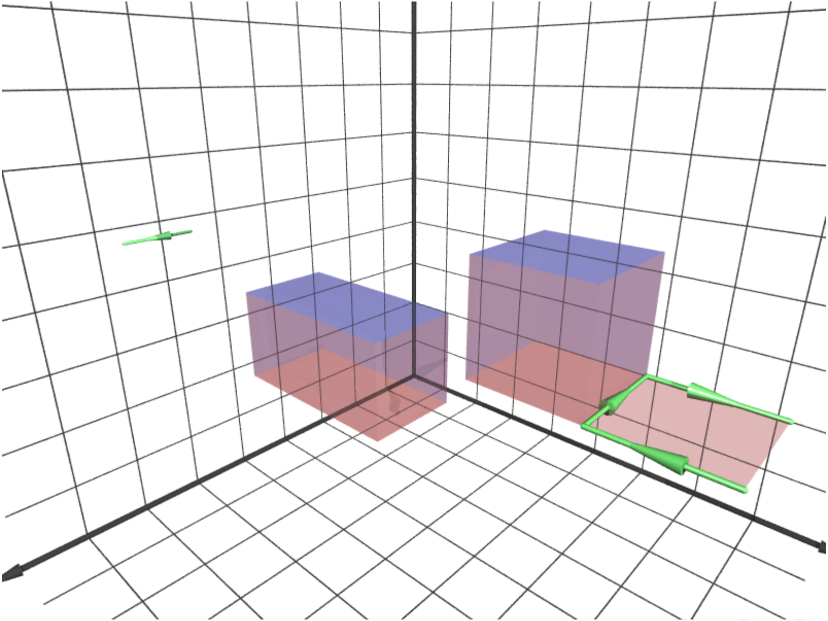

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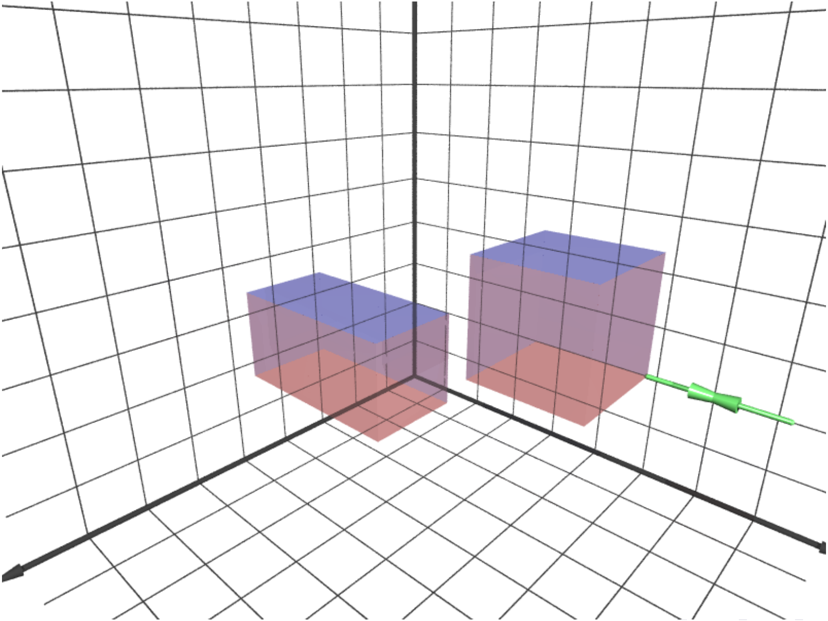

