Chiral Symmetry of Graphene and Strong Coupling Lattice Gauge Theory

arXiv:1003.1769 [cond-mat.str-el]

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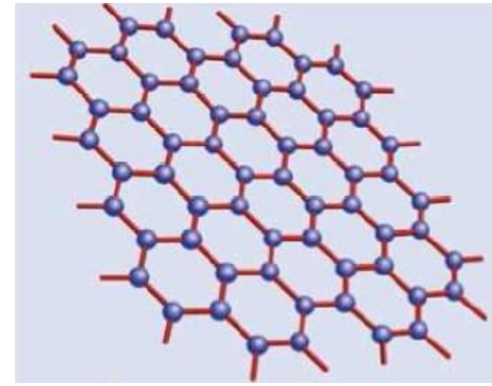
What is graphene?

- Graphene = Single atomic layer of carbon atoms



Honeycomb lattice structure

⇒ Building block of many kinds of carbon materials.

[Castro Neto et al., 2009]

- Electronic features of graphene:

Linear dispersion around two “Dirac points”. [Wallace, 1947]

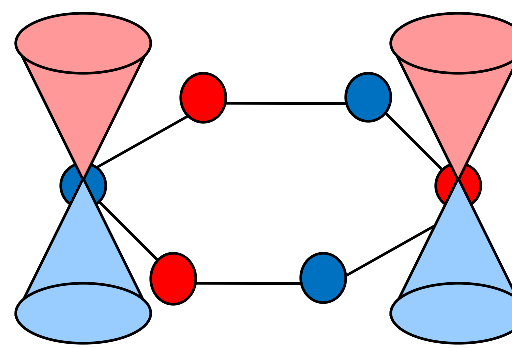
$$E(\mathbf{K}_\pm + \mathbf{p}) = \pm v_F |\mathbf{p}| + O((p/K)^2)$$ (gapless)

“Fermi velocity” ~c/300

⇓

Description as
Massless Dirac fermions

[Semenoff, 1984]



“Chiral symmetry” = sublattice symmetry

Effective gauge theory

- Euclidean action with Coulomb (U(1) gauge) interaction:
  ("braneworld", "reduced QED") [Gorbar *et al.*,2002]

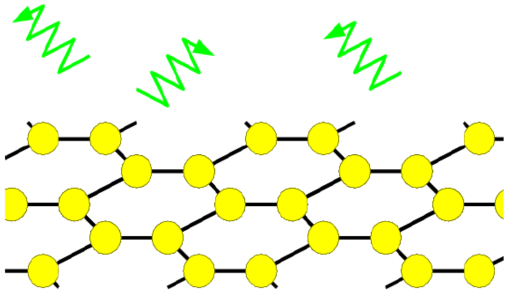
・Fermions in 3-dim.

Induced by lattice distortion

$$S_F = \sum_f \int dx^{(3)} \bar{\psi}_f \left[\gamma_4(\partial_4 + iA_4) + (\gamma_1\partial_1 + \gamma_2\partial_2) + m_* \right] \psi_f$$

・U(1) gauge field in 4-dim.

$$S_G = \frac{\beta}{2} \sum_{j=1,2,3} \int dx^{(4)} (\partial_j A_4)^2$$



- Small Fermi velocity ⇨ Effectively strong Coulomb coupling

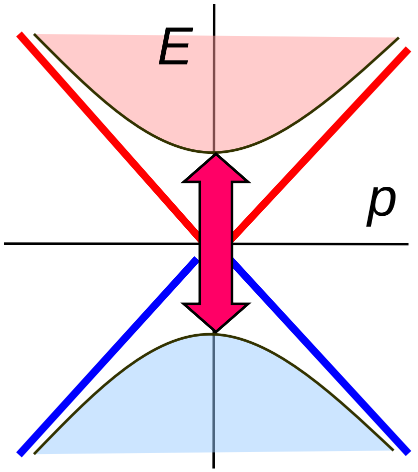
$$g_*^2 \equiv \frac{1}{\beta} = \frac{g_{\mathrm{QED}}^2}{v_F} (\sim 300 g_{\mathrm{QED}}^2)$$

(β~0.04)

⇨ Strong coupling expansion around β=0 will work well.

(If not screened by substrate)

Physics at strong coupling



Strong coupling ⇒ spontaneous chiSB

e.g.) [Drut & Lahde, 2008-2010]

<u>Graphene:</u> Dynamical gap generation

(Semimetal-insulator transition)

Similar mechanism ⇕

<u>QCD:</u> Dynamical quark mass

<u>This work:</u>

Strong coupling expansion of U(1) lattice gauge theory ("reduced QED")

⇒ analytic calculations of

- Fermion dynamical gap at/around

  $\beta=0$ (strong coupling), $m=0$ (chiral limit), $V=\infty$ (infinite volume)

- Collective excitations

  e.g.) (pseudo-)NG mode (~ pion)

Regularization on a square lattice

► Fermions:

[Hands & Strouthos, 2008]
[Drut & Lahde, 2008-2010]

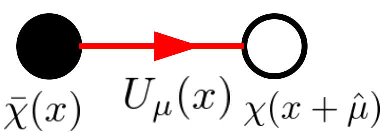
- Described by a single staggered fermion χ

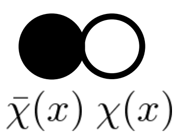
$$S_F = \sum_{x^{(3)}} \left[\frac{1}{2} \sum_{\mu=1,2,4} \left(V_\mu^+(x) - V_\mu^-(x) \right) + m_* M(x) \right]$$

hopping (kinetic) term

Bare (external) mass

(2+1) dim.





► Gauge field:

U(1) Link variables: $U_4(x) = e^{i\theta(x)}$ $(-\pi \leq \theta < \pi)$ $\vec{U}(x) = 1$ (instantaneous approx.)

- Two types of formulation:

Compact: $S_G^{\mathrm{C}} = \beta \sum_{x^{(4)}} \sum_{j=1,2,3} \left[1 - \mathrm{Re} \left(U_4(x) U_4^\dagger(x+\hat{j}) \right) \right]$

plaquette

Non-compact: $S_G^{\mathrm{NC}} = \frac{\beta}{2} \sum_{x^{(4)}} \sum_{j=1,2,3} \left[\theta(x) - \theta(x+\hat{4}) \right]^2$

⇨ difference …?

Strong coupling expansion

Expansion parameter: $\beta \equiv 1/g_*^2$ (inverse effective coupling)

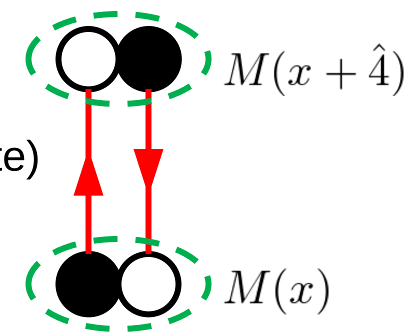
⇨ Link integration is performed order by order: $[S_G \sim O(\beta)]$

$$Z = \int [d\chi d\bar{\chi}][d\theta] \left[\sum_{n=0}^{\infty} \frac{(-S_G)^n}{n!} e^{-S_F} \right] = \int [d\chi d\bar{\chi}] e^{-S_\chi}$$

► 4-fermi couplings are induced by the link integration.

