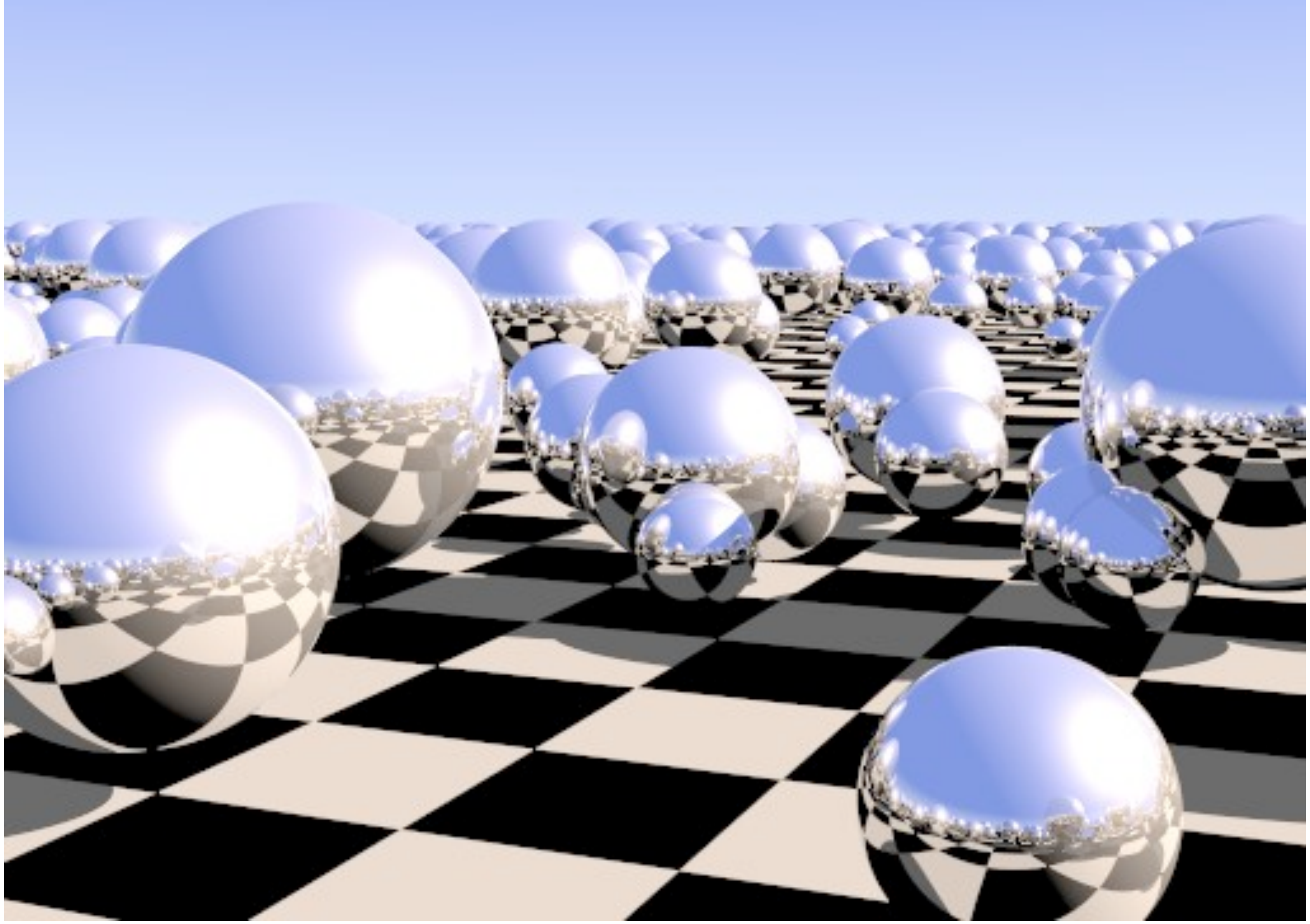

New developments in multi-meson systems

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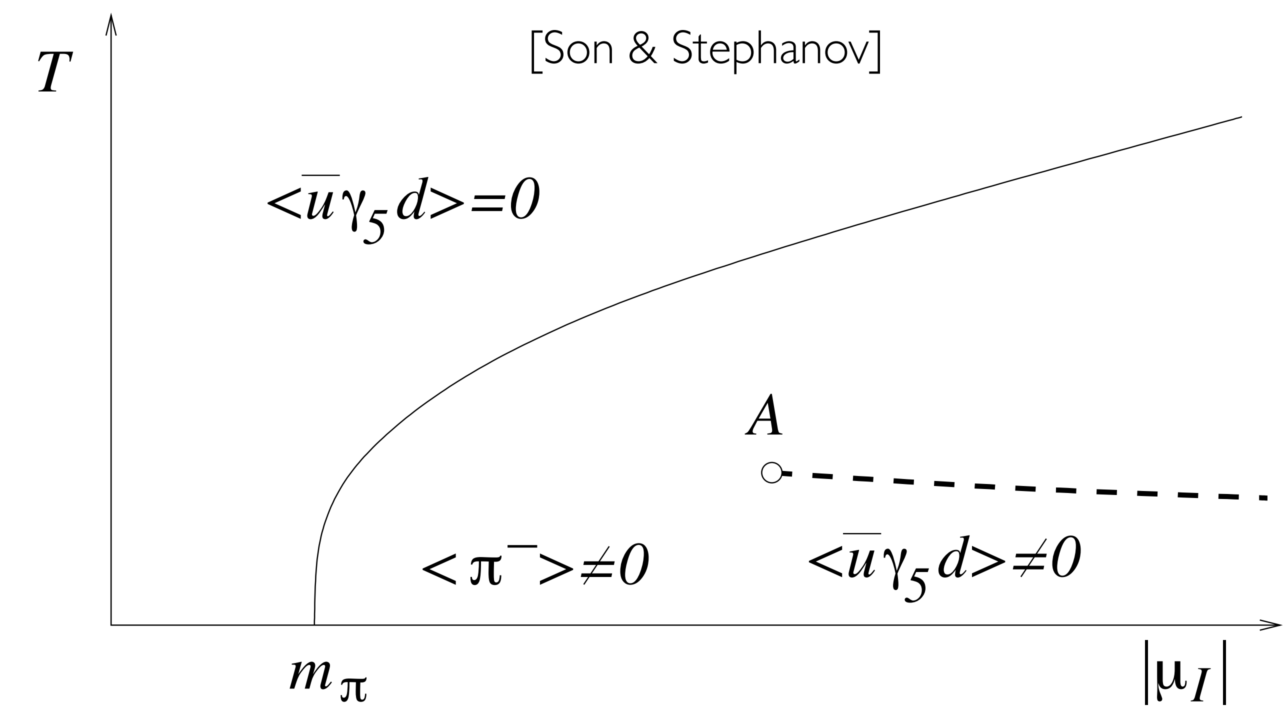
Lattice 2010, Villasimius, Sardinia, June 15th 2010

- Summary of $n < 13$ pion or kaon systems
- Recent developments
  - Really large systems: $^{208}\pi^+$
  - Mixed species systems

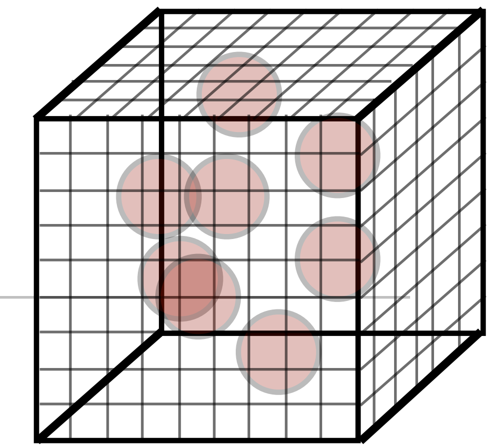
(Brief)Motivation

- Multi hadron interactions
- Meson condensates: interesting state of matter with a complex phase diagram
  - finite μ_I : BEC-BCS crossover
  - Spontaneous rotational symmetry breaking for $\mu_I \geq m_\rho$



- Kaon condensation may be phenomenologically relevant in n-stars [Kaplan/Nelson]
- Complexity frontier: precursor to nuclear systems

Bosons in a box



- Large volume expansion of ground state energy of n meson system to $1/L^7$
  - 2 & 3 body interactions (N body: $L^{-3(N-1)}$)
  - n=2: reproduces expansion of Lüscher result

$$\Delta E_n = \frac{4\pi\overline{a}}{M\ L^3}\,{}^nC_2\left\{1-\left(\frac{\overline{a}}{\pi\ L}\right)\mathcal{I}+\left(\frac{\overline{a}}{\pi\ L}\right)^2\left[\mathcal{I}^2+(2n-5)\mathcal{J}\right]\right.$$
$$\left.-\left(\frac{\overline{a}}{\pi\ L}\right)^3\left[\mathcal{I}^3+(2n-7)\mathcal{I}\mathcal{J}+(5n^2-41n+63)\mathcal{K}\right]\right\}$$
$$+\,{}^nC_3\frac{1}{L^6}\hat{\eta}_3^L+\mathcal{O}(L^{-7})$$

Scattering length

Geometric coefficients

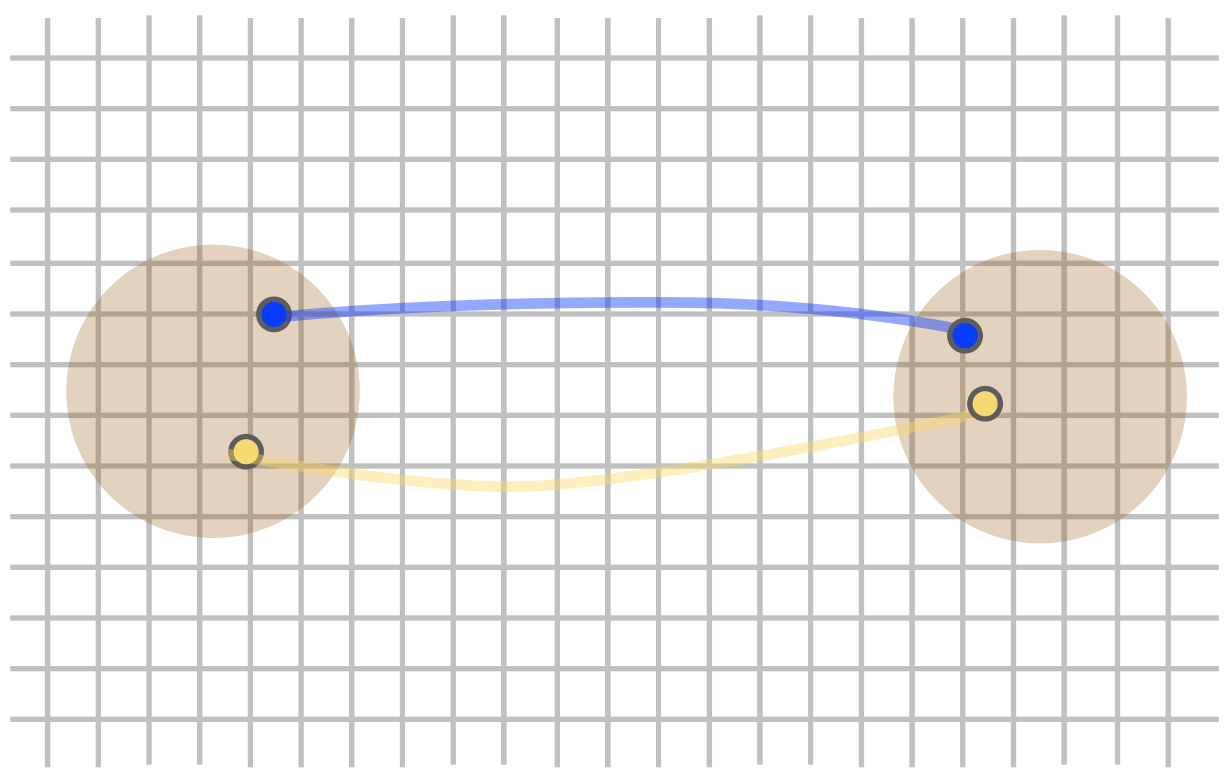
Three body Interaction

[Bogoliubov ‘47][Huang, Yang ‘57][Beane, WD, Savage PRD76;074507, 2007; WD+Savage PRD77:057502,2008]

Many mesons in LQCD

- Consider $\boldsymbol{\pi}^+$ correlator (m_u=m_d)

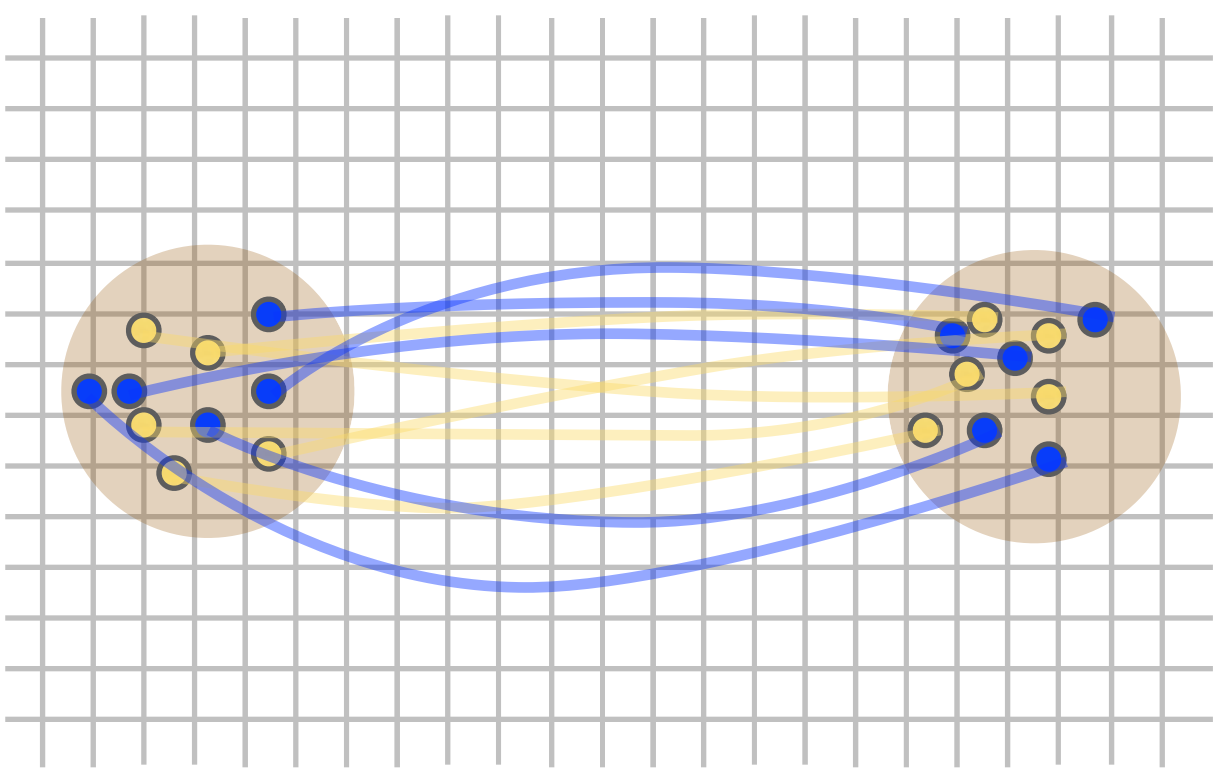
$$C \;\; (t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \overline{d}\gamma_5 u(\mathbf{x}, t)\overline{u}\gamma_5 d(\mathbf{0}, 0) \right] \;\; \right| 0 \right\rangle$$
$$\rightarrow A \; e^{-E \;\; t}$$



Many mesons in LQCD

- Consider n $\boldsymbol{\pi}^+$ correlator (m_u=m_d)

$$C_n(t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \overline{d}\gamma_5 u(\mathbf{x},t)\overline{u}\gamma_5 d(\mathbf{0},0) \right]^n \right| 0 \right\rangle$$
$$\rightarrow A\ e^{-E_n t}$$



Many mesons in LQCD

- Consider *n* π^+ correlator ($m_u = m_d$)

$$C_n(t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \overline{d}\gamma_5 u(\mathbf{x},t)\overline{u}\gamma_5 d(\mathbf{0},0) \right]^n \right| 0 \right\rangle$$
$$\rightarrow A\ e^{-E_n t}$$

- $n!^2$ Wick contractions: $(12!)^2 \sim 10^{17}$

$$C_3(t) = \mathrm{tr}\left[\Pi\right]^3 - 3\ \mathrm{tr}\left[\Pi\right]\mathrm{tr}\left[\Pi^2 + 2\ \mathrm{tr}\left[\Pi^3\right]\right.$$

$$\Pi = \sum_{\mathbf{x}} \gamma_5 S(\mathbf{x},t;0)\gamma_5 S^\dagger(\mathbf{x},t;0)$$