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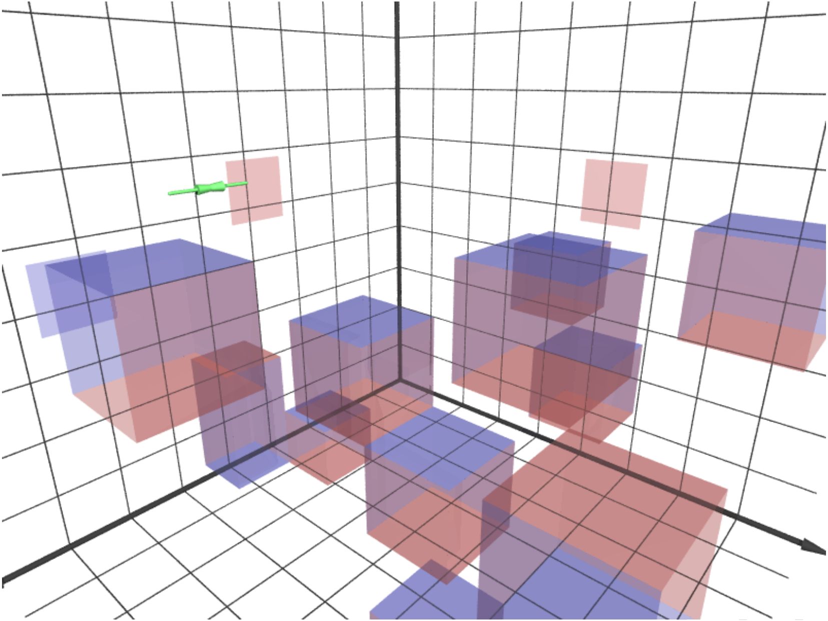

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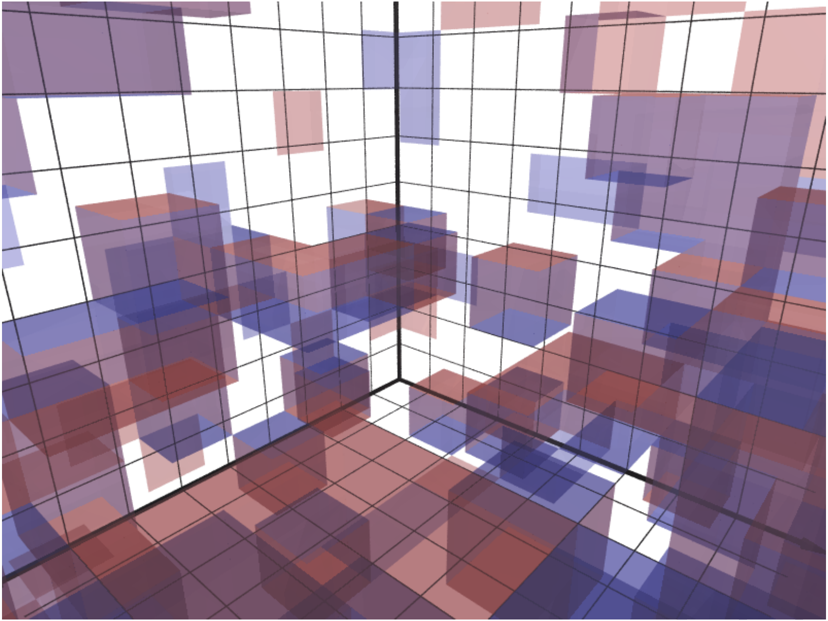
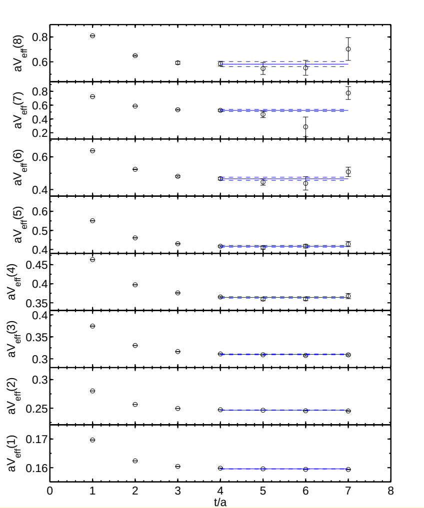


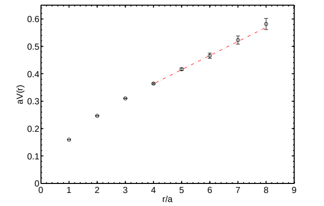
Derived observables

Example $D = 3$, $\beta = 2.08$, $L/a = 16$

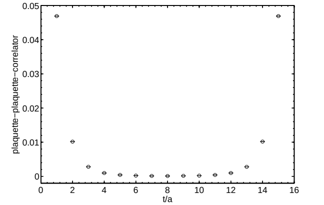
Wilson-loops $W(r,t) \sim \exp[-V(r)\,t]$