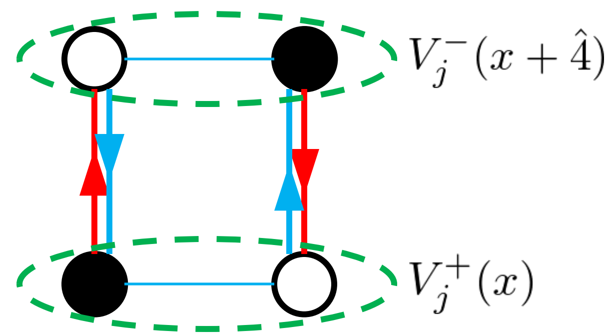
LO[O(1)]

(no plaquette)



NLO[O(β)]

(1 plaquette)



► Introduce auxiliary fields: $\phi(x) \equiv \phi_\sigma + i\epsilon(x)\phi_\pi$

(order parameter of chiSB)

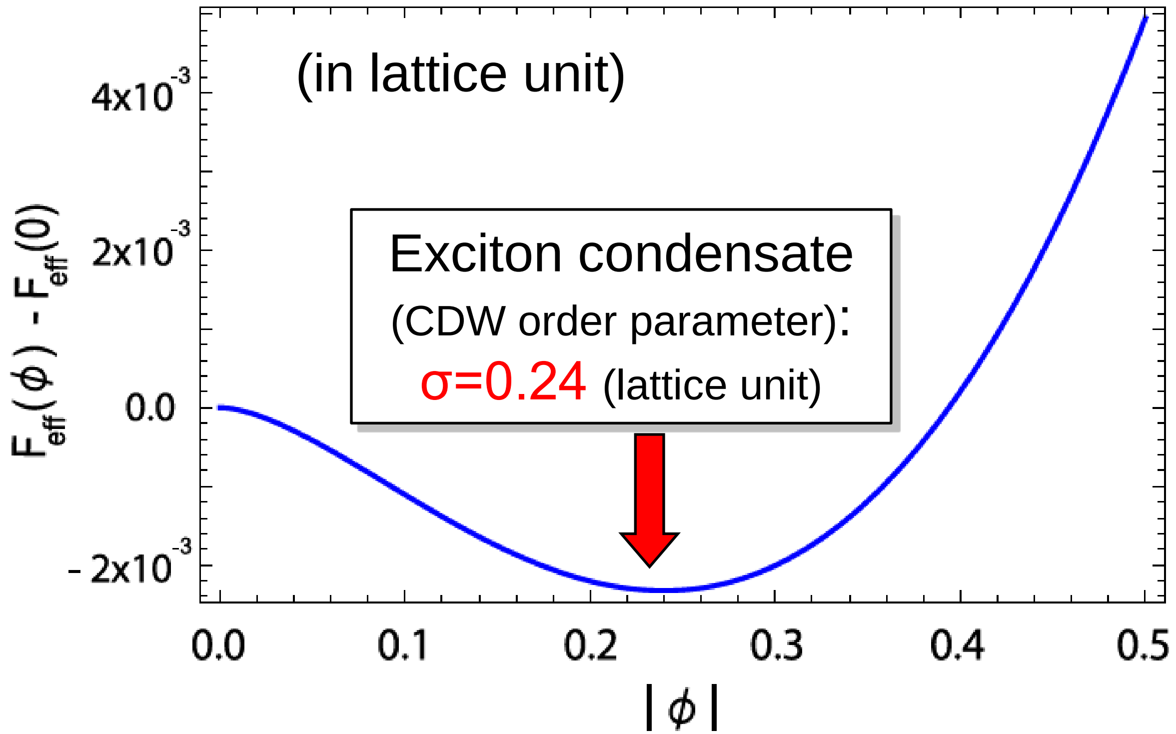
Scalar
$M(x) = \bar{\chi}(x)\chi(x)$

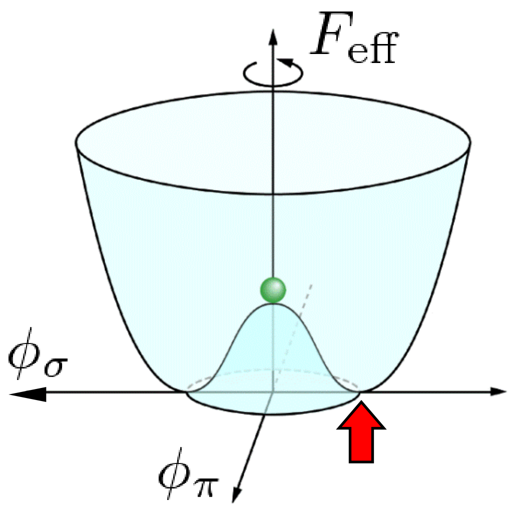
Pseudoscalar
$P(x) = \bar{\chi}(x) i\epsilon(x)\chi(x)$

Strong coupling limit (LO term)

► Strong coupling limit (β=0) & chiral limit (m=0)

▷ The gauge term does not contribute at β=0: $\sigma^{\mathrm{C}}(\beta=0)=\sigma^{\mathrm{NC}}(\beta=0)$