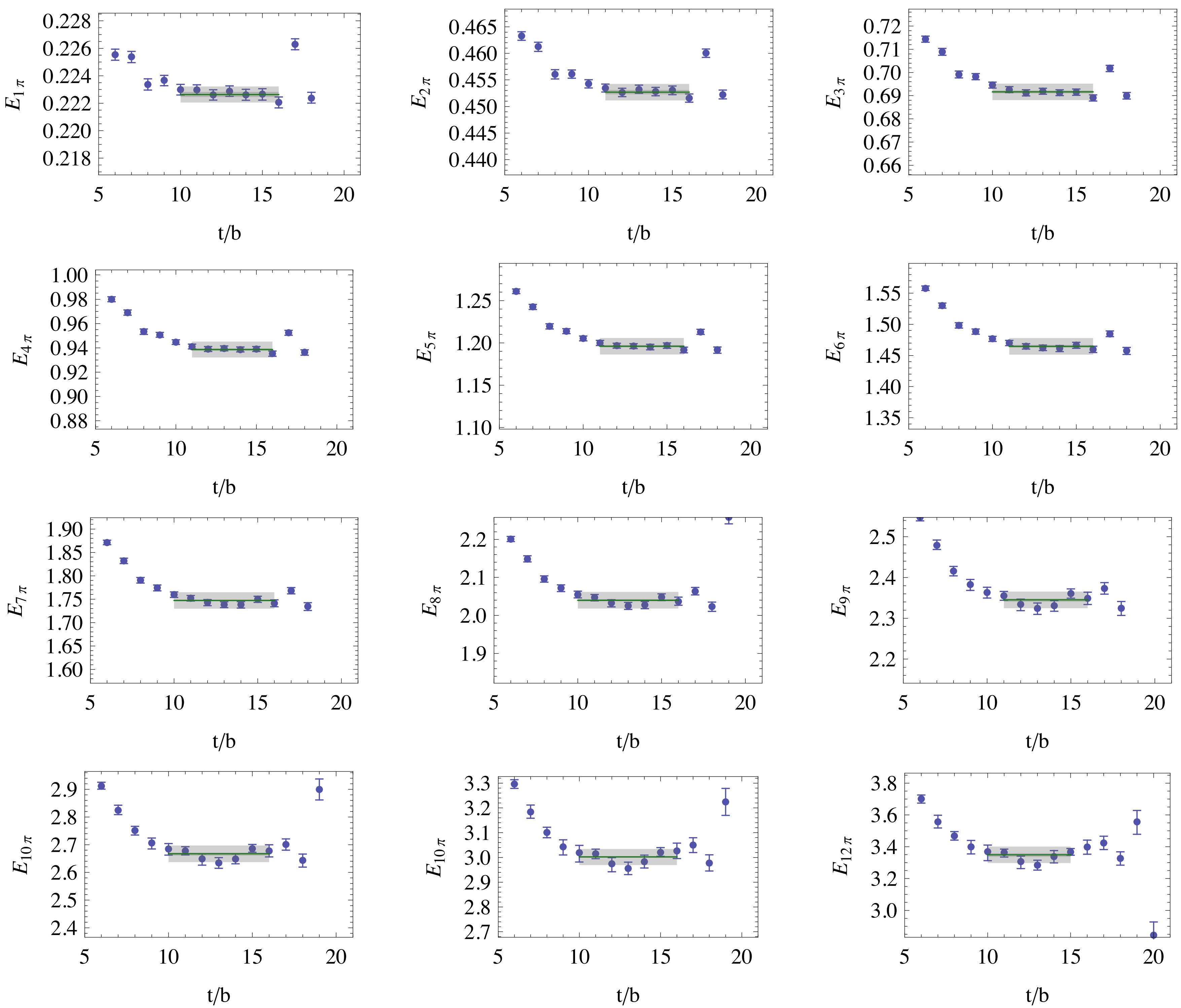
- n=13 has ~150 terms

Lattice details

- Calculations use MILC gauge configurations
  - L=2.5 fm, a=0.12 fm, *rooted* staggered
  - also L=3.5 fm and a=0.09 fm
- NPLQCD: domain-wall quark propagators
  - $m_\pi \sim$ 291, 318, 352, 358, 491, 591 MeV
  - 24 propagators / lattice in best case
- I_z=n=1,...,12 pion and (S=n) kaon systems

n-meson energies

- Effective energy plots: log[$C_n(t)/C_n(t+1)$]



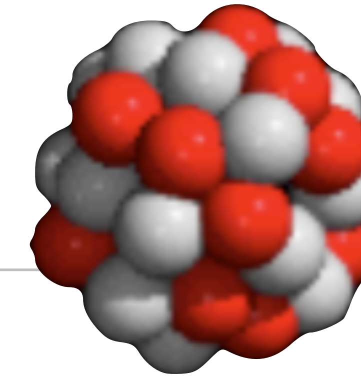
m_π = 352 MeV

Pions and kaon BECs

- Analysis of these energies has allowed
  - Precise extraction of 2-pion or kaon interactions
  - First LQCD measurements of three body interactions
  - Calculation of $\mu_I(\rho_I)$ and $\mu_Y(\rho_Y)$
    - Agreement with LOχPT important for n-star physics
  - Determine how pion gas screens $Q\bar{Q}$ potential
  - ...

[Phys. Rev. Lett. 100:082004, 2008, Phys Rev D78:014507, 2008 Phys Rev D78:054514, 2008, Phys. Rev. Lett. 102:032004, 2009]

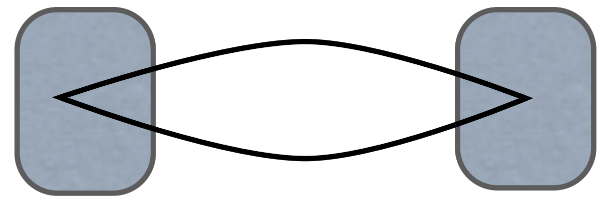
Complex systems

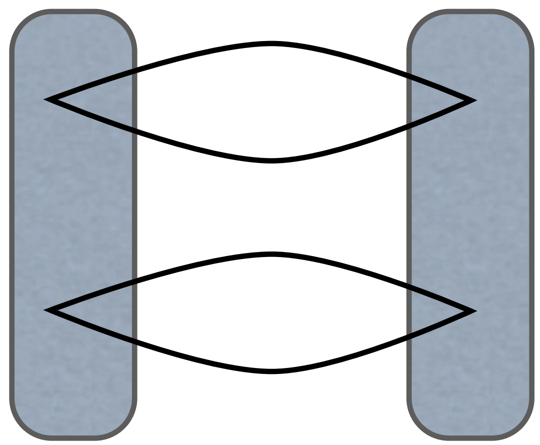
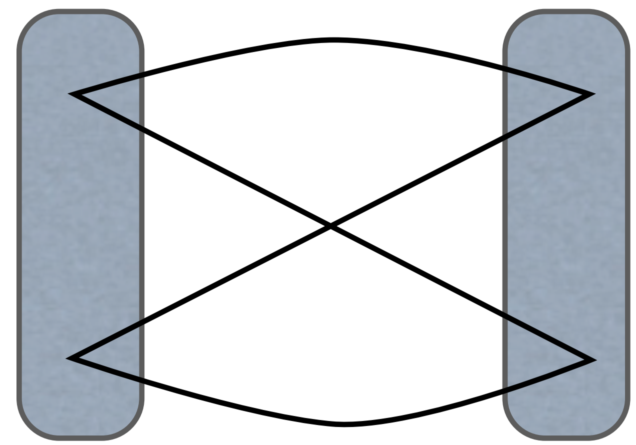


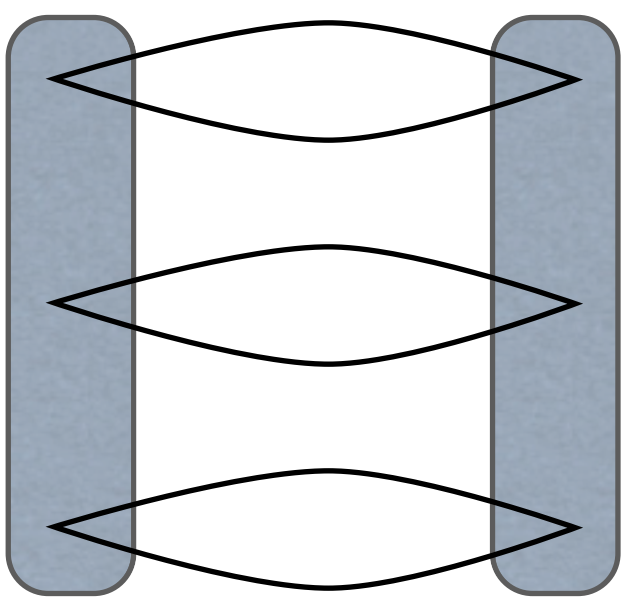
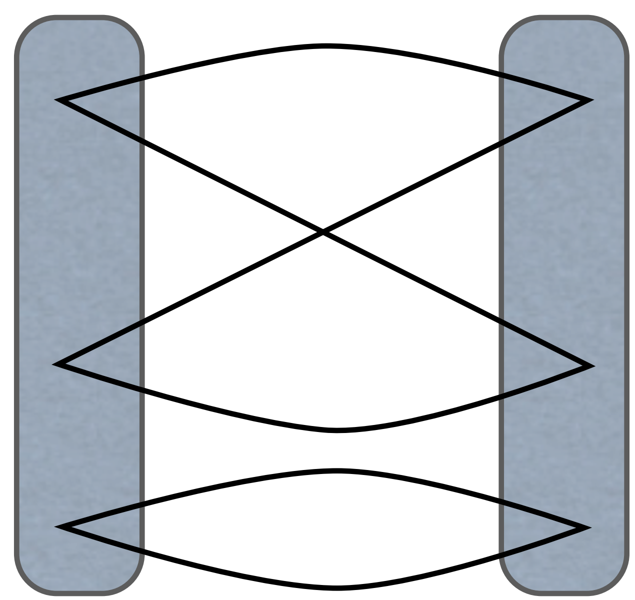
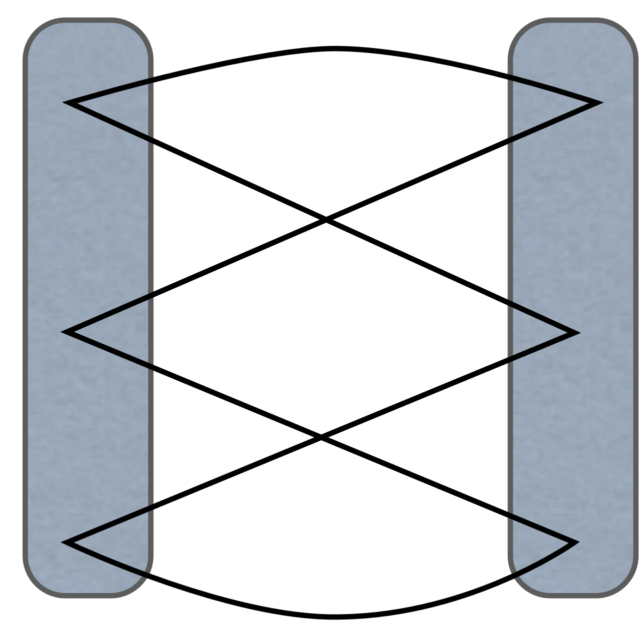
[WD, M Savage 1001.2768]

- How do we deal with complexity of contractions?
  - One species: $N_{\rm terms} \sim e^{\pi\sqrt{2n/3}}/\sqrt{n}$ [Ramanujan & Hardy]
  - Two-species is harder, more is not feasible
- How do we go beyond n=12?
  - Previous method fails because of Pauli principle
  - Avoid by using multiple propagator sources but this leads to contraction complexity

Few pion contractions

$C_{1\pi}(t) =$ 

$C_{2\pi}(t) =$  $-$ 

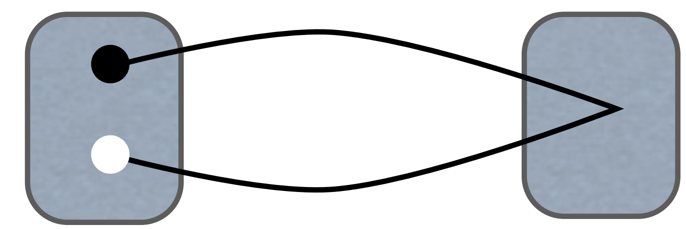
$C_{3\pi}(t) =$  -3  -2 

Blocks

- Define a partly contracted pion correlator

$$\Pi \equiv R_1 = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_0)\gamma_5 S_d(x_0; \mathbf{x}, t)\gamma_5 = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_0) S_d^\dagger(\mathbf{x}, t; x_0)$$

  - Time-dependent 12x12 matrix (spin-colour indices)



- Correlators

$$C_1(t) = \langle \Pi \rangle, \quad C_2(t) = \langle \Pi \rangle^2 - \langle \Pi^2 \rangle, \ldots$$

- Functional definition

$$\Pi_{ij} = \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_1(t)$$

- Generalises to

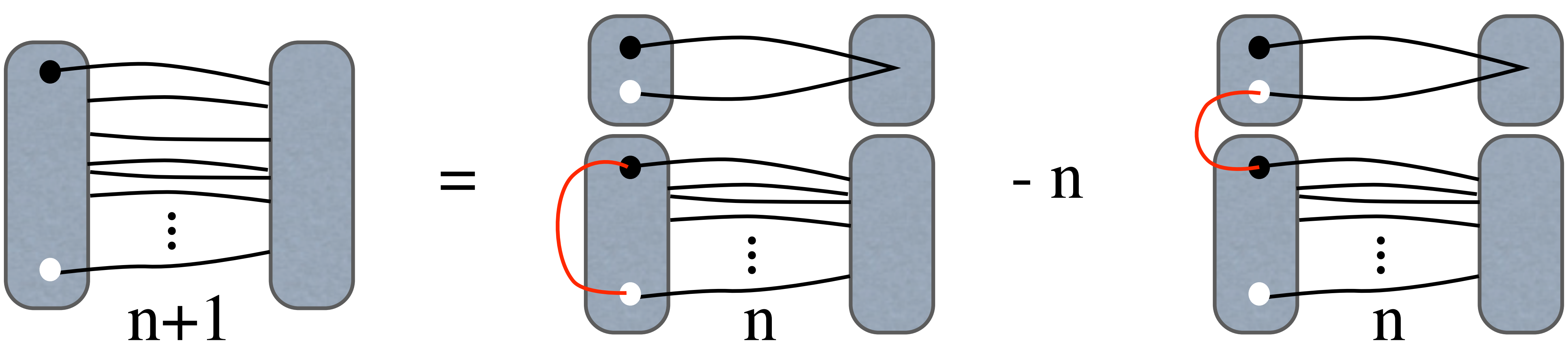
$$(R_n)_{ij} \equiv \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_n(t)$$

Recursion relation

- The block objects <u>are</u> simply related
- Recursion relation

$$R_{n+1} = \langle R_n \rangle \; R_1 - n \; R_n \; R_1$$

- Initial condition is that $R_1 = \Pi, \qquad R_j = 0, \forall j < 1$
- Can also construct a descending recursion as we know that $R_{13}=0$



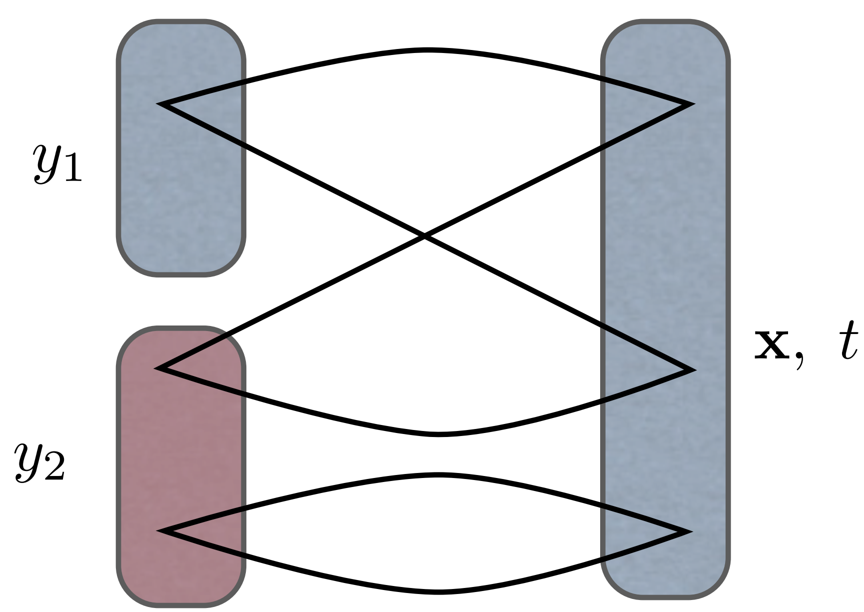
Multi-source systems

- To get beyond n=12, need to consider multi-source systems

- Consider two sources first

$$C_{(n_1\pi_1^+\ ,\ n_2\pi_2^+)}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x},t) \right)^{n_1+n_2} \left(\pi^-(\mathbf{y_1},0) \right)^{n_1} \left(\pi^-(\mathbf{y_2},0) \right)^{n_2} \right\rangle$$

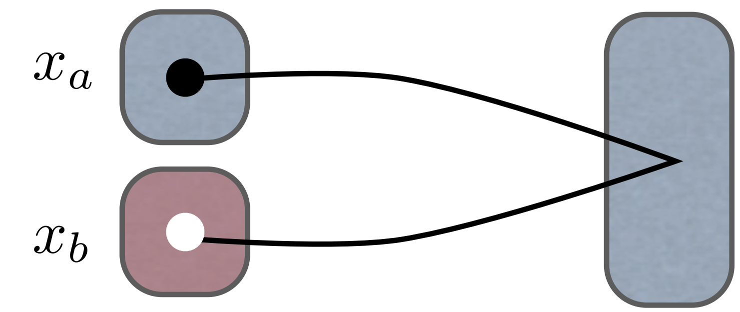
- $C_{(2,1)}(t)$ contains contractions like



Multi-source systems

- Multiple types of blocks needed

$$A_{ab} = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_a) S_d^\dagger(\mathbf{x}, t; x_b)$$



- Two species case has a simple recursion relation: First define

$$P_1 = \left(\begin{array}{c|c} A_{11}(t) & A_{12}(t) \\ \hline 0 & 0 \end{array} \right) \quad , \quad P_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline A_{21}(t) & A_{22}(t) \end{array} \right)$$

Then $Q_{(n1,n2)}$ (generalisations of the R_n) satisfy

$$Q_{(n_1+1,n_2)} = \langle\, Q_{(n_1,n_2)} \,\rangle\, P_1 \; - \; (n_1+n_2)\, Q_{(n_1,n_2)} \;\; P_1$$
$$+\langle\, Q_{(n_1+1,n_2-1)} \,\rangle\, P_2 \; - \; (n_1+n_2)\, Q_{(n_1+1,n_2-1)} \;\; P_2$$

Extensions

- Recursions also constructed for
  - *m*-source systems
  - *k*-species systems: $\boldsymbol{\pi}$'s, K's, D's, B's, ...
  - *m*-source, *k*-species systems

$$T_{\mathbf{n}+\mathbf{1}_{rs}} = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle\, T_{\mathbf{n}+\mathbf{1}_{rs}-\mathbf{1}_{ij}} \,\rangle\, P_{ij} \;-\; \overline{\mathcal{N}}\, T_{\mathbf{n}+\mathbf{1}_{rs}-\mathbf{1}_{ij}} \;\; P_{ij}$$

  where subscripts are matrices

- Implemented as matrix multiplications - computationally tractable

Extensions

- Factorial cost reduced for N mesons
  - Each iteration involves essentially two-body contractions
  - Memory requirements
- Recursions also exist for baryon contractions but are messier
  - Different choices of basis objects for recursion
  - Allow calculations of B=4,5,... systems

Mixed species systems

[work in preparation with B Smigielski (W&M/UW)]

- Weakly interacting two species systems: pions and kaons
- $E_{n,m}$ of *n* pions and *m* kaons depends on three 2-body and four 3-body interaction parameters at L^{-8}
  - Analytic form has been calculated but ugly ($1/L^6$) *[Smigielski & Wasem '08]*
- Matching to lattice energies allows for extraction of interaction parameters
- Reduced symmetry: contractions significantly more complex – n=6 pions, m=6 kaons: 1500 terms!

