Step-scaling with off-shell renormalisation

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RBC and UKQCD Collaborations

Work in collaboration with P. A. Boyle

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Outline

- Rome-Southampton renormalization - best practice.
- Removing O(4) breaking effects.
- Control perturbative errors by raising scale.
- Step scaling with RI/MOM

Rome-Southampton non-perturbative renormalization

- Based on “A General Method For Non-Perturbative Renormalization of Lattice Operators”, Martinelli et al. hep-lat/9411010.
- Physically defined momentum scheme
  $\rightarrow$ regularization independent (RI/MOM) scheme.
- Why Bother? Scheme dependent quantities from lattice.

$$\text{eg. OPE} \rightarrow \sum_i C_i \langle O_i \rangle$$

Rome-Southampton non-perturbative renormalization

- Consider bilinears only in this talk $O(z) = \bar{q}(z)\Gamma q(z)$.
- $G_O(p_1, p_2) = \frac{1}{V}\sum_{x,y,z} e^{(ip_1 x - p_2 y)}\langle q_\alpha(x) O(z) \bar{q}_\beta(y)\rangle$.
- Project (with P) operator vertex onto tree level
  different momenta and projectors define different schemes.
- $\Lambda_O(p_1, p_2) = \frac{1}{12} tr\left[\langle S^{-1}(p_1)\rangle G_O(p_1, p_2)\langle S^{-1}(p_2)\rangle P\right] = \frac{Z_q}{Z_O}$

Point sources

- The original method used point sources.
- Solve $\sum_z D(x,z)S(z,y) = \delta(x-y)$
- However we don't need $S(x,y)$ only $S(x,p)$

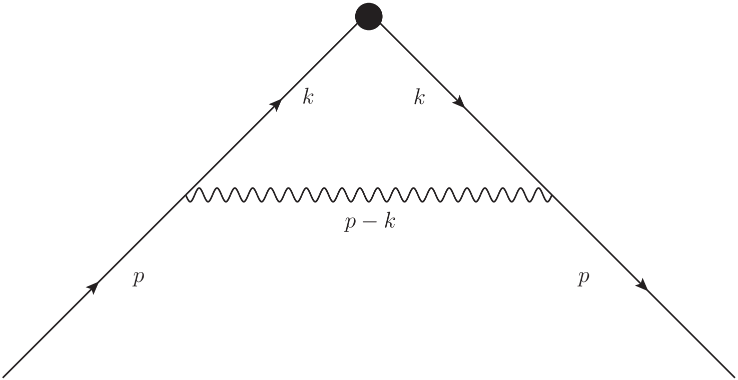
Volume sources

- Solve $\sum_z D(x,z)S(z,p) = e^{ipx}$ instead
  M. Gockeler et al., hep-lat/9807044
- $\sum_y S(x,y)e^{ipy} = S(x,p)$ sum over all point source positions

Improves statistics and reveals systematic errors.

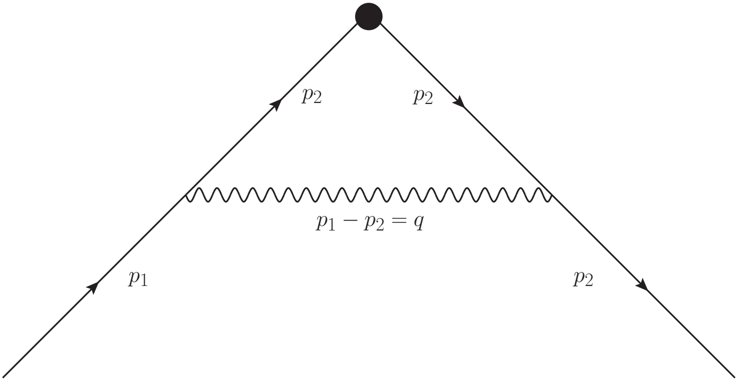
- Chiral Symmetry $\rightarrow Z_A = Z_V \ , Z_S = Z_P$.
- Not observed due to spontaneous chiral symmetry breaking.