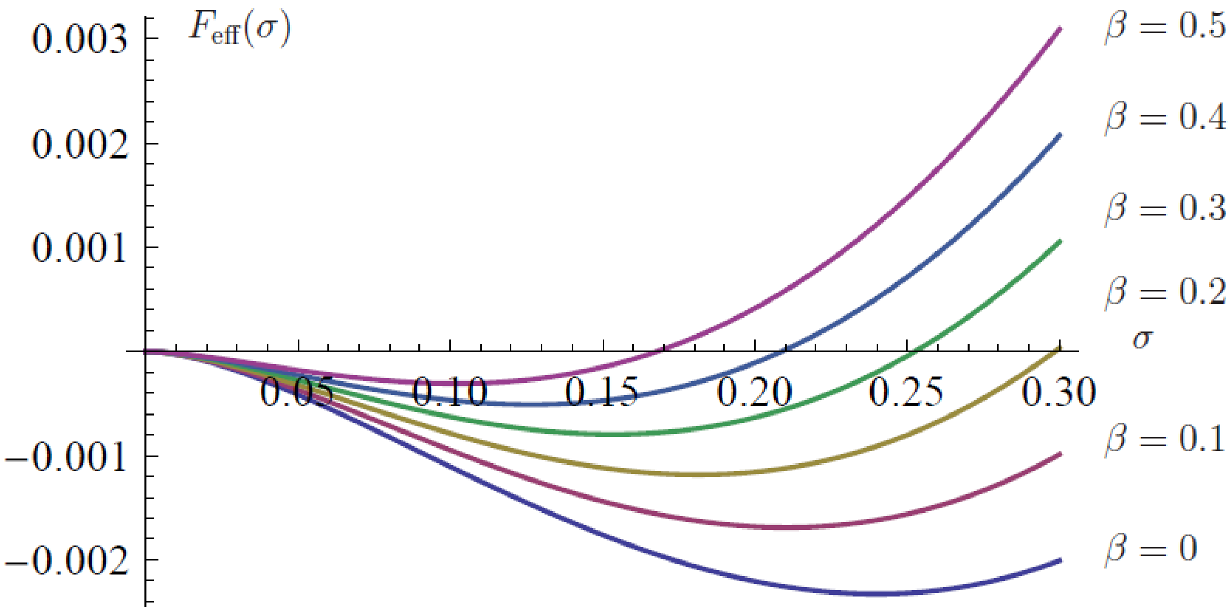


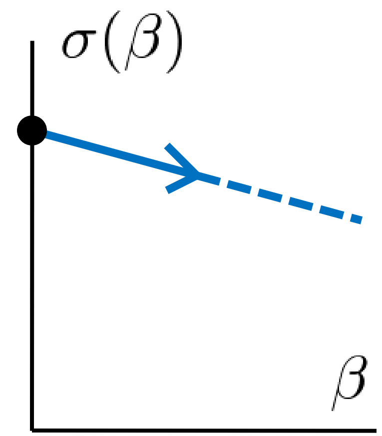
Dynamical mass gap:

$$M_F=\frac{\sigma}{2}\quad\left[\rightarrow\frac{v_F}{a}\frac{\sigma a^2}{2}\right]$$

►“Chiral symmetry” is spontaneously broken in the strong coupling limit.
(sublattice symmetry)

NLO effect: compact formulation





► Exciton condensate (in lattice unit):

$$|\langle\bar{\chi}\chi\rangle| = \sigma = 0.240 - 0.297\beta$$

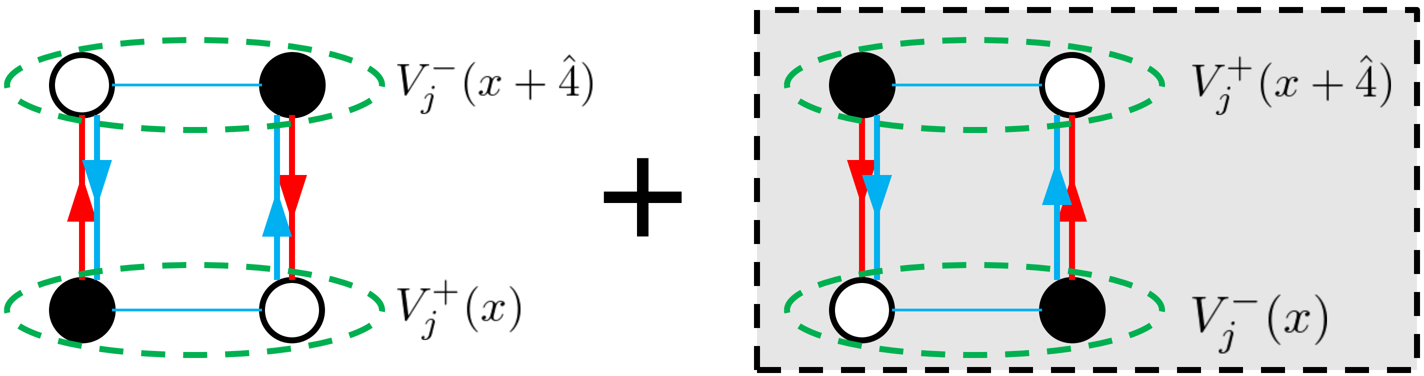
► By setting the lattice spacing $a \sim a_{\text{honeycomb}} = 1.42\text{Å}$

Dynamical gap $M_F \equiv \frac{v_F}{a}\frac{\sigma a^2}{2} \simeq (0.523 - 0.623\beta)\text{eV}$

NLO effect: non-compact form.

In the link integration,

One plaquette also works as its c.c.



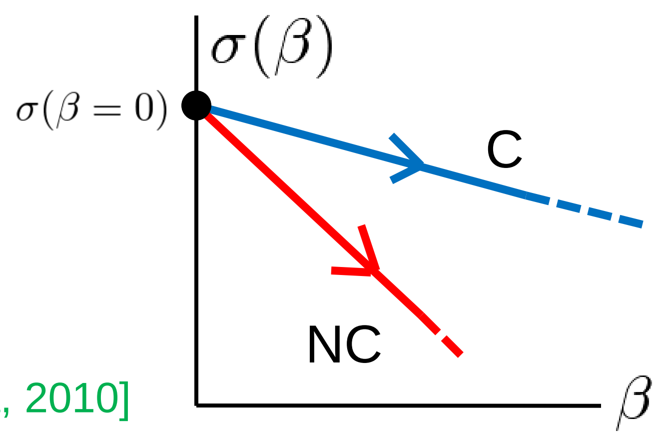
The NLO effect is doubled: $\Delta S_\chi^{NC} = 2 \times \Delta S_\chi^{C}$

⇨ $\sigma^{NC}(\beta)$ drops twice as fast as $\sigma^{C}(\beta)$.

$$\sigma^{NC}(\beta) < \sigma^{C}(\beta)$$

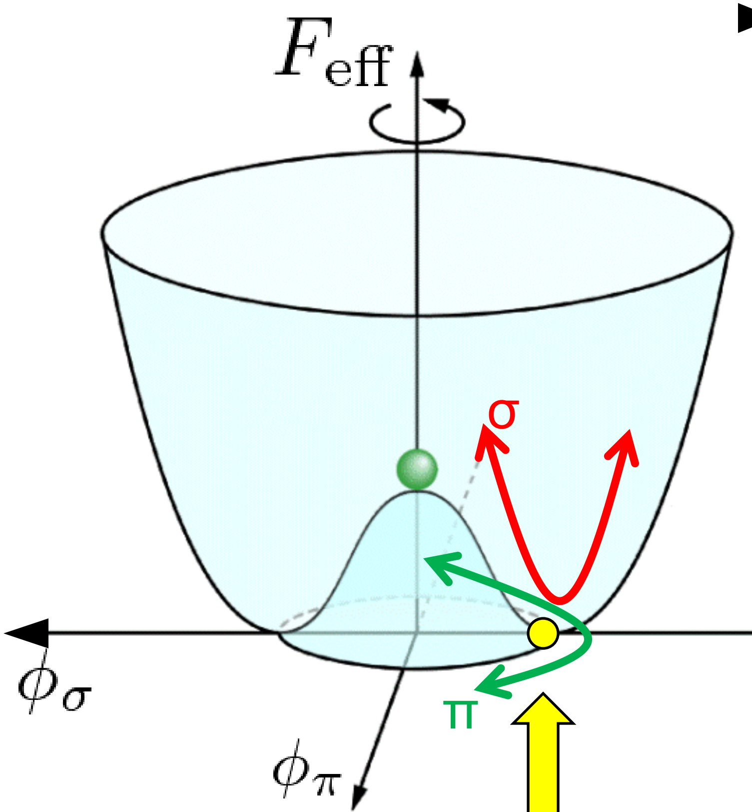
Also seen in lattice Monte Carlo results (extrapolated to β~0).