LQCD calculations

- One ensemble of anisotropic clover lattices
  - Dynamical N_f=2+1 lattices from JLab/HSC
  - m_π=390 MeV, a_s=0.123 fm, ξ=3.5, 20^3x128
  - ~30K measurements: ~75 sources on ~400 cfgs
- Anti-periodic BCs for quarks (periodic for mesons)
  - Correlators have complicated time dependence
- Correlators for all sets of {*n*,*m*} with *n*+*m*<*13*

LQCD correlators

- Extend single species construction

$$C_{N\pi,MK}(t) = \left\langle \left(\sum_{\mathbf{x}} \mathcal{O}_d^\dagger(x) \right)^N \left(\sum_{\mathbf{x}} \mathcal{O}_s^\dagger(x) \right)^M \left(\mathcal{O}_d(0) \right)^N \left(\mathcal{O}_s(0) \right)^M \right\rangle$$

where

$$\mathcal{O}_q(x) = \bar{u}(x)\gamma_5 q(x) \text{ and } x = (\mathbf{x}, t)$$

- Can show the expected behaviour is

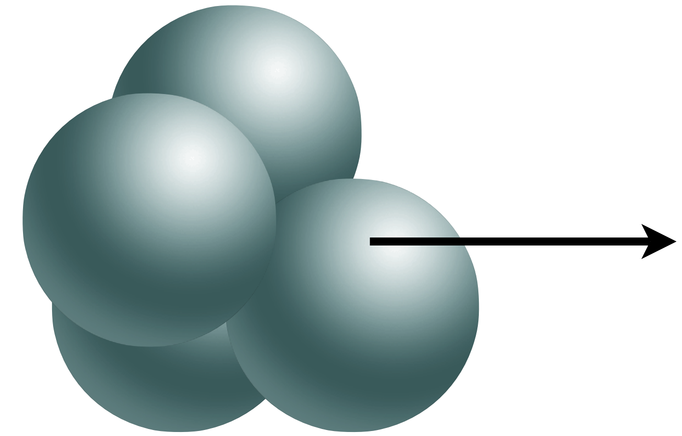
$$C_{N\pi,MK}(t) = \frac{1}{2} \sum_{m=0}^{M} \sum_{n=0}^{N} Z_{n,m}^{N-n,M-m} e^{-\left(E_{N-n,M-m}+E_{n,m}\right)T/2} \times$$

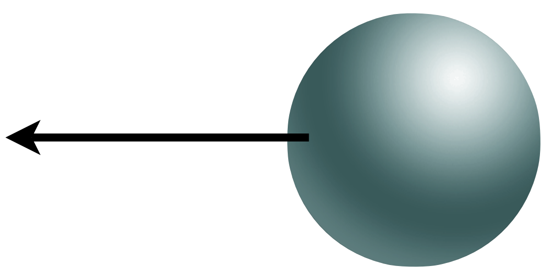
t_T=t-T

$$\cosh\left(\left(E_{N-n,M-m} - E_{n,m}\right)t_T\right) + \frac{1}{2} Z_{\frac{N}{2},\frac{M}{2}}^{\frac{N}{2},\frac{M}{2}} e^{E_{N/2,M/2}T/2} \; \delta_{N,2l}\delta_{M,2k}$$

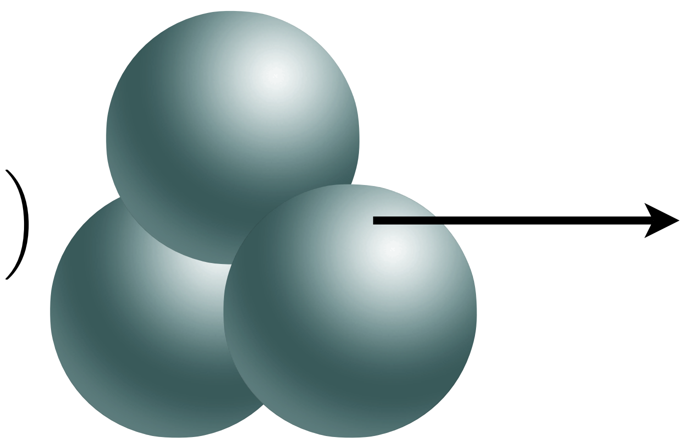
(1/2) Four pion correlation

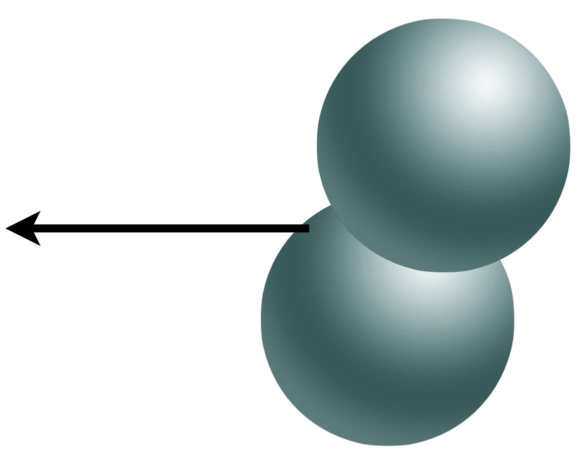
$$Z_{4\pi}\left(e^{-4E_{4\pi}t}+e^{-4E_{4\pi}(T-t)}\right)$$



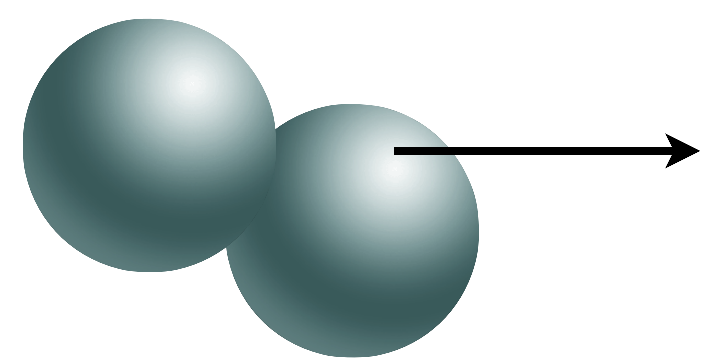


$$Z_{3/1\pi}\left(e^{-E_{3\pi}t}e^{-E_{1\pi}(T-t)}+e^{-E_{3\pi}(T-t)}e^{-E_{1\pi}t}\right)$$





$$Z_{2/2\pi}e^{-E_{2\pi}t}e^{-E_{2\pi}(T-t)}=Z_{2/2\pi}e^{-E_{2\pi}T}$$



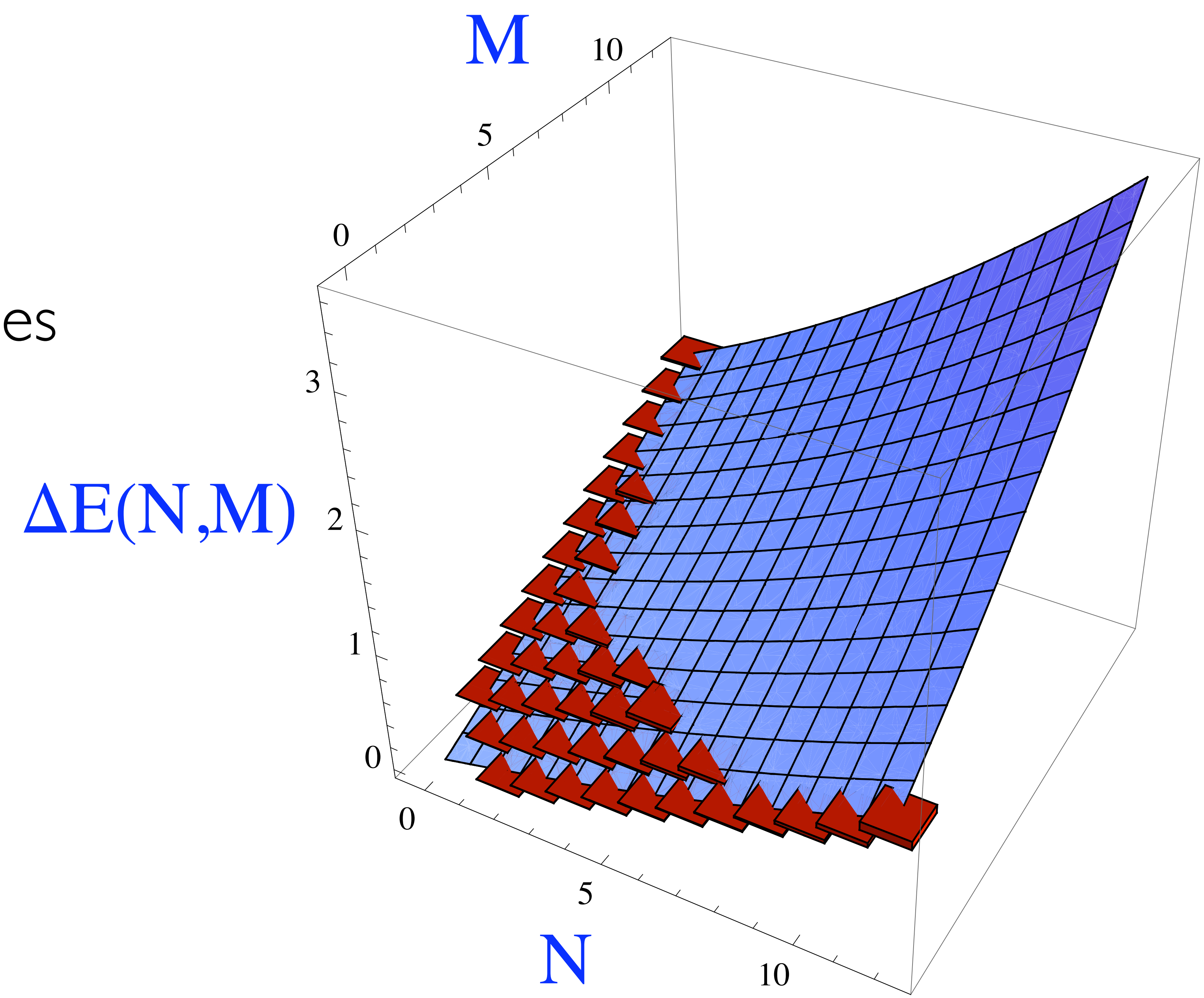
t=0

Analysis

- Extracting the eigen-energies (or interaction parameters) from these correlators is involved
- Huge parameter space; O(90) correlators
  - $C_{4\pi,2K}$ involves 18 parameters
  - Correlations between different {*n*,*m*} important
  - *Variable projection*: minimize the χ^2 function analytically for analytic parameters
  - Augment χ^2 via Bayesian priors

Extracted energies

- Boxes correspond to extracted energies
- Surface is fitted dependence on 2- and 3-body parameters



Interaction parameters

- Extracted interaction parameters

$$
\begin{aligned}
m_K \bar{a}_{KK} &= 0.444 \pm 0.0123 \pm 0.005 \\
m_\pi \bar{a}_{\pi\pi} &= 0.243 \pm 0.016 \pm 0.007 \\
m_{\pi K} \bar{a}_{\pi K} &= 0.136 \pm 0.015 \pm 0.007 \\
m_\pi \bar{\eta}_{3\pi} f_\pi^4 &= 1.002 \pm 0.303 \pm 0.149 \\
m_K \bar{\eta}_{3K} f_K^4 &= 0.800 \pm 0.314 \pm 0.127 \\
\left(\frac{m_K m_\pi}{m_K + 2m_\pi}\right) \ \bar{\eta}_{3,\pi KK} \ f_{\pi KK}^4 &= 1.123 \pm 0.379 \pm 0.162 \\
\left(\frac{m_K m_\pi}{m_\pi + 2m_K}\right) \ \bar{\eta}_{3,\pi\pi K} \ f_{\pi\pi K}^4 &= 0.769 \pm 0.348 \pm 0.150
\end{aligned}
$$

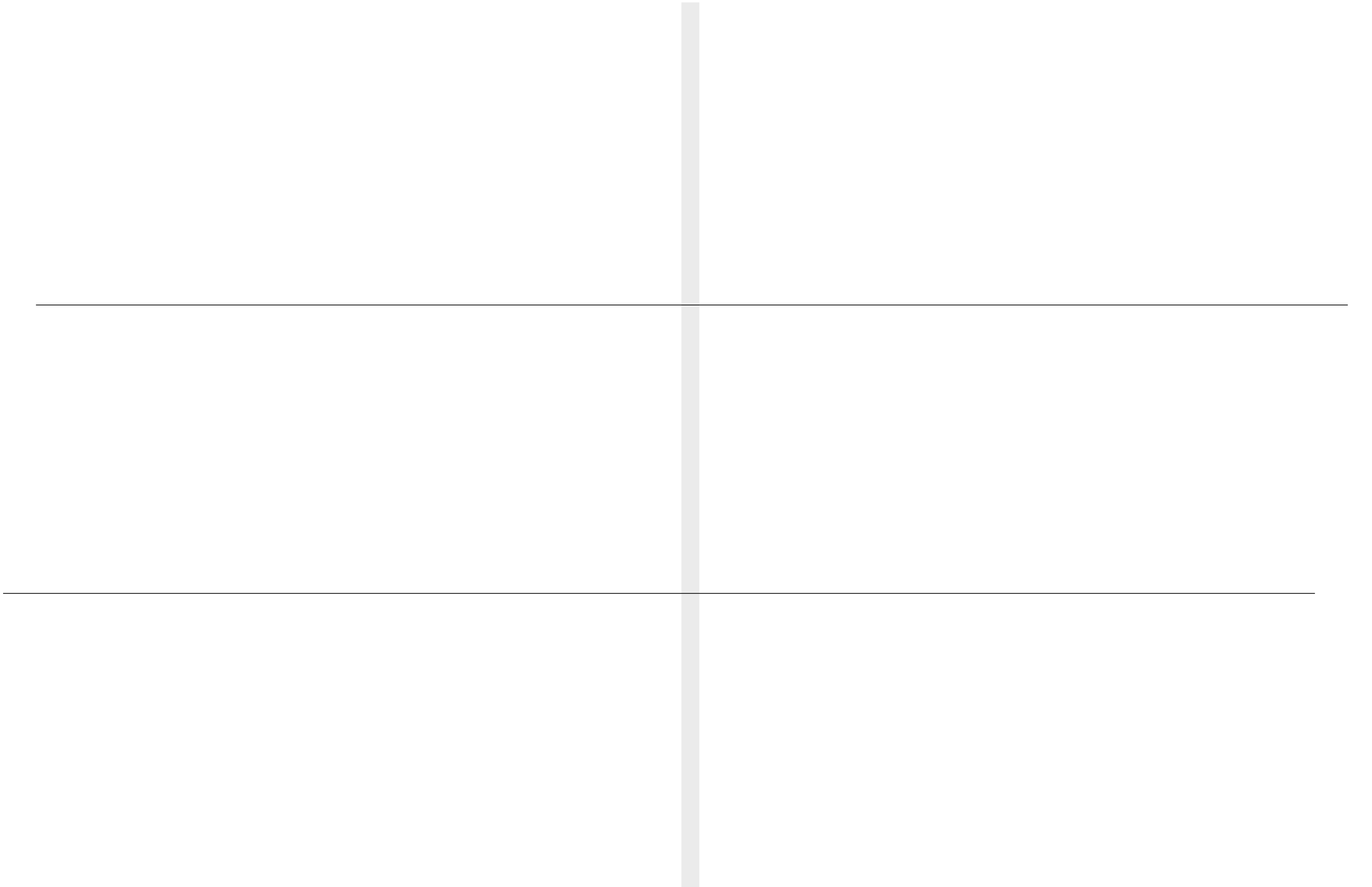
- Single species parameters consistent with previous calculation but with better precision in 3-body case

Summary

- LQCD is making progress in many-body systems
  - Explore novel types of QCD matter
  - Properties and effects of meson condensates
- New algorithms to ameliorate contraction complexity
- (Light) Nuclei?