Exceptional Momenta



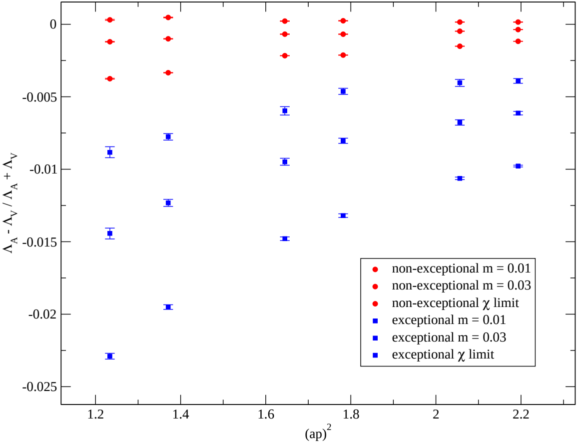
Aoki et al. hep-lat/0712.1061

Non-Exceptional Momenta

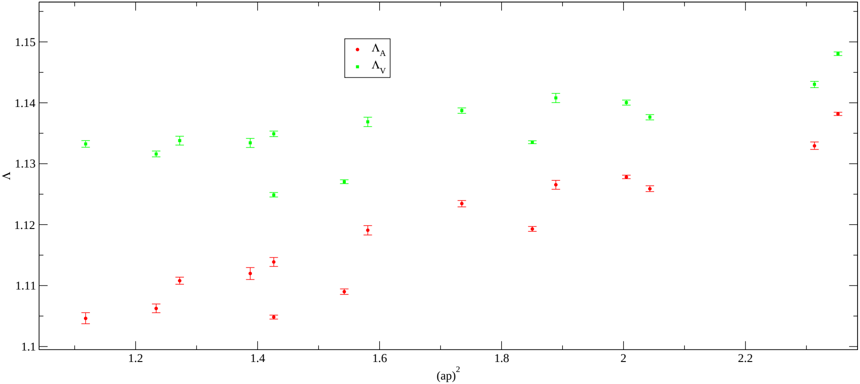


Aoki et al. hep-lat/0712.1061

$$\frac{\Lambda_A - \Lambda_V}{\Lambda_A + \Lambda_V}$$



O(4) Breaking



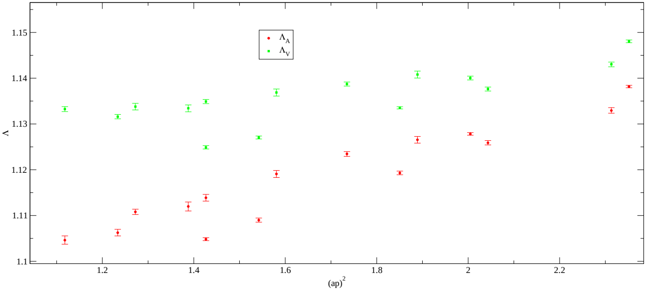
With exceptional kinematics

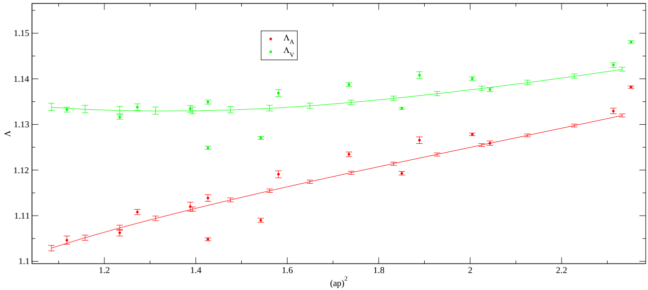
O(4) Breaking

- To access different momenta we use different Fourier modes
- But different fourier modes are different observables with different lattice artefacts.
- Symanzik expansion differs, no controlled continuum extrapolation possible.