[Drut, Lahde & Suoranta, 2010]



Collective excitations (excitons)

► Two excitation modes: fluctuations of the order parameter φ



$$\langle \phi_\sigma \rangle = -\sigma$$

► π-mode (phase fluctuation mode)

$$M_\pi \simeq \frac{2v_F}{a}\sqrt{\frac{m_{\rm bare}}{M_F(m=0)}} \quad \xrightarrow{m=0} 0$$

$$\left(= 8.40\sqrt{\frac{m}{M_F(m=0)}}\mathrm{eV}\right)$$

⇨ "NG boson" from chiSB

Light mode (pion-like)

Similar to pions in QCD (GMOR relation):

$$M_\pi = \sqrt{m_{\rm bare}\sigma}/f_\pi$$

► σ-mode (amplitude fluctuation mode)

$$M_\sigma \simeq (5.47 - 1.97\beta)\mathrm{eV}$$

Quite a heavy mass (Higgs-like)

Conclusion

- ► Monolayer graphene ⇨ "reduced QED" model (gapless)
  ⇨ Lattice strong coupling expansion (analytic)
- ► Strong coupling: spontaneous chiSB
  ⇨ Dynamical spectral gap: $M_F = \frac{\sigma}{2} \quad \left[\to \frac{v_F}{a} \frac{\sigma a^2}{2} \right]$
- ► Compact vs. Non-compact formulation:
  ⇨ Difference starts from NLO: $\sigma^{NC}(\beta) < \sigma^{C}(\beta)$
- ► Collective excitation modes:
  π (phase fluctuation) ⇨ Pion-like behavior (GMOR rel.)

Future prospects:

- NNLO effect? (now being investigated)
- Finite T? (KT transition in (2+1)-D?)
- Taste breaking? (compare with overlap fermions, …)
- Anisotropy effects? (graphene under uniaxial strain)
- Gauge theory on the original honeycomb lattice? etc.

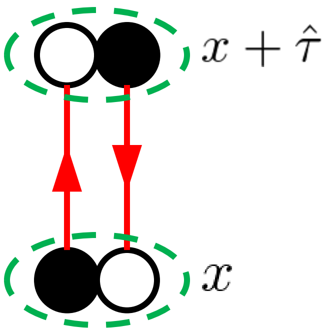
Thank you.

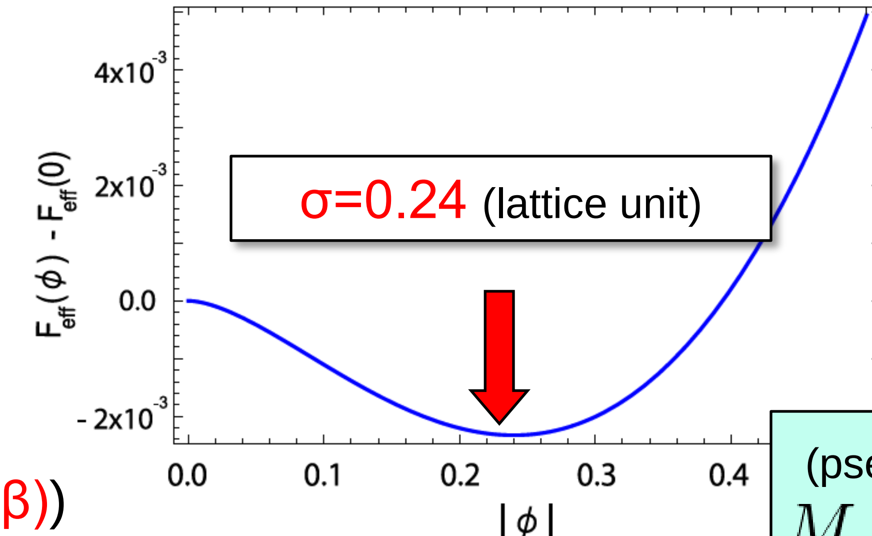
Results from Strong coupling expansion

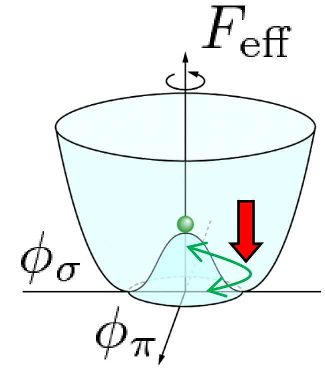
Y. Araki & T. Hatsuda, arXiv:1003.1769

► Strong coupling limit (β=0)

induced 4-fermi:



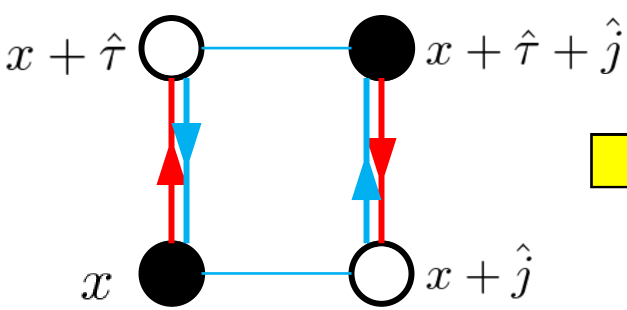


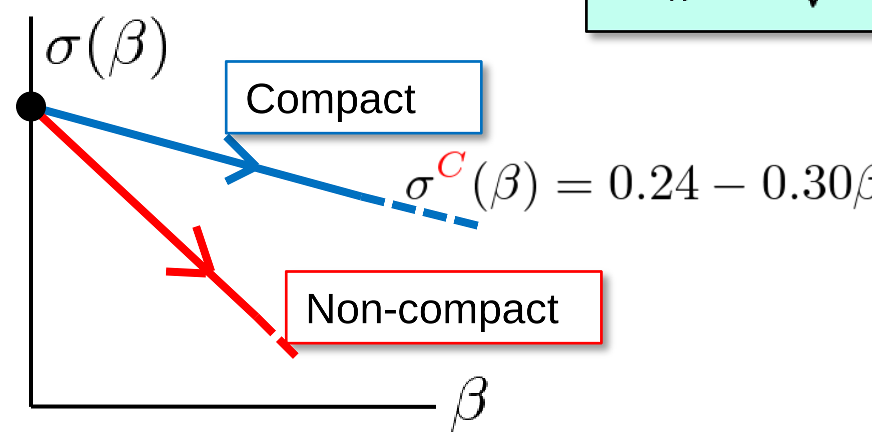


(pseudo-)NG mode

$$M_\pi \propto \sqrt{m_{\rm bare}}$$

► Next-leading order ($O(\beta)$)





NC drops twice faster than C at O(β).