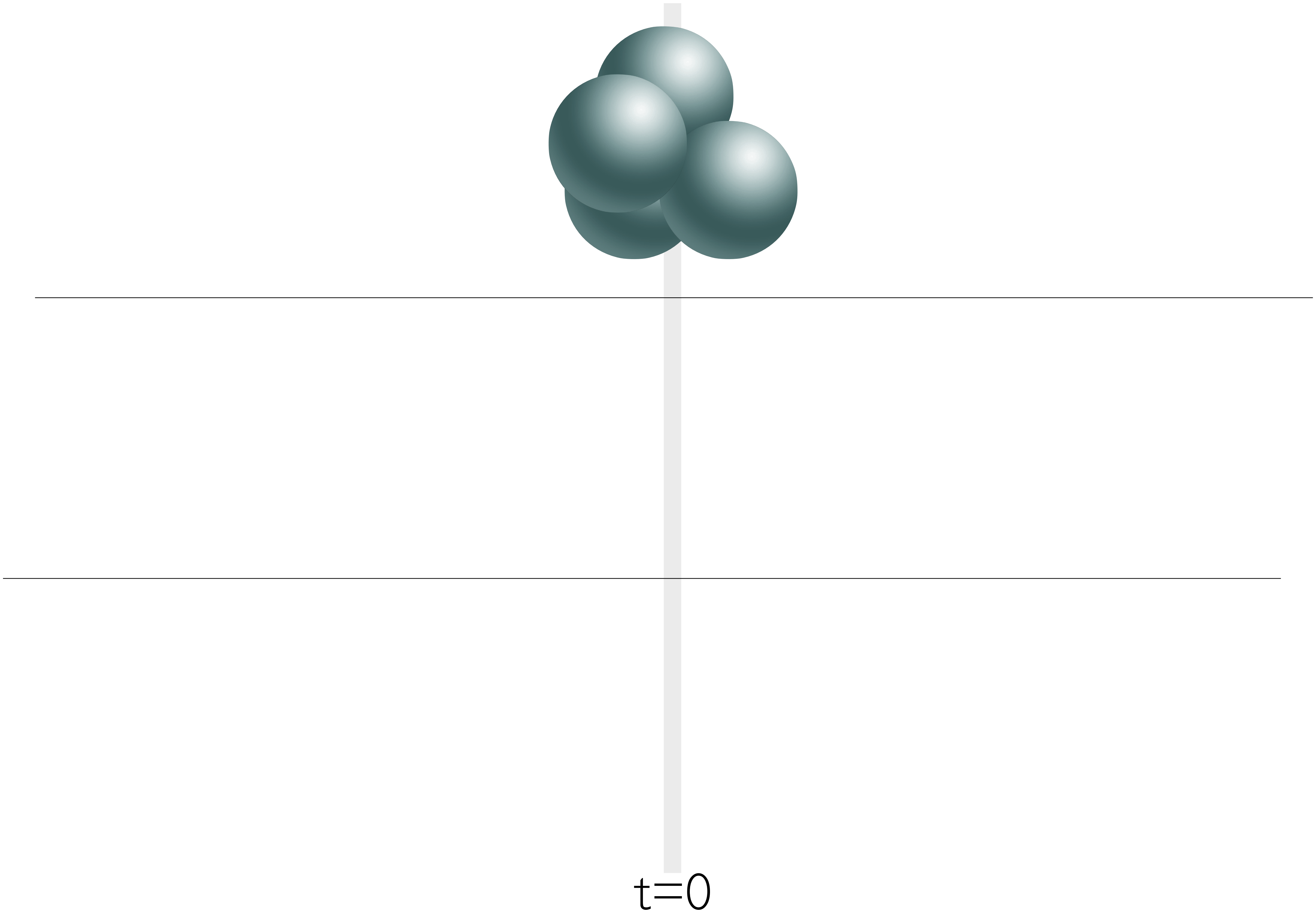
[FIN]

Four pion correlation

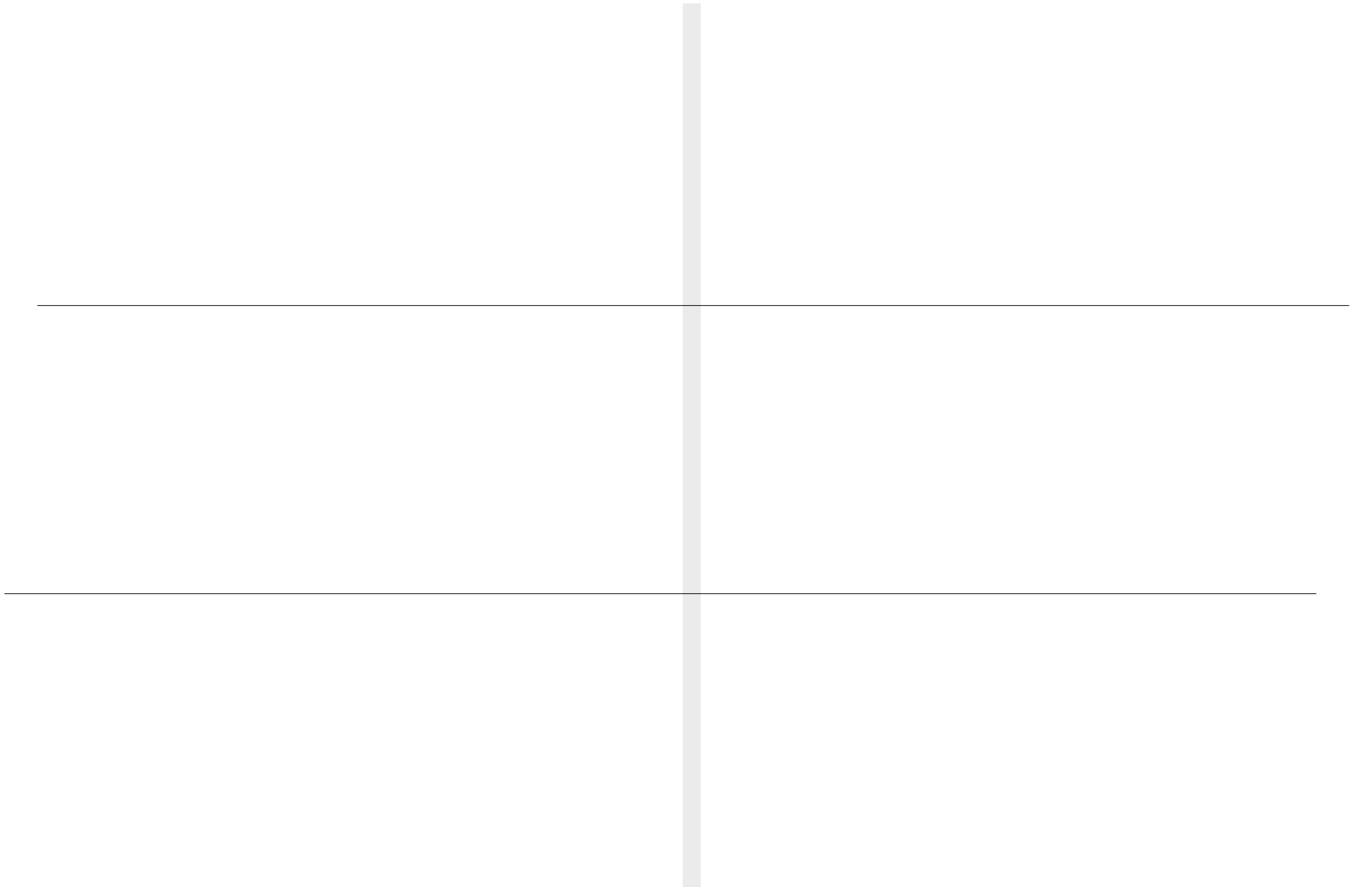


t=0

Four pion correlation

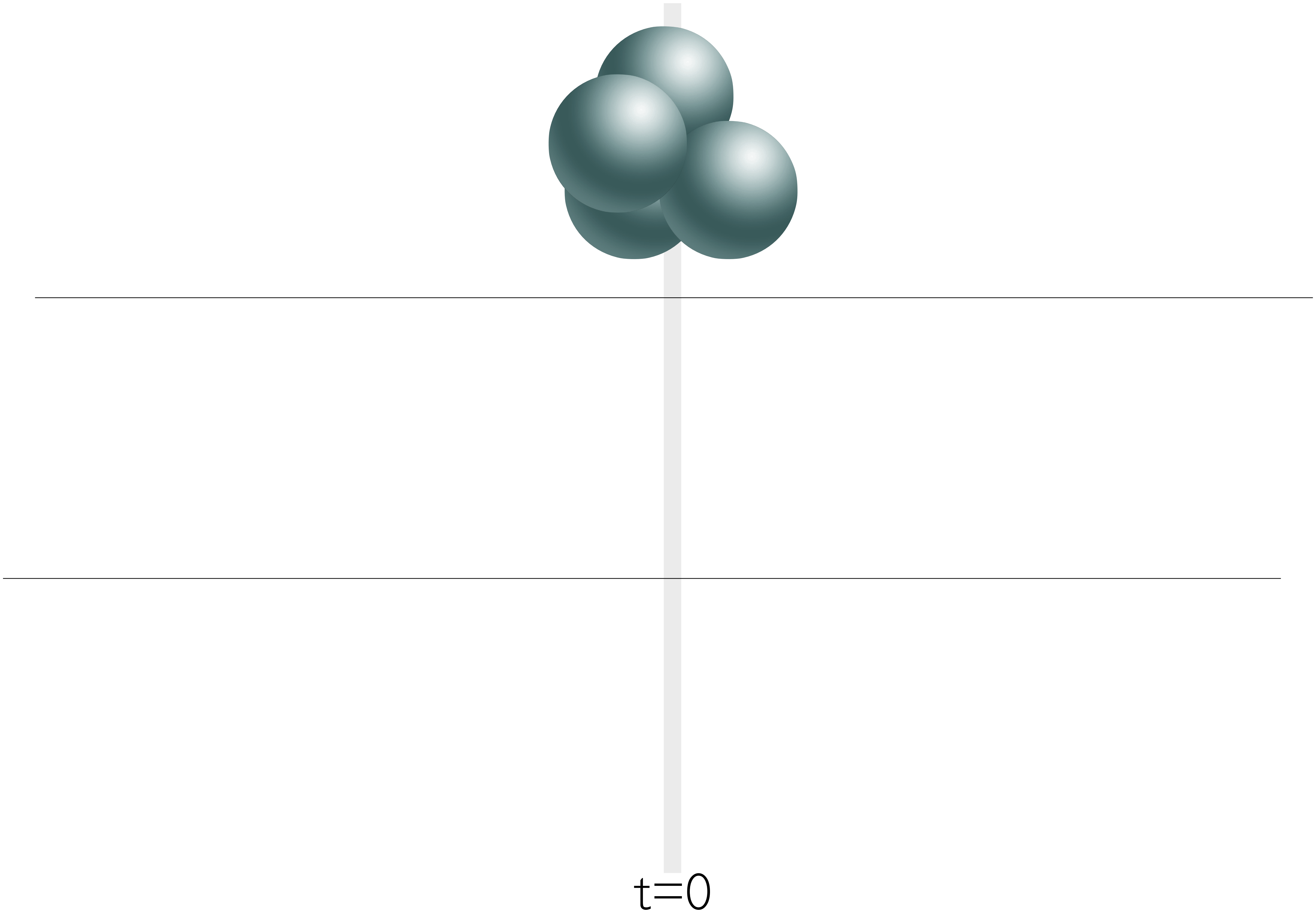


Four pion correlation



t=0

Four pion correlation



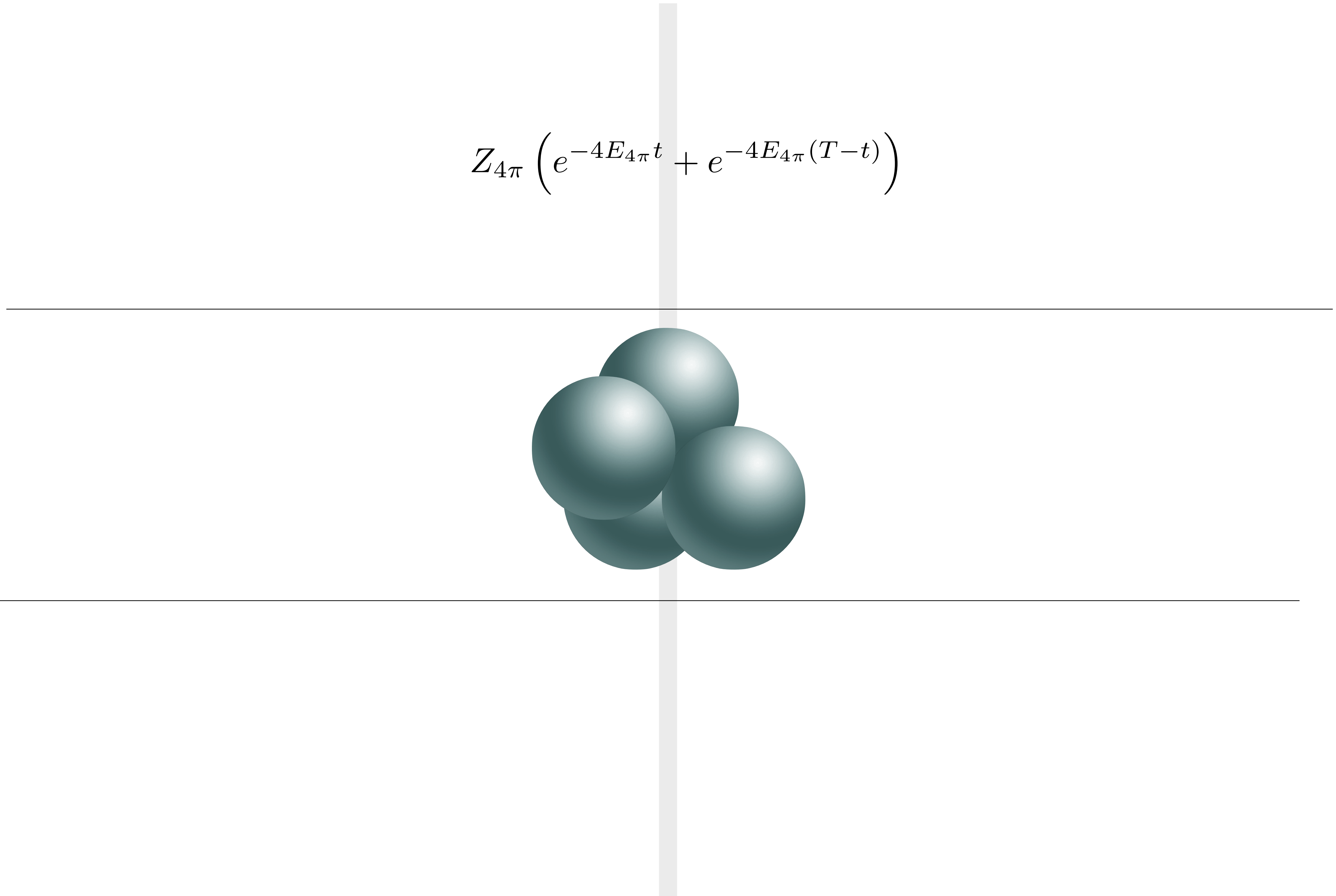
Four pion correlation

$$Z_{4\pi}\left(e^{-4E_{4\pi}t} + e^{-4E_{4\pi}(T-t)}\right)$$

t=0

Four pion correlation

$$Z_{4\pi}\left(e^{-4E_{4\pi}t}+e^{-4E_{4\pi}(T-t)}\right)$$



t=0

Four pion correlation

$$Z_{4\pi}\left(e^{-4E_{4\pi}t}+e^{-4E_{4\pi}(T-t)}\right)$$

t=0

Four pion correlation

$$Z_{4\pi}\left(e^{-4E_{4\pi}t}+e^{-4E_{4\pi}(T-t)}\right)$$

$$Z_{3/1\pi}\left(e^{-E_{3\pi}t}e^{-E_{1\pi}(T-t)}+e^{-E_{3\pi}(T-t)}e^{-E_{1\pi}t}\right)$$

t=0

Four pion correlation

$$Z_{4\pi}\left(e^{-4E_{4\pi}t}+e^{-4E_{4\pi}(T-t)}\right)$$

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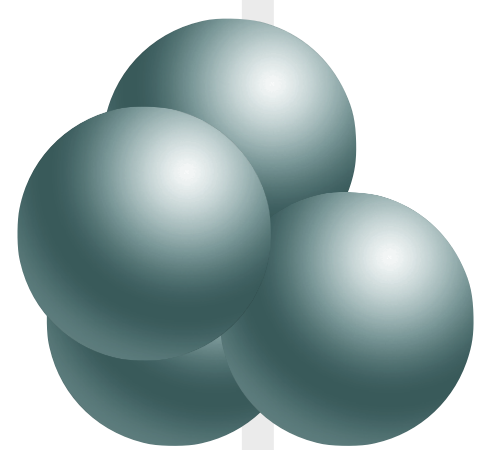

t=0

Four pion correlation

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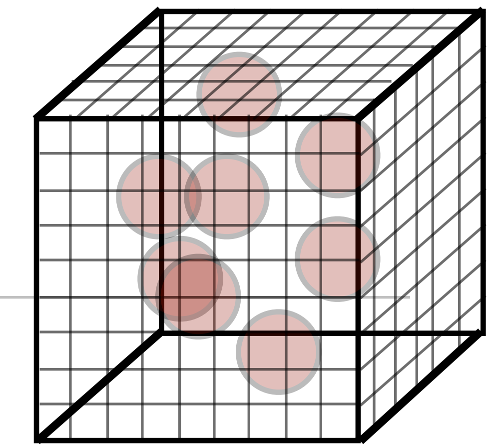
$$Z_{2/2\pi}e^{-E_{2\pi}t}e^{-E_{2\pi}(T-t)}=Z_{2/2\pi}e^{-E_{2\pi}T}$$

t=0

Bosons in a box



- Large volume expansion of GS energy of n meson system to $1/L^7$
  - 2 & 3 body interactions (N body: $L^{-3(N-1)}$)
  - n=2: reproduces expansion of Lüscher

$$\Delta E_n = \frac{4\pi \bar{a}}{M\ L^3}\, {}^nC_2 \Big\{ 1 - \left(\frac{\bar{a}}{\pi\ L}\right)\mathcal{I} + \left(\frac{\bar{a}}{\pi\ L}\right)^2 \left[\mathcal{I}^2 + (2n-5)\mathcal{J}\right] - \left(\frac{\bar{a}}{\pi\ L}\right)^3 \left[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + (5n^2-41n+63)\mathcal{K}\right] \Big\} + {}^nC_3 \frac{1}{L^6}\hat{\eta}_3^L + \mathcal{O}(L^{-7})$$