Static quark potential $V(r) \to c + \sigma r$



plaquette-plaquette correlator



Scaling of the algorithm

- Use glueball masses estimation from [M.Loan, M.Brunner, C.Sloggett, C.Hamer (2002)]

$$mL = c_1 \frac{L}{a} \sqrt{8\pi^2\beta} \exp[-\pi^2\beta\, 0.2527]$$

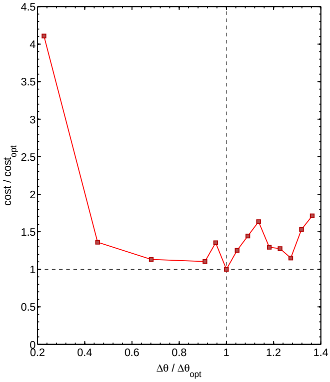
$c_1 = 5.23(11)$ with Wilson action

- Keep volume constant while increasing the resolution
  - $mL = 6$
  - $L/a \in \{8, 16, 24, 32, 40\}$
- Monitor autocorrelation times, variances and a cost indicator (CPU-time $\times$ error squared) of various observables
  - Average plaquette
  - Rectangular $L/4 \times L/4$ Wilson loop
  - Static quark potential at separation $L/4$
  - String tension
  - Effective mass from a plaquette-plaquette correlator at distance $L/4$
- Compare with a standard Metropolis algorithm

Tuning of the parameters

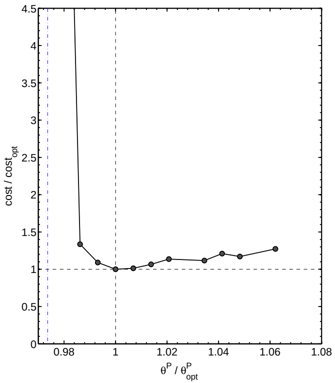
Metropolis algorithm

- Interval [-$\Delta\theta$, $\Delta\theta$] from which to draw random phase changes
- Sweeps / measurement

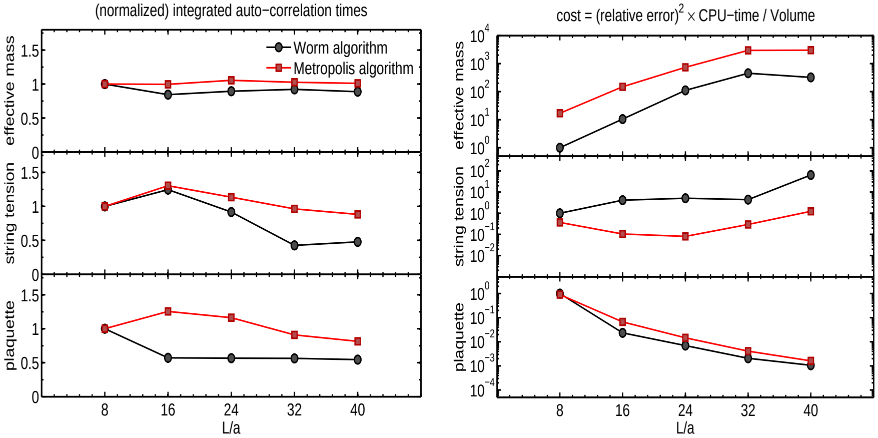


Worm algorithm

- Loop-tension θ^P
- Below a threshold (blue dashes), ergodicity is lost



Performance



- No critical slowing down observed over the investigated range of correlation lengths with neither algorithm
- The variance of some observables varies strongly with the lattice-volume

Summary and outlook

Summary

- The idea behind worm algorithms has been extended to gauge theories
- An update scheme for the all-order strong coupling expansion of $U(1)$ gauge theory has been presented
- All standard observables can be measured in the proposed ensemble

Outlook

- Further performance tests
- Refinements of the algorithm
- Improved estimators?
- Information in defect configurations should be exploited!
- Inclusion of matter fields
- Extension to non-Abelian gauge groups
- Physical applications