Low-energy effective theory

Electrons/holes on graphene show linear dispersion around two "Dirac points":

$$E(\vec{K}_\pm + \vec{p}) = \pm v_F\,|\vec{p}| + O((p/K)^2)$$

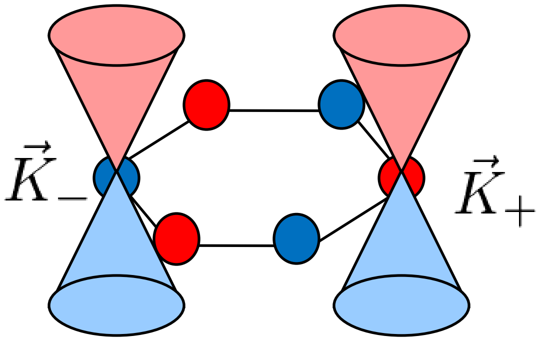
[Wallace, 1947]

"Fermi velocity"

$$v_F = (3/2)a_{\rm hc}t \sim c/300$$

a_{hc}=1.42 Å (interatomic spacing)

t = 2.8eV (hopping parameter)



Electrons/holes are described as massless Dirac fermions.

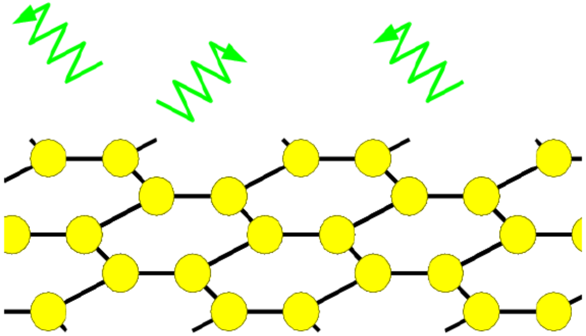
4	=	2	x	2
components		DPs		sublattices

$$\psi_\sigma(\vec{p}) = \begin{pmatrix} a_\sigma(\vec{K}_+ + \vec{p}) \\ b_\sigma(\vec{K}_+ + \vec{p}) \\ b_\sigma(\vec{K}_- + \vec{p}) \\ a_\sigma(\vec{K}_- + \vec{p}) \end{pmatrix}$$

(Spin up/down is considered as "flavor": N_f=2)

Mixed dimension model

Incorporate Coulomb interaction.



Low-energy effective action   [Son, 2007]

(2+1)-dim. fermionic field:

$$S_F = \sum_f \int dx^{(3)} \bar{\psi}_f \left[\gamma_4(\partial_4 + iA_4) + v_F(\gamma_1\partial_1 + \gamma_2\partial_2) + m\right]\psi_f$$

(3+1)-dim. U(1) gauge (electric) field:

$$S_G = \frac{1}{2g^2} \sum_{j=1,2,3} \int dx^{(4)} (\partial_j A_4)^2$$

$g^2 = \frac{e^2}{\epsilon_0}$ (for suspended graphene)

Assume "instantaneous" Coulomb interaction → Magnetic components (A_i) neglected

Effectively strong coupling

Scale transformation in the temporal direction:

$$\tau \rightarrow \tau / v_F \qquad A_4 \rightarrow v_F A_4$$

$$S_F = \sum_f \int dx^{(3)} \bar{\psi}_f \left[\gamma_4(\partial_4 + iA_4) + (\gamma_1 \partial_1 + \gamma_2 \partial_2) + m_* \right] \psi_f$$

$$S_G = \frac{1}{2{g_*}^2} \sum_{j=1,2,3} \int dx^{(4)} (\partial_j A_4)^2$$

$$m_* = m / v_F$$

Effectively strong Coulomb coupling.

$$g_*^2 = \frac{e^2}{v_F \epsilon_0} = \frac{g_{QED}^2}{v_F} (\sim 300 g_{QED}^2)$$

Perturbative approach cannot be applied.

Treated as strong coupling U(1) gauge theory.

Strong coupling nature

What will occur in the strong coupling regime?

Strong Coulomb coupling

⇩

Spontaneous chiSB ...?

e-h pair: exciton condensate $\langle\bar{\psi}\psi\rangle$

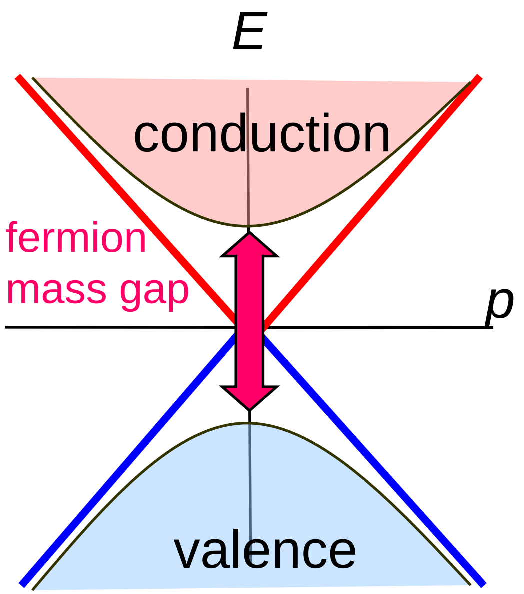
⇩

Dynamical mass gap ...?

⇩

Insulator ...?

?



Semimetal-insulator transition (in graphene)

⇕ *Similar mechanism*

Dynamical mass generation (of quarks [QCD])

Correspondence

Strong Coupling gauge theory (e.g. QCD)	*Graphene (low-energy)*
Gauge field A_μ^a	Electric field A_μ
Coupling constant g^2	Effective coupling g_*^2
Quarks $q(= u, d, \ldots)$	Quasi-electrons ψ
Chiral condensate $\langle \bar{q} q \rangle$	Exciton condensate $\langle \bar{\psi} \psi \rangle$
Flavors	Layers x Spin d.o.f.