Scattering length

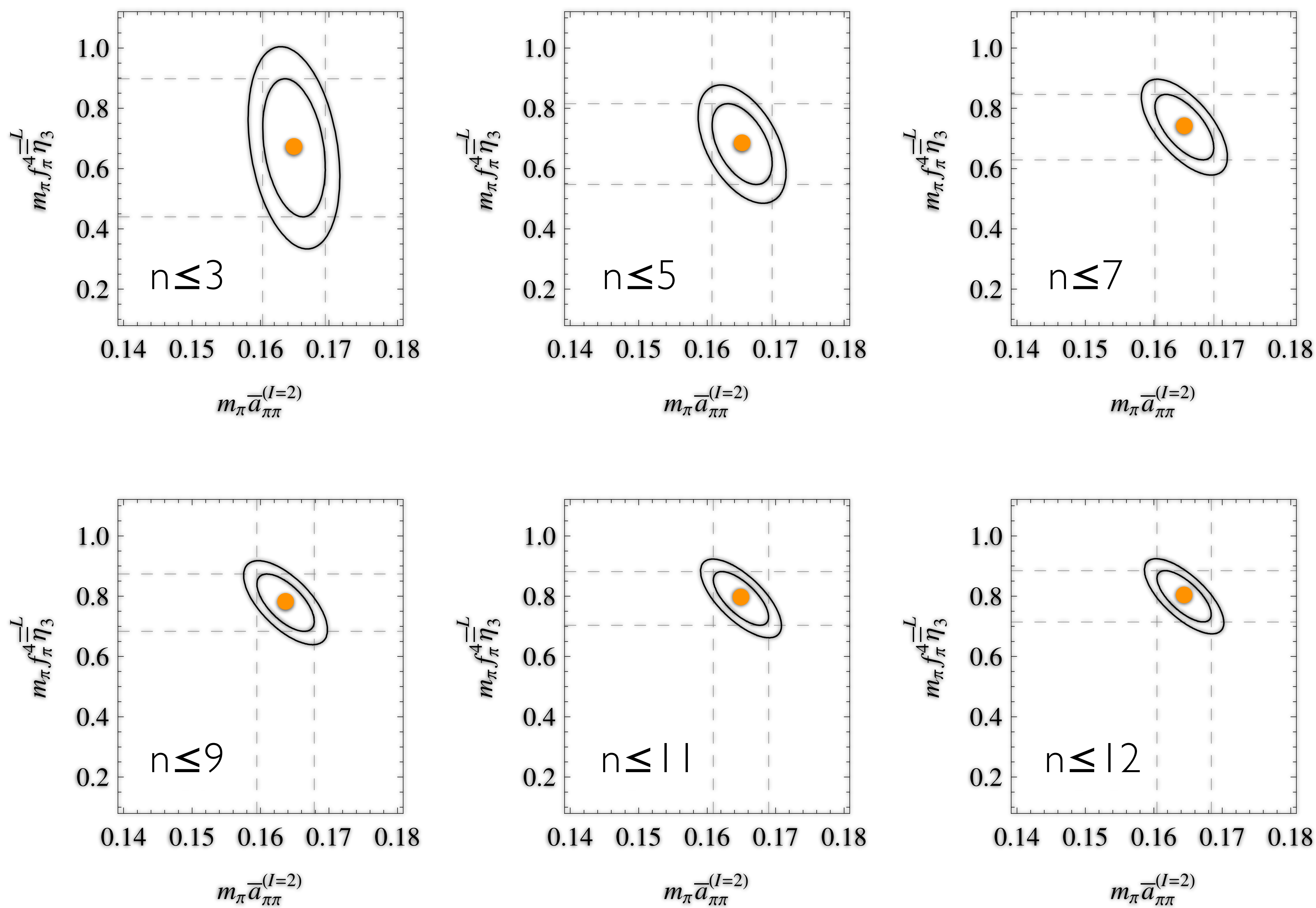
Geometric coefficients

Three body Interaction

[Bogoliubov '47][Huang,Yang '57][Beane, WD, Savage PRD76;074507, 2007; WD+Savage PRD77:057502,2008]

n correlations

Three-body



Two body

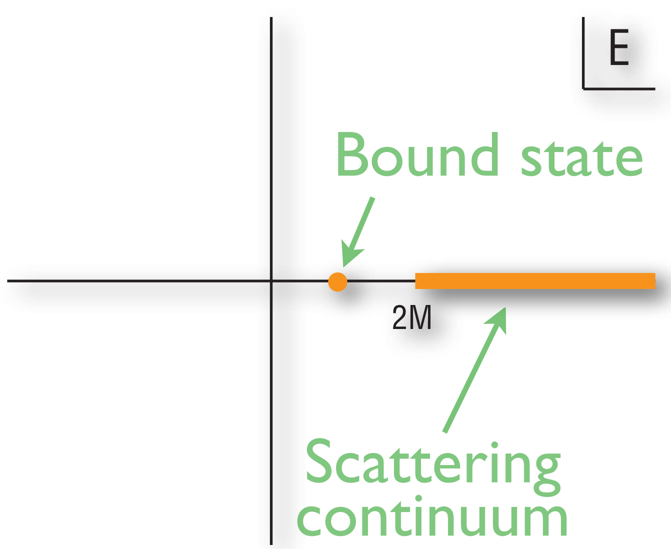
Hadron scattering

- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels $\Rightarrow$ scattering phase-shift up to inelastic threshold

$$\Delta E_{(n)} = \sqrt{|\mathbf{q}_{(n)}|^2 + m_A^2} + \sqrt{|\mathbf{q}_{(n)}|^2 + m_B^2} - m_A - m_B$$

$$q_{(n)} \cot \delta(q_{(n)}) = \frac{1}{\pi\ L} S\left(\frac{q_{(n)} L}{2\pi}\right)$$

$$S(\eta) = \lim_{\Lambda\to\infty}\left[\sum_{\vec{n}}^{|\vec{n}|<\Lambda} \frac{1}{|\vec{n}|^2 - \eta^2} - 4\pi\Lambda\right]$$



Hadron scattering

- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels $\Rightarrow$ scattering phase-shift up to inelastic threshold

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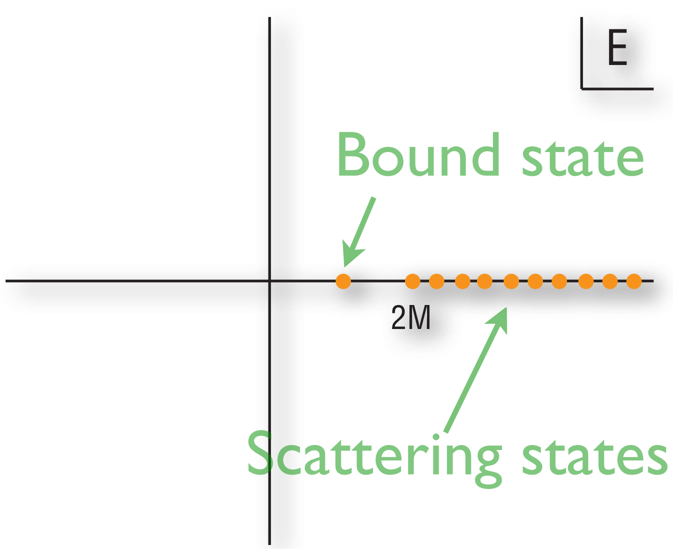
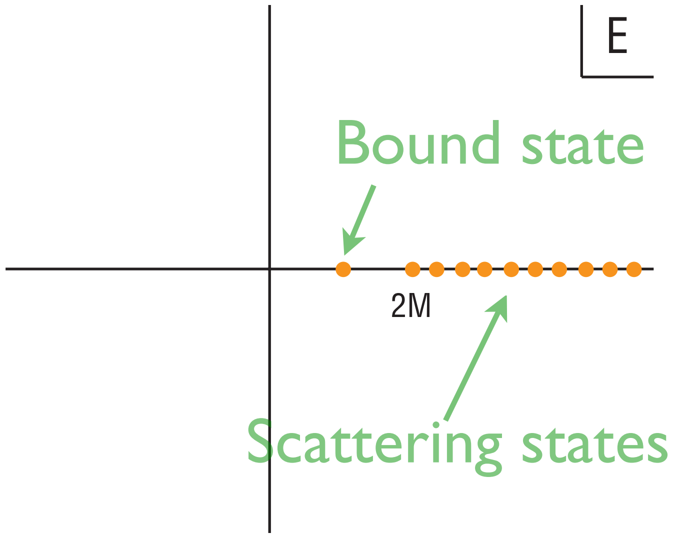
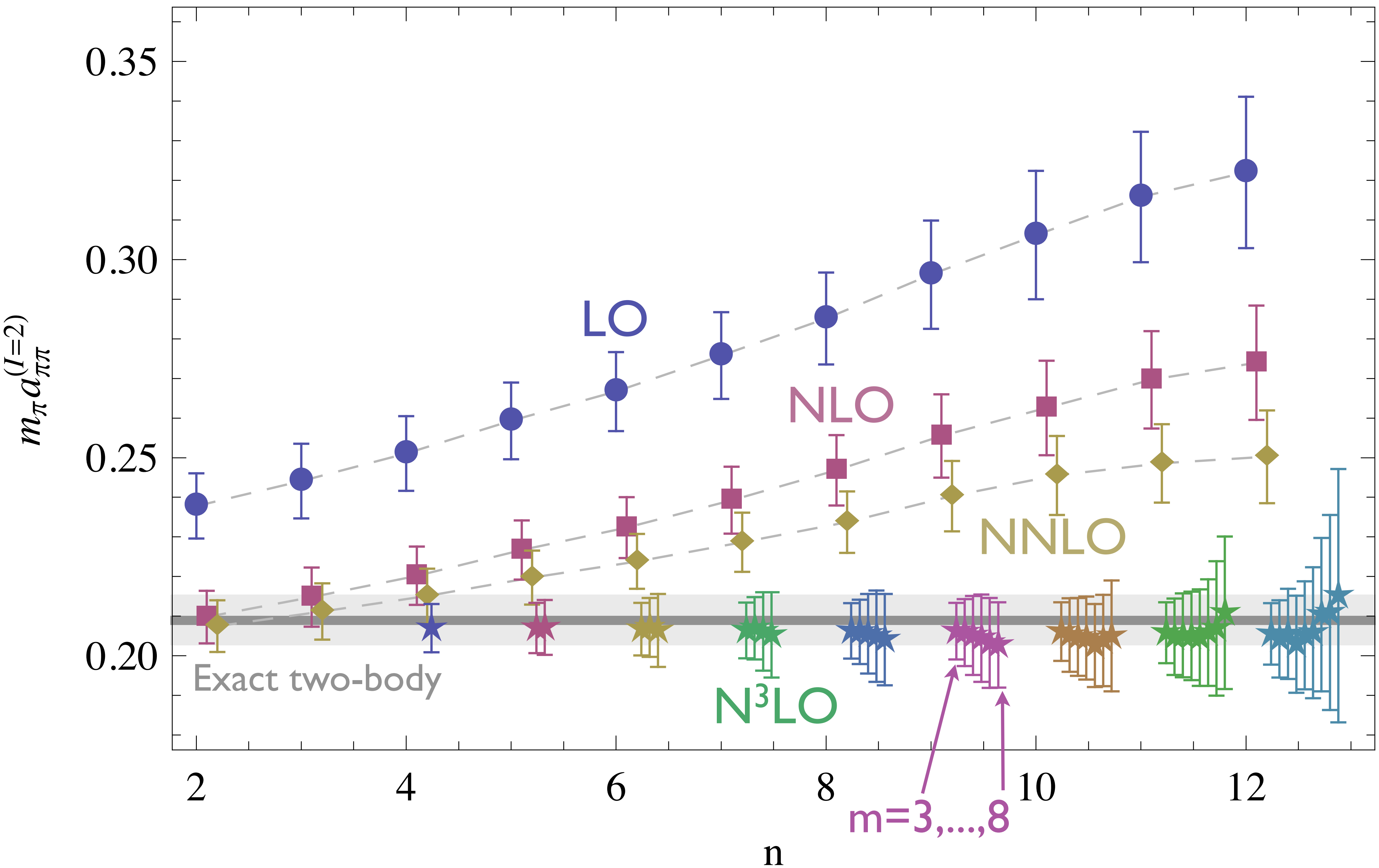


Hadron scattering

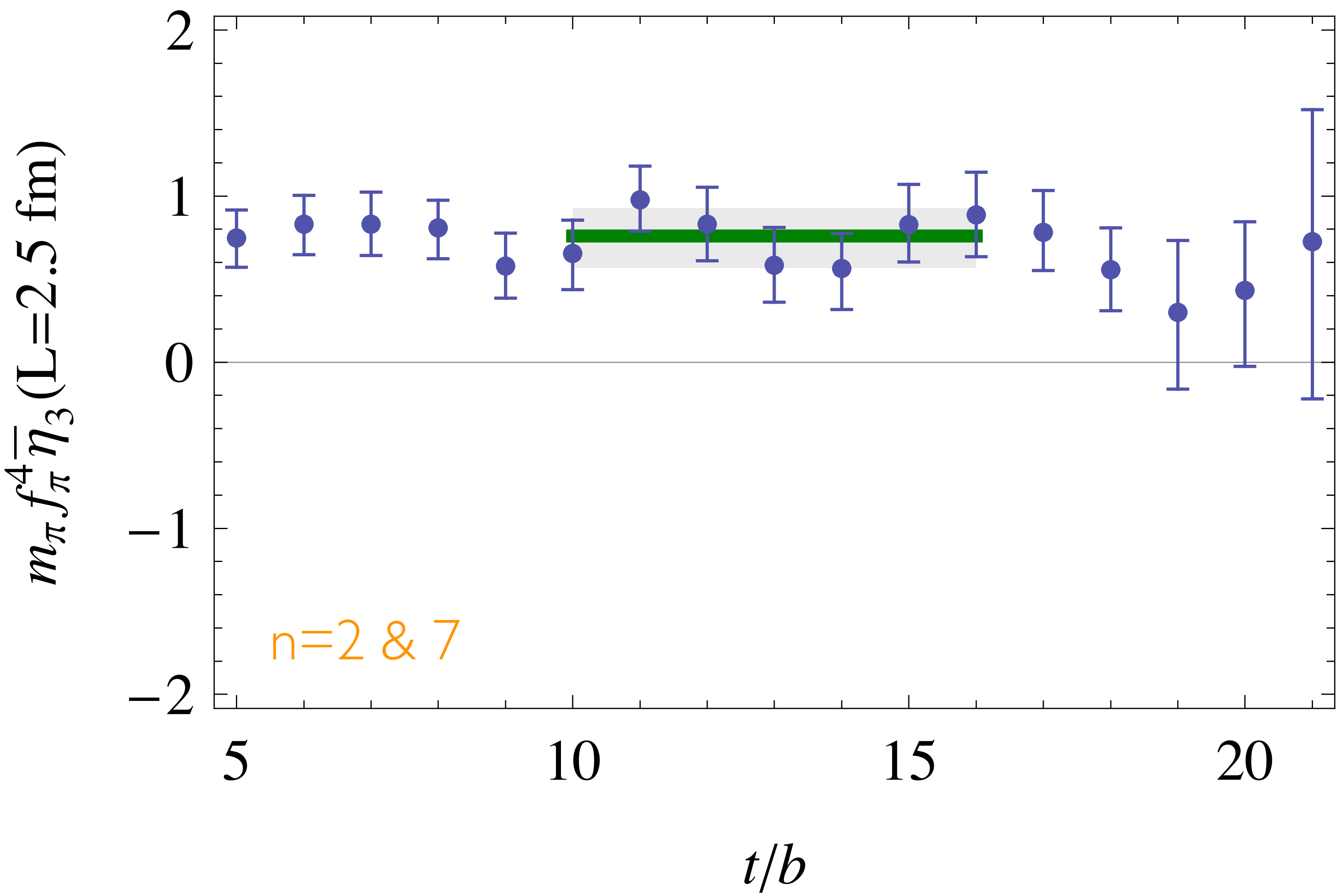
- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels $\Rightarrow$ scattering phase-shift up to inelastic threshold
- Exact relation provided $r \ll L$
- Used for $\pi\,\pi$, KK, NN, ΛN, ...
- What about $n>2$ hadrons?



Pion scattering



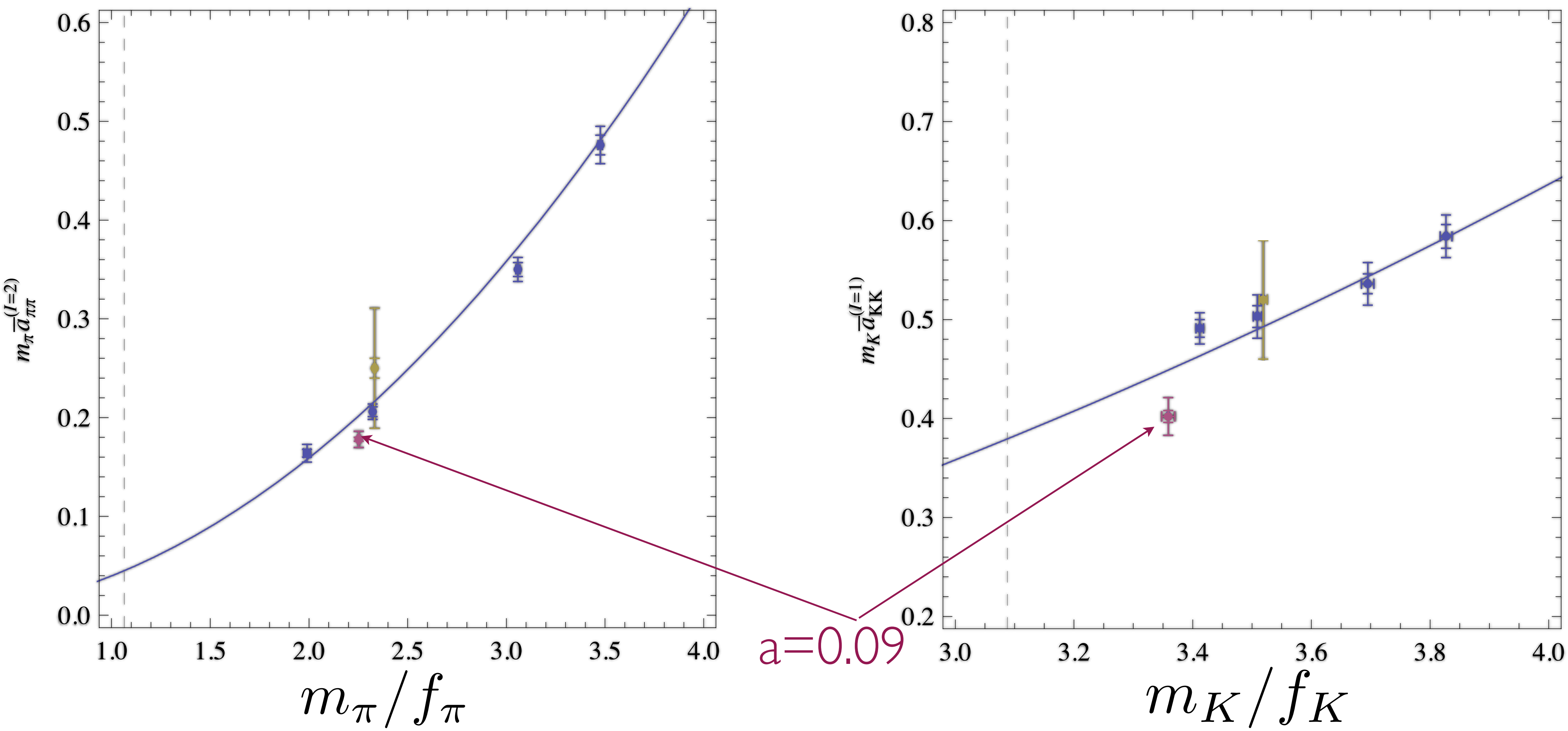
$\pi^+\pi^+\pi^+$ interaction



m_π = 352 MeV

$2\pi^+$ and $2K^-$ interaction

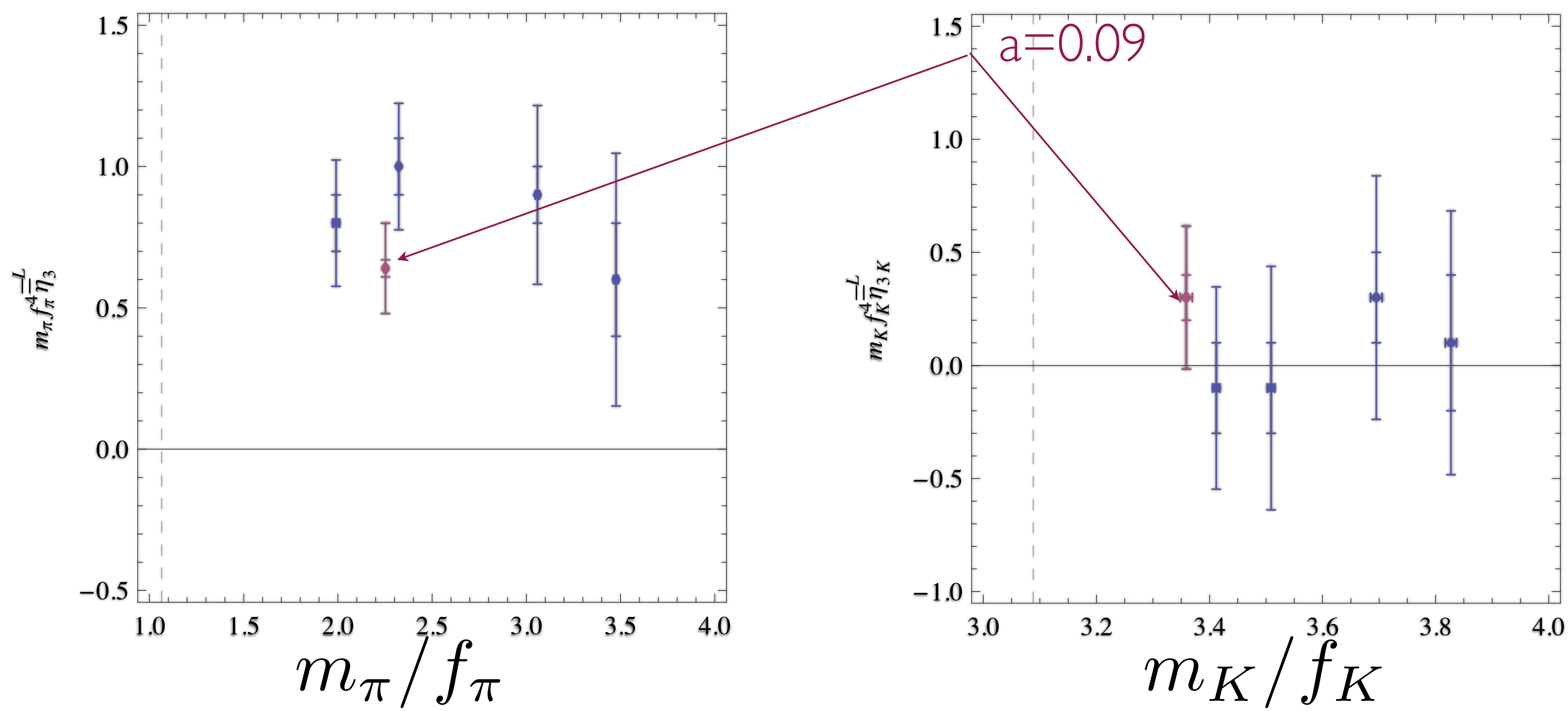
- Scattering lengths



curves: Weinberg

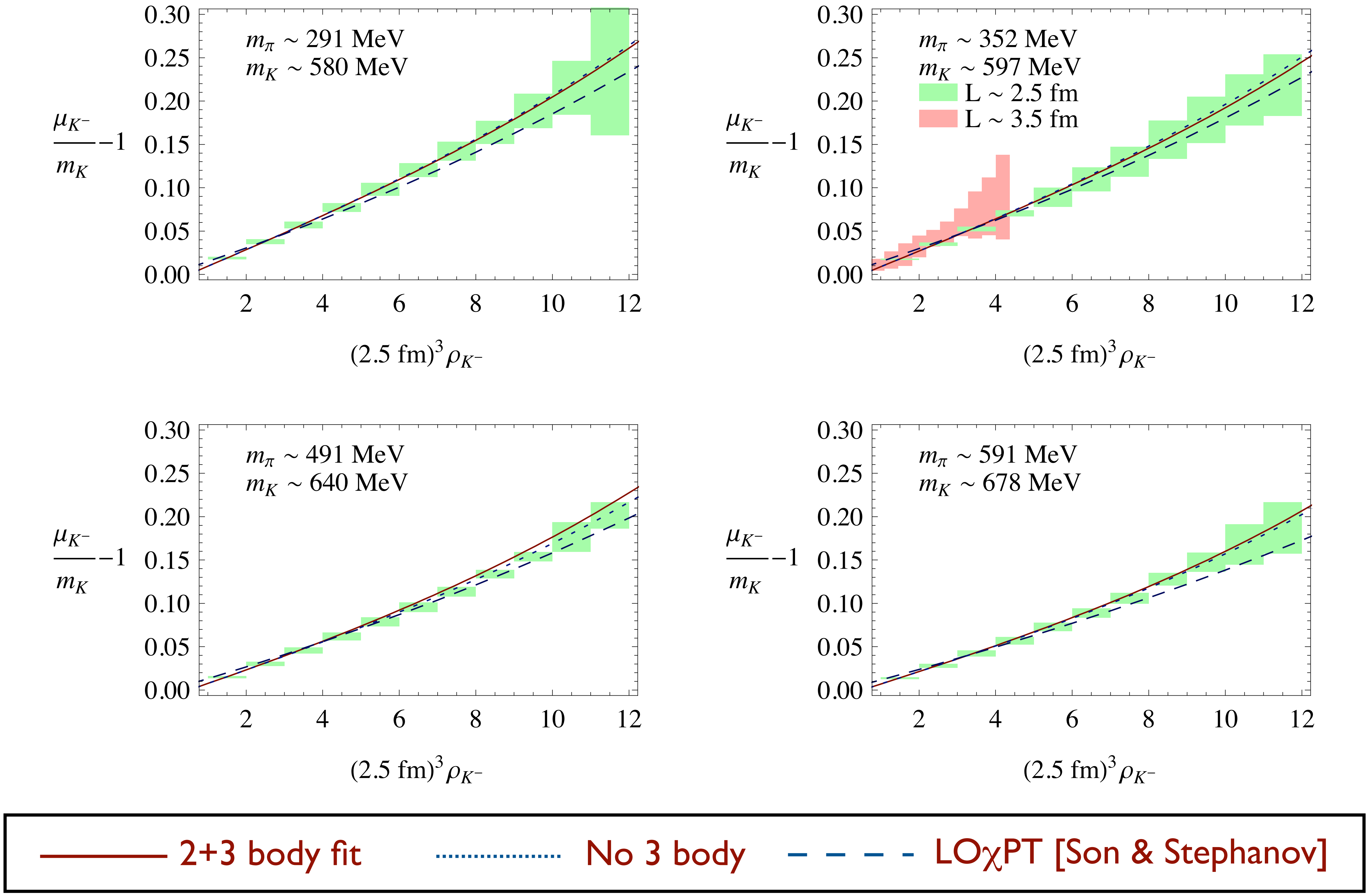
$3\pi^+$ and $3K^-$ interaction

- First QCD three body interaction



Naïve dimension analysis: 1

Kaon Chemical Potential: $\mu = \left.\frac{d\ E}{d\ n}\right|_{V\ \text{const}}$



Colour screening by pions

[WD+ M Savage PRL 09]

- Static quark potential

[Leinweber][Bali *et al.*][Petreczky/Petrov]

Colour screening by pions

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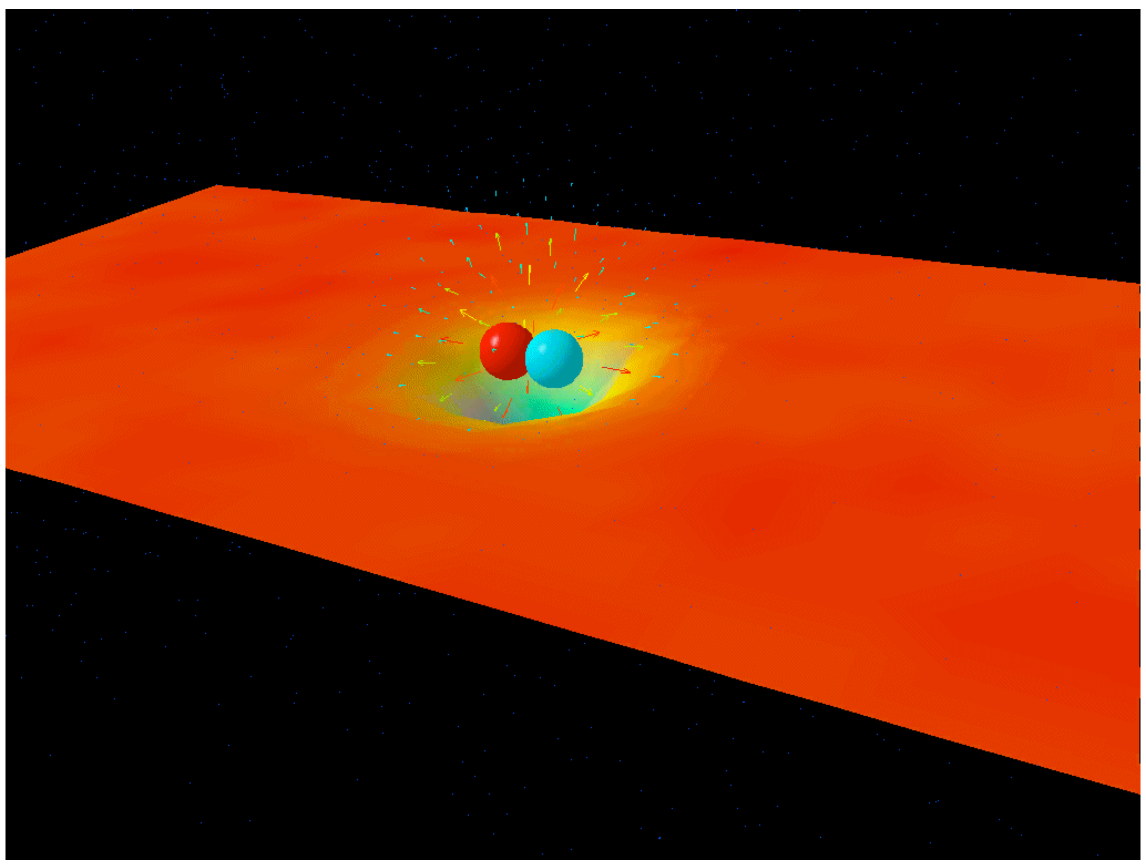

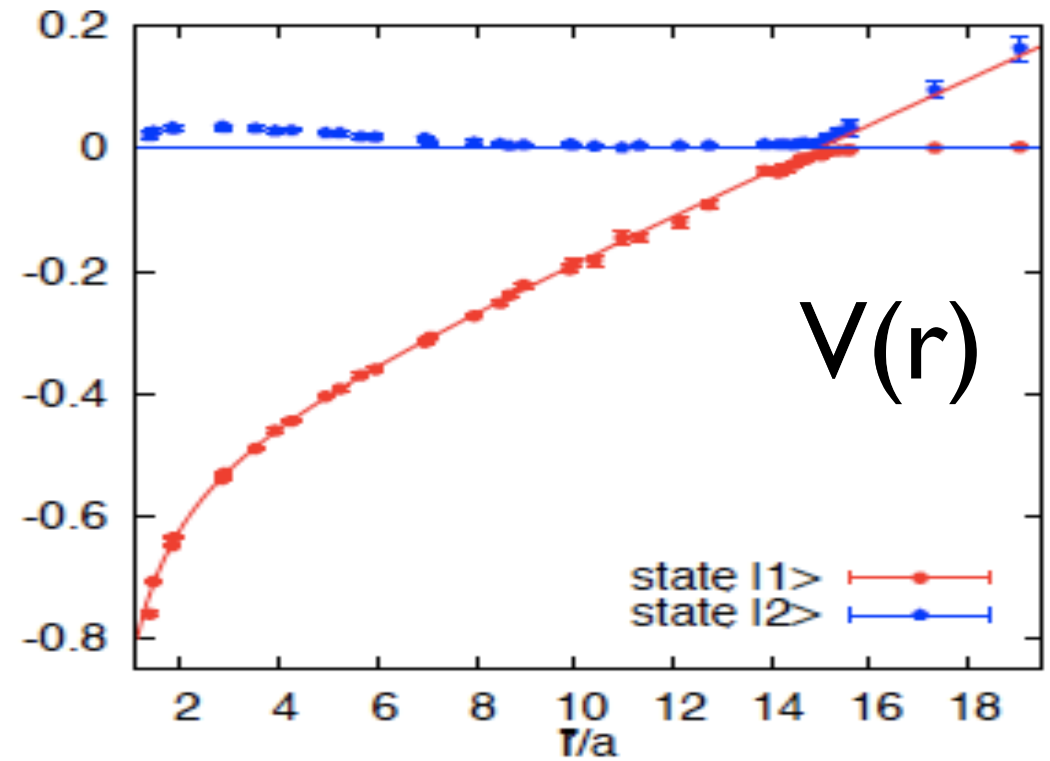

[Leinweber][Bali *et al.*][Petreczky/Petrov]