Our approach

Lattice gauge theory

Strong coupling expansion

(Powerful approach in QCD)

→ Properties of graphene:

Dynamical mass gap

Collective excitations

Symmetries

Continuum theory is invariant under U(4) transf. generated by

$$\{1, \vec{\sigma}\} \otimes \{1, \gamma_3, \gamma_5, \gamma_3\gamma_5\}$$

Subgroups:

global gauge transf.

$$\mathrm{U}(1)_\mathrm{V}: \quad \psi \to e^{i\theta_V}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\theta_V}$$

chiral transf. (at chiral limit *m*=0)

$$\mathrm{U}(1)_\mathrm{A}: \quad \psi \to e^{i\gamma_5\theta_A}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\gamma_5\theta_A}$$

Lattice regularized theory is invariant only under

global gauge transf.

$$\mathrm{U}(1)_\mathrm{V}: \quad \chi(x) \to e^{i\theta_V}\chi(x), \quad \bar{\chi}(x) \to \bar{\chi}(x)e^{-i\theta_V}$$

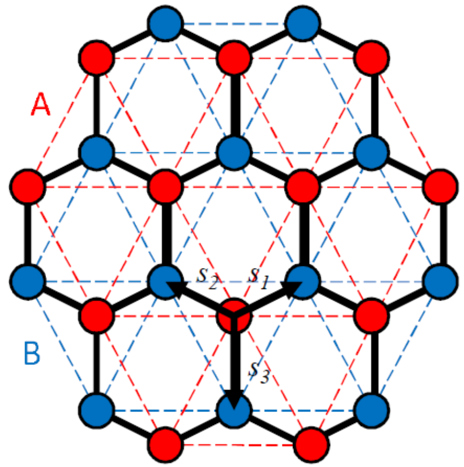
"chiral" transf. (at chiral limit *m*=0)

$$\mathrm{U}(1)_\mathrm{A}: \quad \chi(x) \to e^{i\epsilon(x)\theta_A}\chi(x), \quad \bar{\chi}(x) \to \bar{\chi}(x)e^{i\epsilon(x)\theta_A}$$

Staggered fermion preserves $U(1)_V x U(1)_A$ symmetry.

Electron spectrum

To describe the behavior of the electrons,
we introduce the tight-binding Hamiltonian.

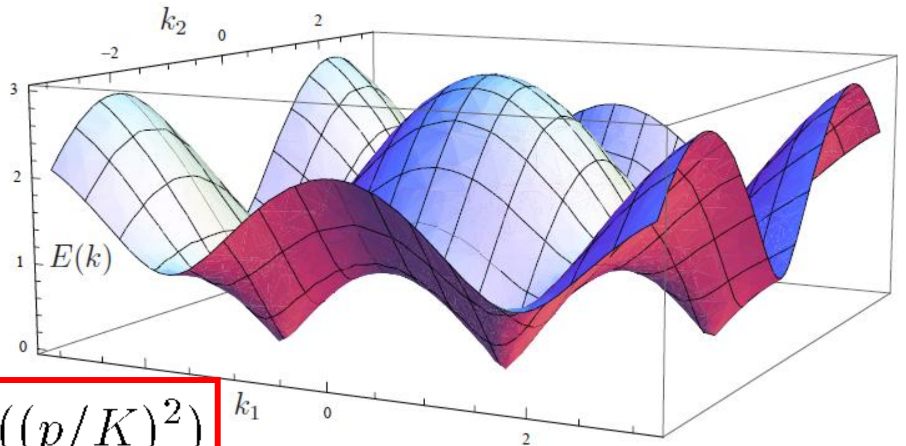


$$H_0 = -t \sum_{\vec{r} \in A} \sum_{i=1,2,3} \left[a^\dagger(\vec{r}) b(\vec{r} + \vec{s_i}) + b^\dagger(\vec{r} + \vec{s_i}) a(\vec{r}) \right]$$

t ~ 2.8eV a~1.42 Å (interatomic spacing)

The energy eigenvalue vanishes at two Fermi (Dirac) points.

$$\vec{K}_\pm = \left(\pm \frac{4\pi}{3\sqrt{3}a}, 0 \right)$$



"Fermi velocity" =(3/2)at ~ 0.00302c

$$E(\vec{K}_\pm + \vec{p}) = \pm v |\vec{p}| + O((p/K)^2)$$

The dispersion relation is linearized around the Dirac points.

Low-energy effective theory

Expanding the operators around the Dirac points,

$$\psi(\vec{p}) \equiv \begin{pmatrix} a(\vec{K}_+ + \vec{p}) \\ b(\vec{K}_+ + \vec{p}) \\ b(\vec{K}_- + \vec{p}) \\ a(\vec{K}_- + \vec{p}) \end{pmatrix}$$

the Hamiltonian is given as

$$H_0 = v \sum_{\vec{p}} \psi^\dagger(\vec{p}) \left(\gamma_0 \vec{\gamma} \cdot \vec{p}\right) \psi(\vec{p})$$

Lagrangian form:

$$L_0 = i \int d^2 r \bar{\psi}(\vec{r}) \left[\gamma^0 \partial_t - v \vec{\gamma} \cdot \vec{\partial}\right] \psi(\vec{r})$$

Low-energy electrons on graphene can be described as (2+1)-D massless Dirac fermions. (except Fermi velocity)

Dirac points → *"pseudospin"*

Spin d.o.f. → *"flavors"*

Strong coupling nature

$$S = i\int dtd^2x\left[\bar{\psi}\gamma^0(\partial_0 - iA_0)\psi - v\bar{\psi}\vec{\gamma}\cdot\vec{\partial}\psi\right] + \frac{1}{2g^2}\int dtd^3x(\partial_i A_0)^2$$

Scale transformation

$$t \to t/v \quad A_0 \to vA_0$$

$$g^2 = e^2/\epsilon_0$$

$$S = i\int dtd^2x\left[\bar{\psi}\gamma^0(\partial_0 - iA_0)\psi - \bar{\psi}\vec{\gamma}\cdot\vec{\partial}\psi\right] + \frac{v}{2g^2}\int dtd^3x(\partial_i A_0)^2$$

Coupling strength ("fine structure constant") becomes

$$\alpha_g = \frac{e^2}{4\pi v\epsilon_0} \sim 300\alpha \sim 2$$

Perturbative approach cannot be applied.

Treated as strong coupling U(1) gauge theory.

Correspondence

SC gauge theory (e.g. QCD)	*Graphene (low-energy)*
Gauge field A_μ^a	Electromagnetic field A_μ
Coupling constant g^2	$e^2/v\epsilon$
Fermions	Electrons
Chiral condensate $\langle \bar{q}q \rangle$	Exciton cond. $\langle \bar{\psi}\psi \rangle$
Flavors N_f	Layers x Spin d.o.f.