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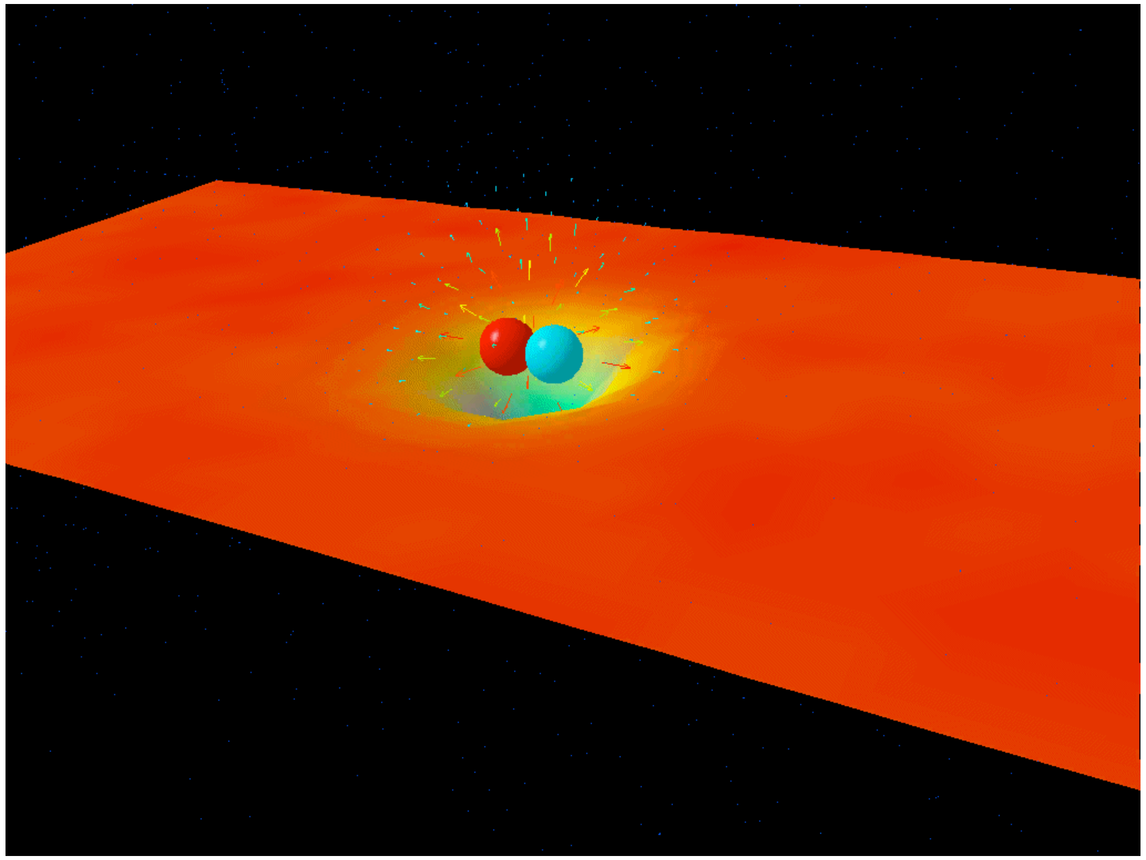
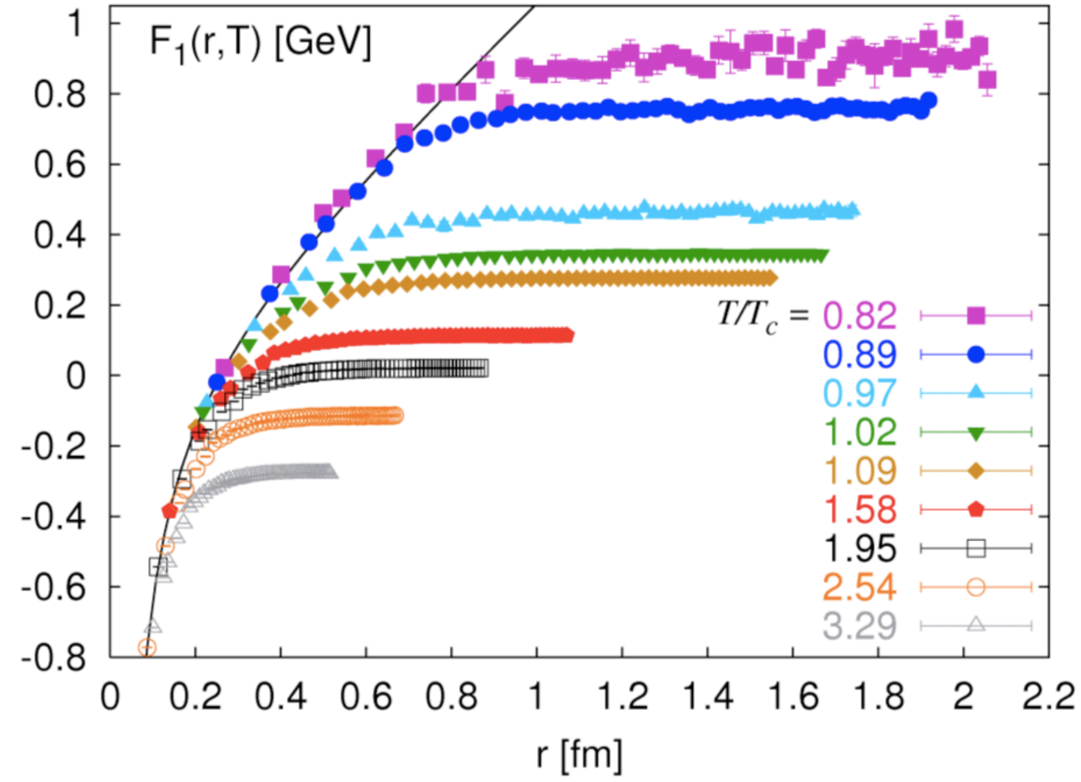




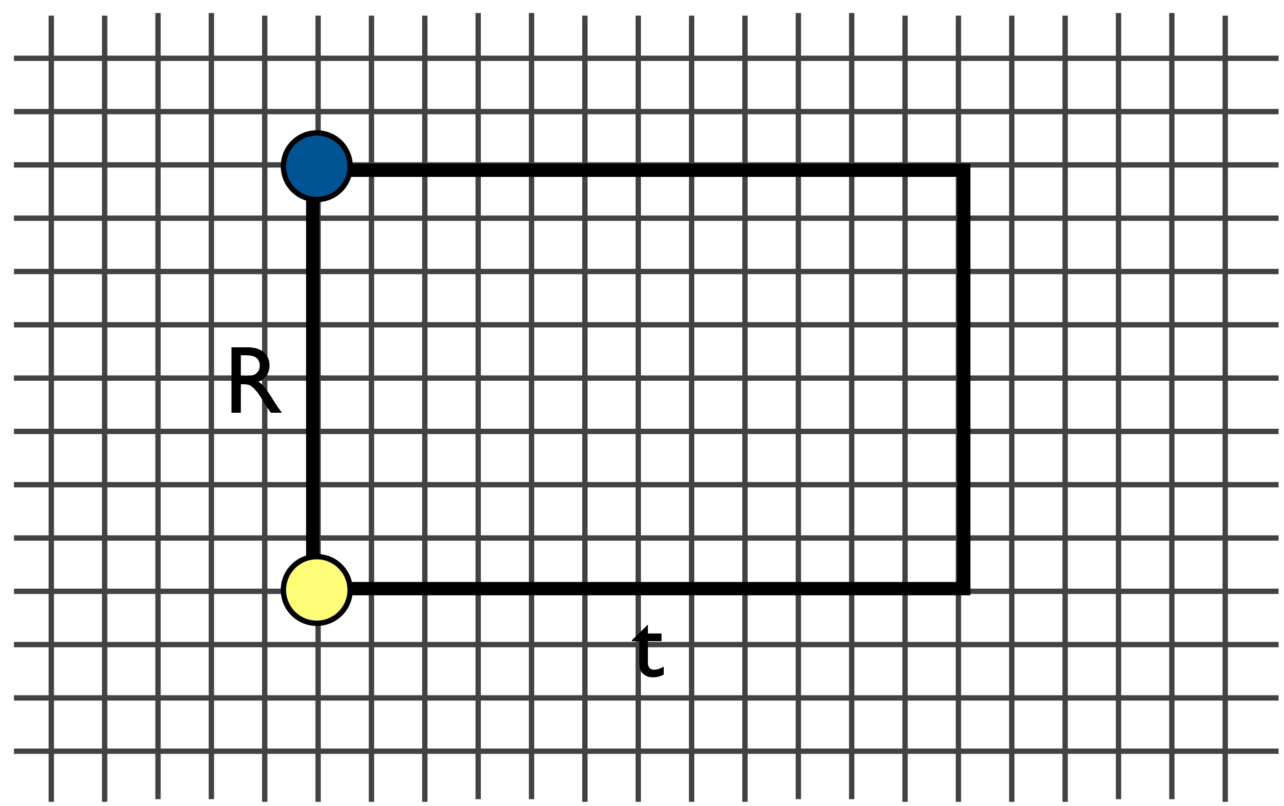
- Screening: evidence for quark-gluon plasma

[Leinweber][Bali *et al.*][Petreczky/Petrov]

Color screening by pion gas

[WD+ M Savage PRL 09]

- Static quark potential



Color screening by pion gas

[WD+ M Savage PRL 09]

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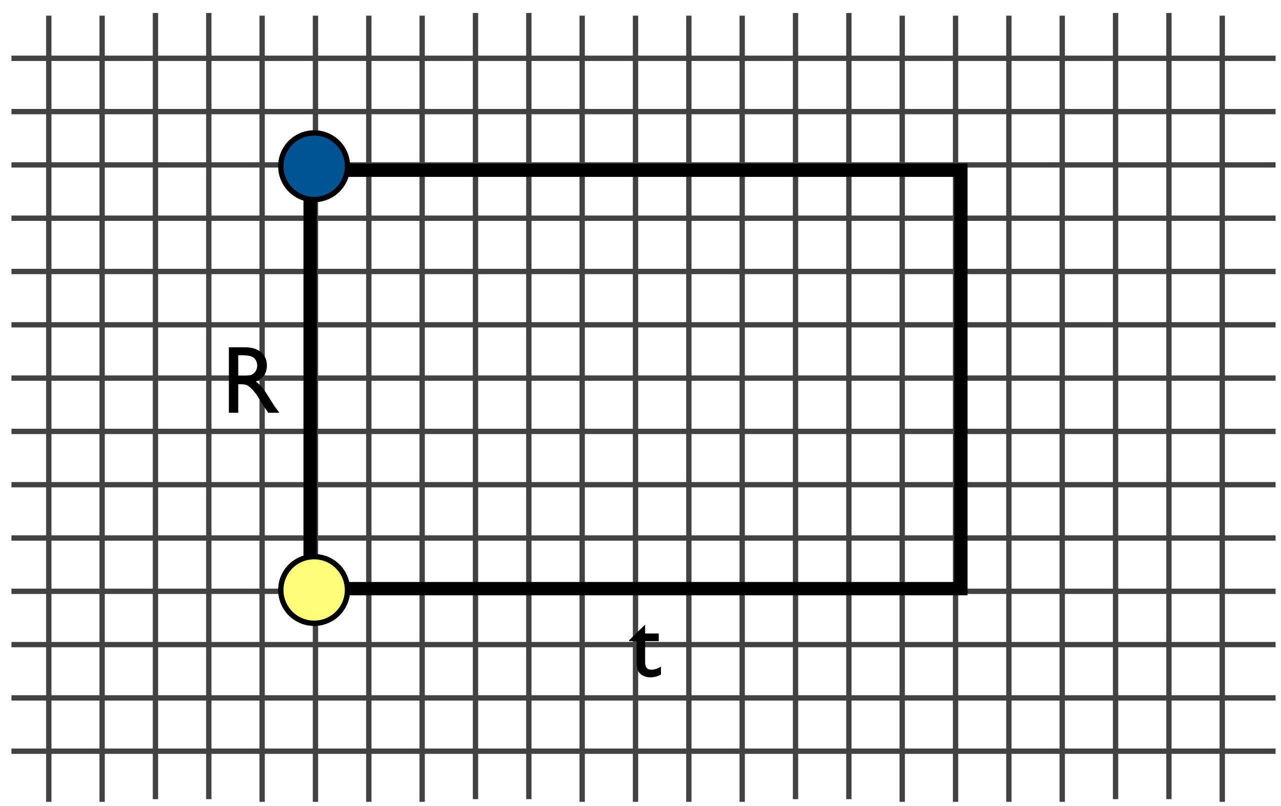
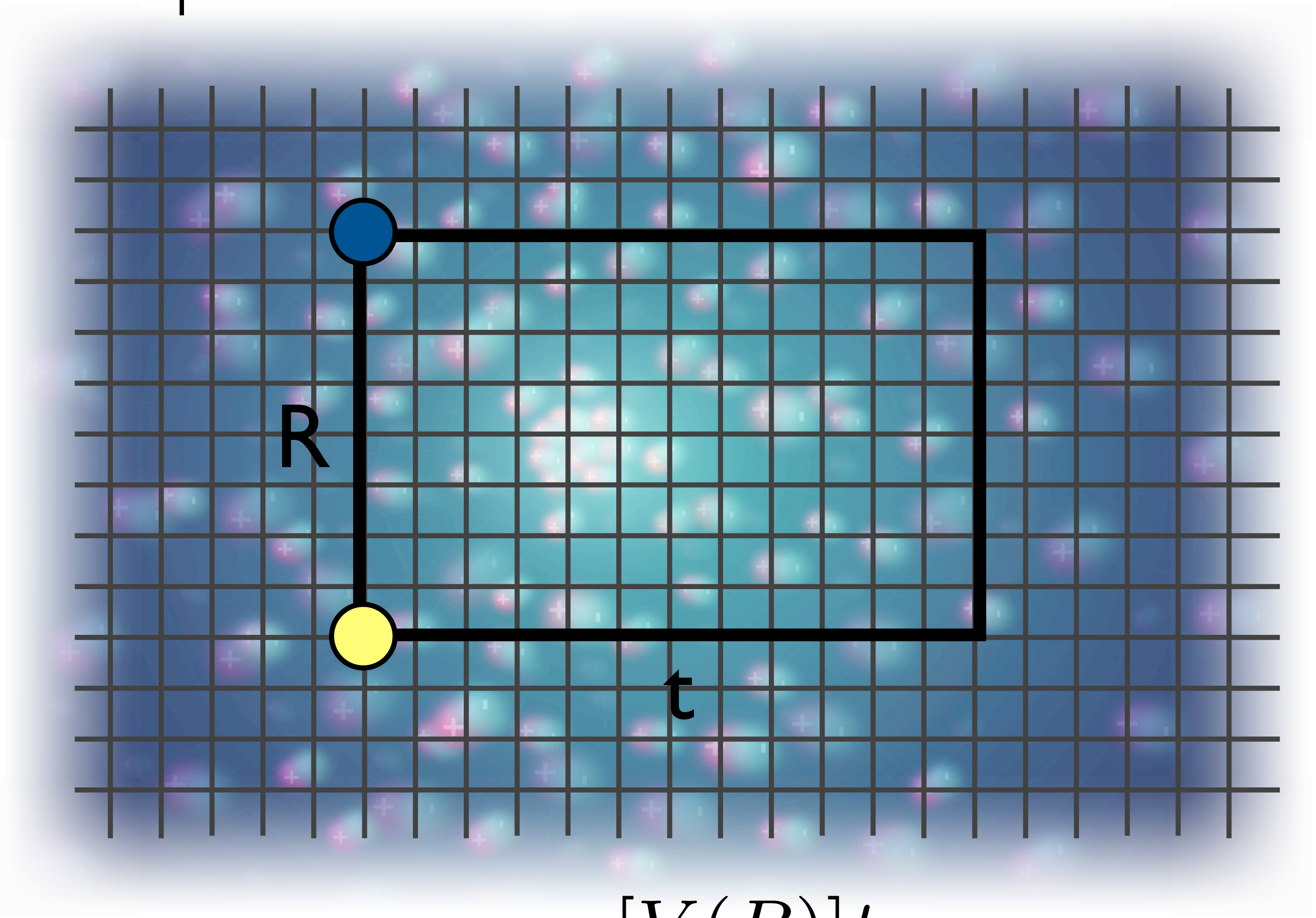


$$\rightarrow \# e^{-[V(R)]t}$$

Color screening by pion gas

[WD+ M Savage PRL 09]

- Static quark potential



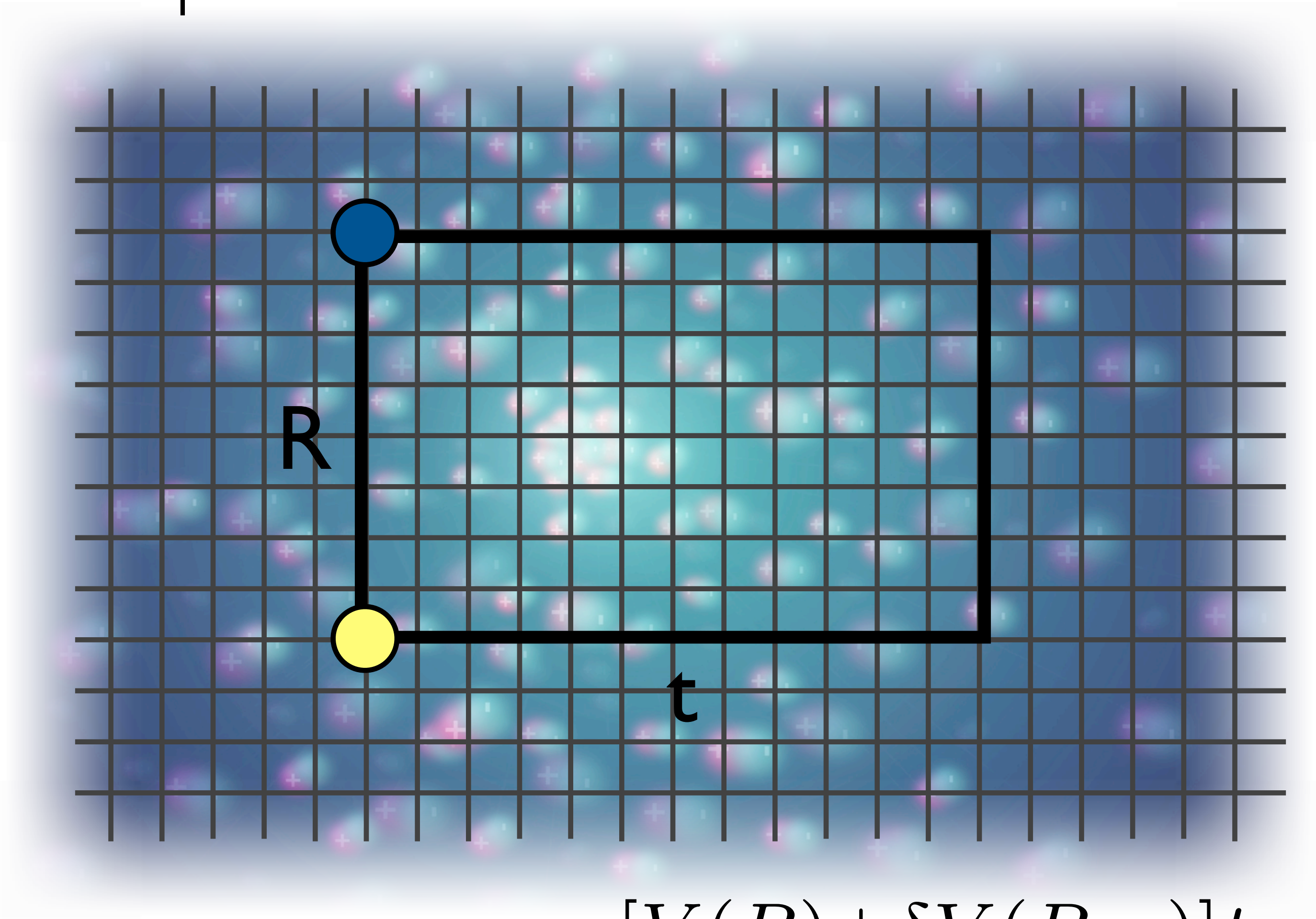
$$\rightarrow \# e^{-[V(R)]t}$$

- Modified by condensate? Hadronic screening?

Color screening by pion gas

[WD+ M Savage PRL 09]

- Static quark potential



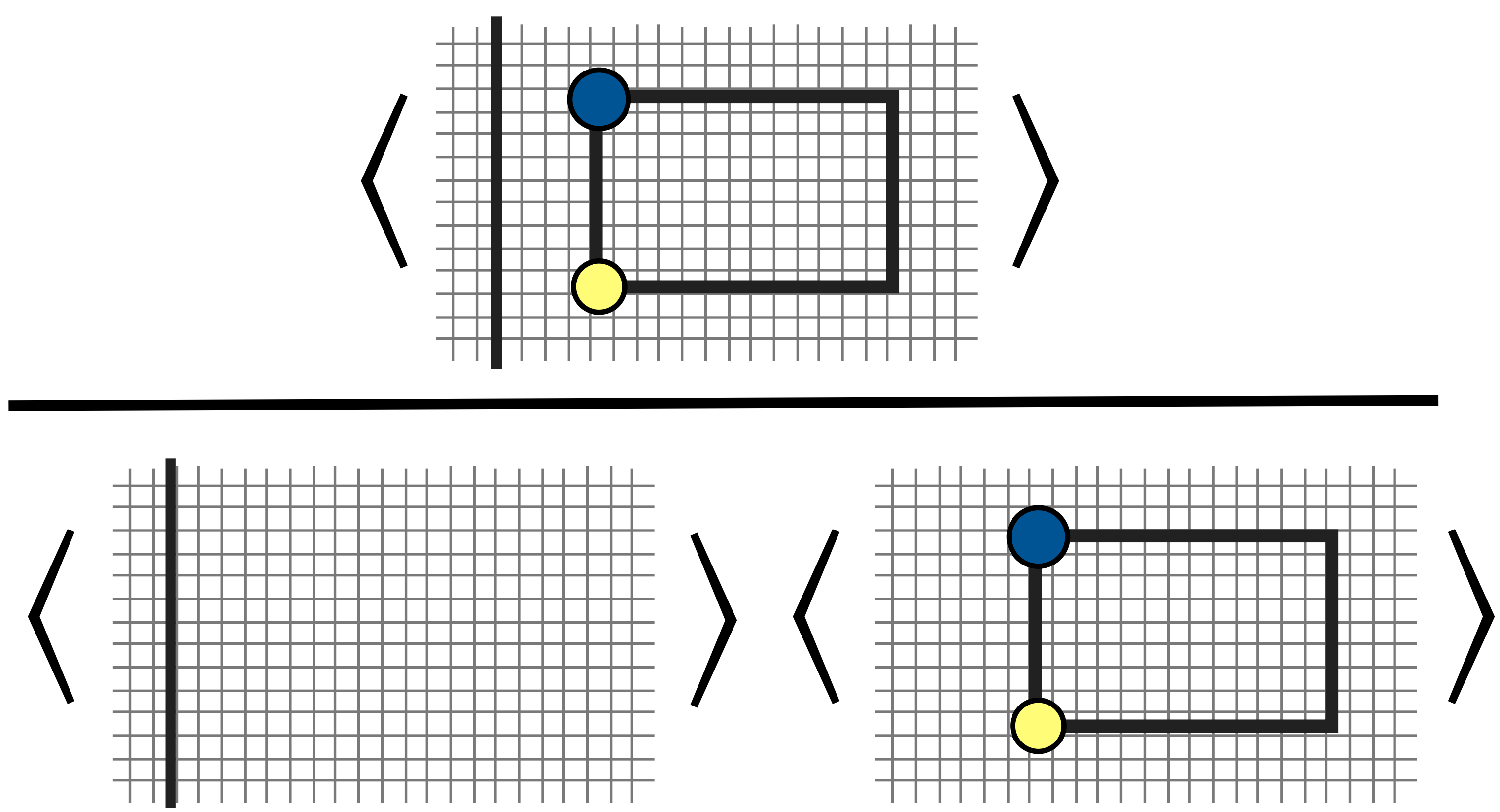
$$\rightarrow \# e^{-[V(R)+\delta V(R,n)]t}$$

- Modified by condensate? Hadronic screening?

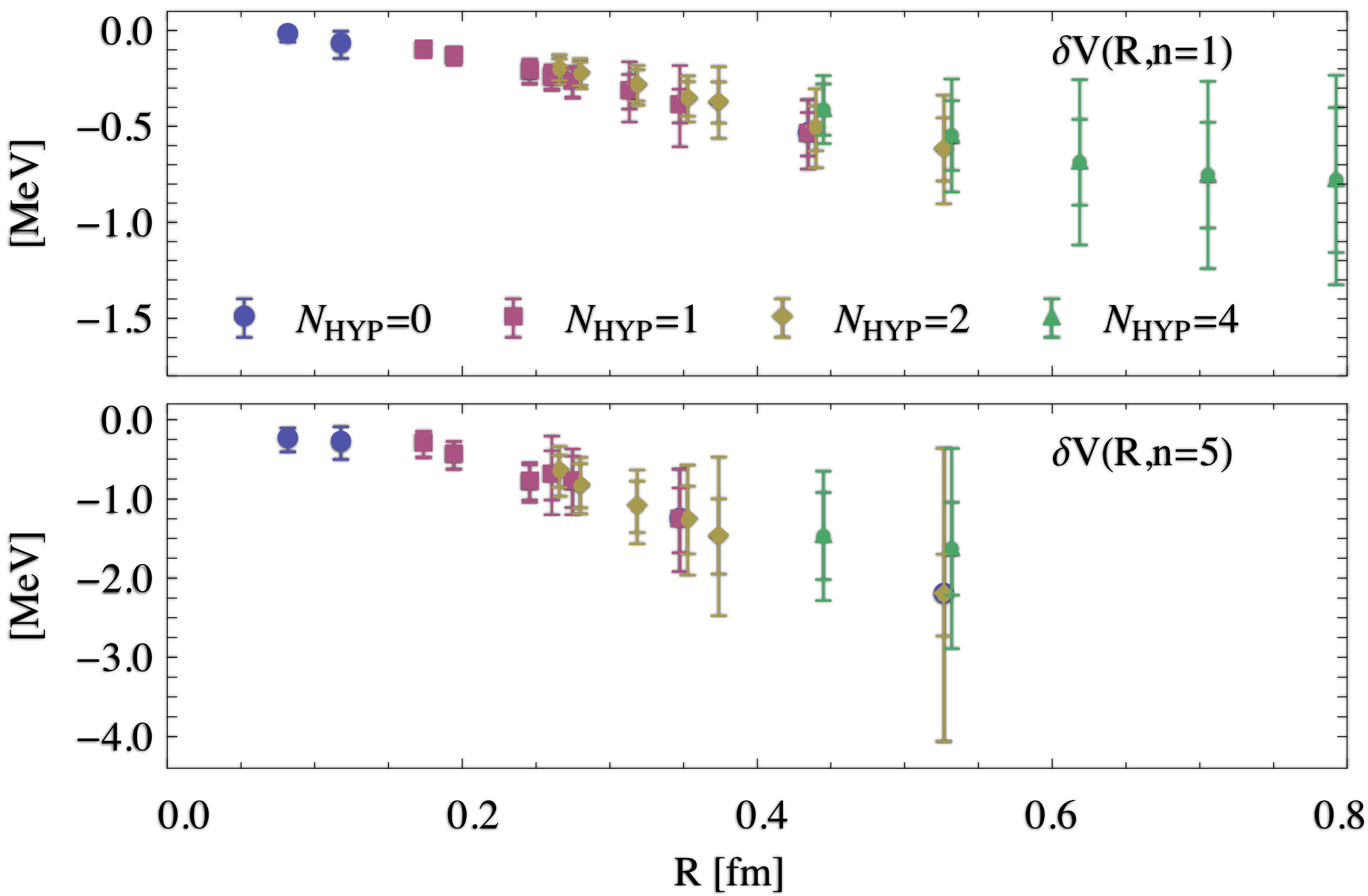
In medium effects

$$G_{n,W}(R,t_\pi,t_W,t) = \frac{C_{n,W}(R,t_\pi,t_W,t)}{C_n(t_\pi,t)C_W(R,t_w,t)}$$

$$\longrightarrow \#\exp\left[-\delta V(R,n)(t-t_w)\right]$$



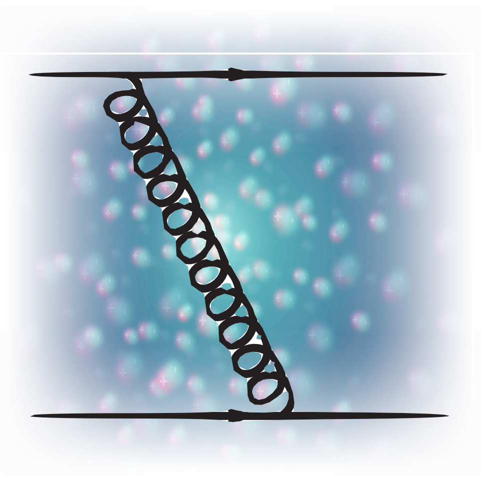
δV(R, n=1 & 5)

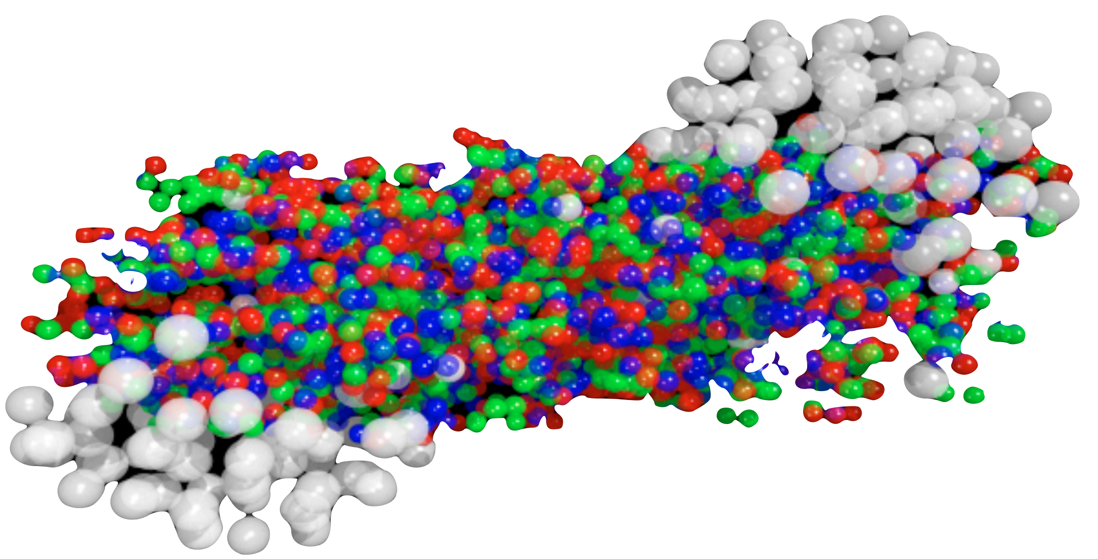


DWF on MILC: a=0.09 fm, 28^3x96, m_π=318 MeV

Pion screening

- r independent shift in $Q\bar{Q}$ force
  - Dielectric medium inside flux tube
  - Small effect: $\delta F(n=1)/F = 0.002$ at large R
- Hadronic medium effect
- Relevance to *J*/ψ suppression @ SPS/RHIC?





Mixed species case

$$
\begin{aligned}
\Delta \tilde{E}_{\pi K}(n,m,L) \quad = \quad & \frac{2\pi \bar{a}_{\pi K} mn}{m_{\pi K} L^3} \left[1 - \left(\frac{\bar{a}_{\pi K}}{\pi L} \right) \mathcal{I} + \left(\frac{\bar{a}_{\pi K}}{\pi L} \right)^2 \right. \\
& \times \left(\mathcal{I}^2 + \mathcal{J} \left[-1 + \frac{\bar{a}_\pi}{\bar{a}_{\pi K}} (n-1) \left(\frac{1}{m_{\pi K}} + \frac{2}{m_\pi} \right) \right. \right. \\
& \left. \left. + \frac{\bar{a}_K}{\bar{a}_{\pi K}} (m-1) \left(\frac{1}{m_{\pi K}} + \frac{2}{m_K} \right) \right] \right) + \left(\frac{\bar{a}_{\pi K}}{\pi L} \right)^3 \\
& \left(-\mathcal{I}^3 + f^{\mathcal{K},\pi K} \left(\frac{\bar{a}_\pi \bar{a}_K}{\bar{a}_{\pi K}^2} \right) \mathcal{K} \right. \\
& \left. \left. + \sum_{i=0}^{2} \sum_{p=\pi,K} \left(f_i^{\mathcal{I}\mathcal{J},p} \mathcal{I}\mathcal{J} + f_i^{\mathcal{K},p} \mathcal{K} \right) \left(\frac{\bar{a}_p}{\bar{a}_{\pi K}} \right)^i \right) \right] \\
& + \frac{nm(n-1)\bar{\eta}_{3,\pi\pi K}(L)}{2L^6} + \frac{nm(m-1)\bar{\eta}_{3,\pi K K}(L)}{2L^6} + \mathcal{O}(L^{-7}).
\end{aligned}
$$

[Smigielski & Wasem '08]