Our approach

Lattice gauge theory

Strong coupling expansion

→ Properties of graphene:

Dynamical mass gap

Collective excitations

(Powerful approach in QCD)

[Nishida, Fukushima & Hatsuda, 2004]

[Miura, Nakano & Ohnishi, 2009]

Lattice regularization

Euclidean Low-energy effective theory [Son,2007]

$$S_F = \int dt d^2x \bar{\psi}_a \left[\gamma_0(\partial_0 + iA_0) + \vec{\gamma}\cdot\vec{\partial} + m_0\right]\psi_a$$

$$S_G = \frac{\beta}{2}\int dt d^3x (\partial_i A_0)^2$$

UV cutoff: $\Lambda = \pi/a \sim 4.36\mathrm{keV}$

Discretize on the square lattice. (with staggered fermions)

$$S_F = -\frac{1}{2}\sum_{x,t}\eta_0(x,t)\left[\bar{\chi}_a(x,t)e^{i\theta(x,t)}\chi_a(x,t+1) - \bar{\chi}_a(x,t+1)e^{-i\theta(x,t)}\chi_a(x,t)\right]$$
$$-\frac{1}{2}\sum_{x,t}\sum_{j=1,2}\eta_j(x,t)\left[\bar{\chi}_a(x,t)\chi_a(x+\hat{j},t) - \bar{\chi}_a(x+\hat{j},t)\chi_a(x,t)\right]$$
$$-m_0\sum_{x,t}\bar{\chi}_a(x,t)\chi_a(x,t)$$

a=1,…*N*: layer index

$$S_G = \beta\sum_{x,t}\sum_{j=1,2,3}\left[1 - \frac{1}{2}\left(e^{i\theta(x,t)}e^{-i\theta(x+\hat{j},t)} + e^{-i\theta(x,t)}e^{i\theta(x+\hat{j},t)}\right)\right]$$

Link variable (compact form): $e^{i\theta} \sim 1 + iA_0$

Formulation of fermions

Tight-binding Hamiltonian:

Creation/annihilation operators $a_\sigma(\vec{r}), \quad b_\sigma(\vec{r})$

$\sigma = \pm$: original spin d.o.f.

Low energy effective theory:

Dirac spinor form

$$\psi_\sigma(\vec{p}) = \begin{pmatrix} a_\sigma(\vec{K}_+ + \vec{p}) \\ b_\sigma(\vec{K}_+ + \vec{p}) \\ b_\sigma(\vec{K}_- + \vec{p}) \\ a_\sigma(\vec{K}_- + \vec{p}) \end{pmatrix}$$

Dirac point d.o.f.

= "*pseudospin*"

Square lattice regularization:

Staggered fermion χ

2^3 doublers = 4 (spinor d.o.f.) x 2 (spin d.o.f.)

Monolayer graphene is described by a single staggered fermion.

Leading order: O(1)

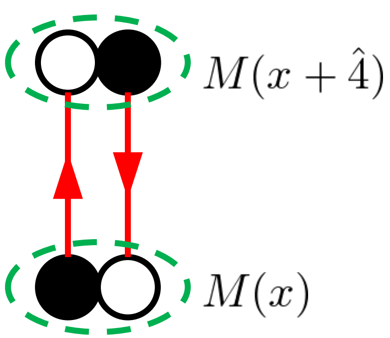
e.g.) Integration over $\theta(x)$

$$\int_{-\pi}^{\pi}\frac{d\theta(x)}{2\pi}\exp\left[-\frac{1}{2}\left(V_4^+(x)-V_4^-(x)\right)\right]$$

$$=\int_{-\pi}^{\pi}\frac{d\theta(x)}{2\pi}\left[1-\frac{1}{2}\bar{\chi}(x)U_4(x)\chi(x+\hat{4})\right]\left[1+\frac{1}{2}\bar{\chi}(x+\hat{4})U_4^\dagger(x)\chi(x)\right]$$

$$=1-\frac{1}{4}\bar{\chi}(x)\chi(x+\hat{4})\bar{\chi}(x+\hat{4})\chi(x)\;=\exp\left[\frac{1}{4}M(x)M(x+\hat{4})\right]$$

$$S_\chi^0=-\frac{1}{4}\sum_{x^{(3)}}M(x)M(x+\hat{4})+\sum_{x^{(3)}}\left[\frac{1}{2}\sum_{j=1,2}\left(V_j^+(x)-V_j^-(x)\right)+m_*M(x)\right]$$



4-fermi term:

Local in spatial directions

Non-local in temporal direction

Next-leading order: O(β)

e.g.) Integration over $\theta(x)$ and $\theta(x+\hat{j})$

from plaquette

$$\int_{-\pi}^{\pi}\frac{d\theta(x)}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta(x+\hat{j})}{2\pi}\exp\left[-\frac{1}{2}\left(V_4^+(x)-V_4^-(x)+V_4^+(x+\hat{j})-V_4^-(x+\hat{j})\right)\right]U_4^\dagger(x)U_4(x+\hat{j})$$

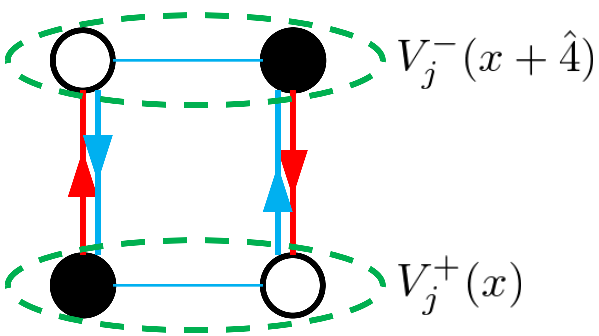
$$=\int_{-\pi}^{\pi}\frac{d\theta(x)}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta(x+\hat{j})}{2\pi}U_4^\dagger(x)U_4(x+\hat{j})\times\left[1-\frac{1}{2}\bar{\chi}(x)U_4(x)\chi(x+\hat{4})\right]\left[1+\frac{1}{2}\bar{\chi}(x+\hat{4})U_4^\dagger(x)\chi(x)\right]$$

$$\times\left[1-\frac{1}{2}\bar{\chi}(x+\hat{j})U_4(x+\hat{j})\chi(x+\hat{j}+\hat{4})\right]\left[1+\frac{1}{2}\bar{\chi}(x+\hat{j}+\hat{4})U_4^\dagger(x+\hat{j})\chi(x+\hat{j})\right]$$

$$=-\frac{1}{4}\bar{\chi}(x)\chi(x+\hat{4})\bar{\chi}(x+\hat{j}+\hat{4})\chi(x+\hat{j}) = \frac{1}{4}V_j^+(x)V_j^-(x+\hat{4})$$

$$\Delta S_\chi = \frac{\beta}{8}\sum_{x^{(3)}}\sum_{j=1,2}\left[V_j^+(x)V_j^-(x+\hat{4})+V_j^-(x)V_j^+(x+\hat{4})\right]$$

j=3 does not contribute.



4-fermi term:

Non-local both in spatial and temporal directions

Mean-field approximation

Mean-field approximation:

$$\phi(x) \equiv \phi_\sigma + i\epsilon(x)\phi_\pi$$

Scalar

$$M(x) = \bar{\chi}(x)\chi(x)$$

Pseudoscalar

$$P(x) = \bar{\chi}(x)i\epsilon(x)\chi(x)$$

Integrate out fermion bilinear.

Effective potential:

$$F_{\text{eff}}(\phi,\lambda) = -\frac{1}{N_s^2 N_\tau}\ln Z(\phi,\lambda)$$

$$= \frac{1}{4}|\phi|^2 + \frac{\beta}{2}|\lambda|^2 - \frac{1}{2}\int_{\vec{k}} \ln\left[\left|\frac{\phi}{2} - m_*\right|^2 + \left(1 + \frac{\beta\lambda_2}{2}\right)^2 \sum_{j=1,2} \sin^2\left(k_j - \frac{\beta\lambda_1}{2}\right)\right]$$

Stationary condition

By solving the stationary condition for λ , it is rewritten as a function of φ :

$$\frac{\partial F_{\text{eff}}(\phi,\lambda)}{\partial \lambda_1} = 0 \;\Rightarrow\; \lambda_1 = 0 + O(\beta)$$

$$\frac{\partial F_{\text{eff}}(\phi,\lambda)}{\partial \lambda_2} = 0 \;\Rightarrow\; \lambda_2 = \int_{\vec{k}} \frac{\sin^2 k_i}{|\phi/2 - m_*|^2 + \sum_j \sin^2 k_j} + O(\beta)$$

$$\Downarrow$$

$$G^{-1}(\vec{k};\phi) \equiv \sum_{j=1,2} \sin^2 k_j + \underbrace{\left| m_* - \frac{\phi}{2} \right|^2}_{\text{"effective mass"}}$$

Effective potential:

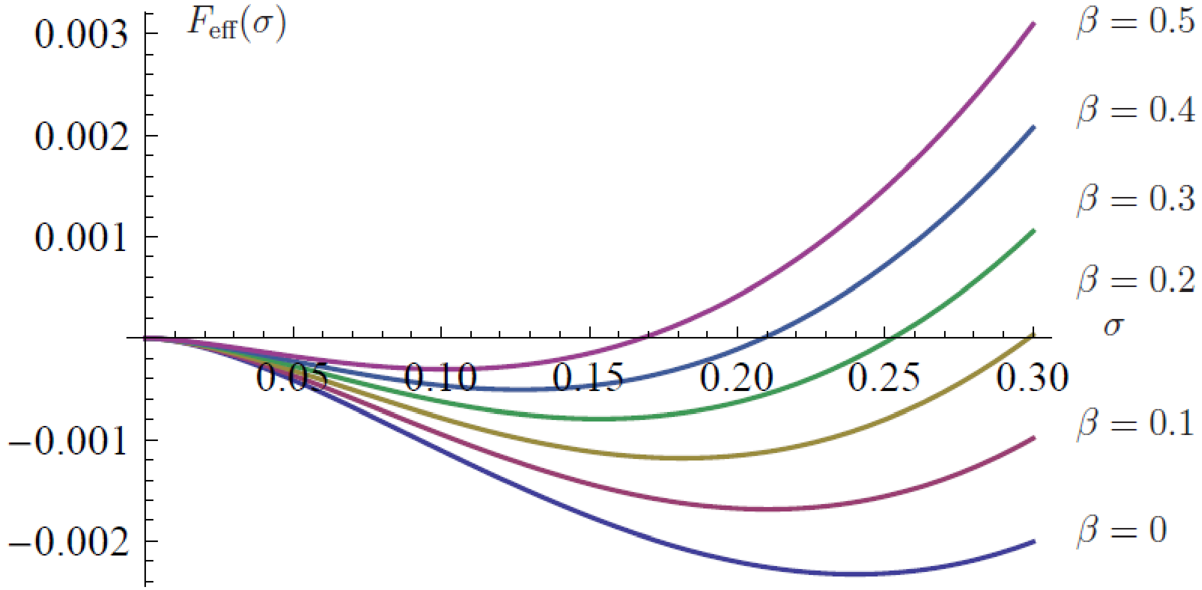
$$F_{\text{eff}}(\phi) = -\frac{1}{N_s^2 N_\tau} \ln Z(\phi)$$

$$= \frac{1}{4}|\phi|^2 - \frac{1}{2}\int_{\vec{k}} \ln\left[G^{-1}(\vec{k};\phi)\right] - \frac{\beta}{4}\sum_{j=1,2}\left[\int_{\vec{k}} G(\vec{k};\phi)\sin^2 k_j\right] + O(\beta^2)$$

Effective potential with varying β

At the chiral limit, the effective potential is

$$F_{\text{eff}} = \frac{\sigma^2}{4} - \frac{2}{L^2}\sum_k^{\pi} \ln\left[\frac{\sigma^2}{4} + \sum_j \sin^2 k_j\right] - \frac{\beta}{2}\left[\frac{4}{L^2}\sum_k^{\pi} \frac{\sin^2 k_1}{(\sigma/2)^2 + \sum_j \sin^2 k_j}\right]^2$$



Solving the stationary condition analytically, we obtain

$$\sigma(\beta) = 0.24 - 0.30\beta + O(\beta^2)$$

(β<<1)

Non-compact link integration

e.g.) Integration over $\theta(x)$ and $\theta(x+\hat{j})$

From non-compact plaquette

$$\int_{-\pi}^{\pi}\frac{d\theta(x)}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta(x+\hat{j})}{2\pi}\exp\left[-\frac{1}{2}\left(V_4^+(x)-V_4^-(x)+V_4^+(x+\hat{j})-V_4^-(x+\hat{j})\right)\right]\theta(x)\theta(x+\hat{j})$$

$$=\int_{-\pi}^{\pi}\frac{d\theta(x)}{2\pi}\theta(x)\left[1-\frac{1}{2}\bar{\chi}(x)e^{i\theta(x)}\chi(x+\hat{4})+\frac{1}{2}\bar{\chi}(x+\hat{4})e^{-i\theta(x)}\chi(x)+\frac{1}{4}M(x)M(x+\hat{4})\right]$$

$$\times\int_{-\pi}^{\pi}\frac{d\theta(x+\hat{j})}{2\pi}\cdots$$

$$\left(\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\theta e^{\pm i\theta}=\pm i\right)$$

$$=\frac{1}{4}\left[V_j^+(x)V_j^-(x+\hat{4})+V_j^-(x)V_j^+(x+\hat{4})\right]+(\text{irrelevant terms})$$

One plaquette also works as its c.c.

⇨ The NLO effect is doubled: $\Delta S_\chi^{NC}=2\times\Delta S_\chi^{C}$

With non-compact formulation, σ(β) drops twice as fast as that with compact formulation.