Twisted Boundary conditions

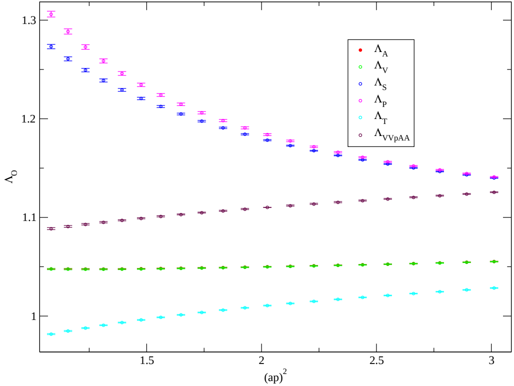
- P.A. Boyle hep-lat/0309100 , Bedaque nucl-th/0402051.
- $q(x+L) = e^{iBL} q(x)$
- $\not{\partial} \to \not{\partial} + \not{B}$
- $S(x,p) \to S(x, p+B)$
- For example: $p = (0,1,1,0)$ $B = \frac{\pi}{L}(0,\theta,\theta,0)$


Λ_A
Λ_V
1.1
1.11
1.12
1.13
1.14
1.15
Λ
1.2
1.4
1.6
1.8
2
2.2
$(ap)^2$



- Still have $O(a^2)$ errors, but with **the same** coefficient.
- Now has a well defined Symanzik expansion.

Bilinear vertex functions.

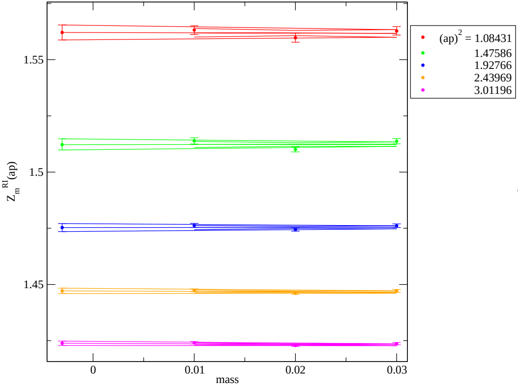


$16^3 \times 32 \times 16$
Domain wall fermion action
Iwasaki gauge action
$am_s = 0.04$.

NPR Summary

- Momentum sources give $< 0.1\%$ stat error.
- Non-Exceptional kinematics reduce effects of spontaneous chiral symmetry breaking.
- Twisted boundary conditions remove $O(4)$ breaking scatter.

Example: quark mass renormalization.



$16^3 \times 32 \times 16$
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$am_s = 0.04$.

Non-exceptional momenta enable a benign chiral extrapolation.

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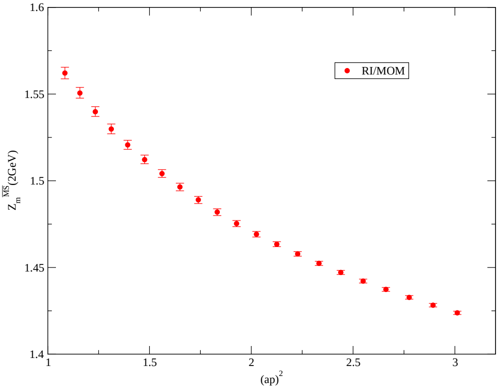

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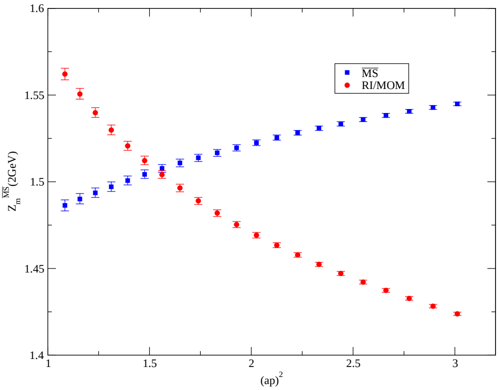

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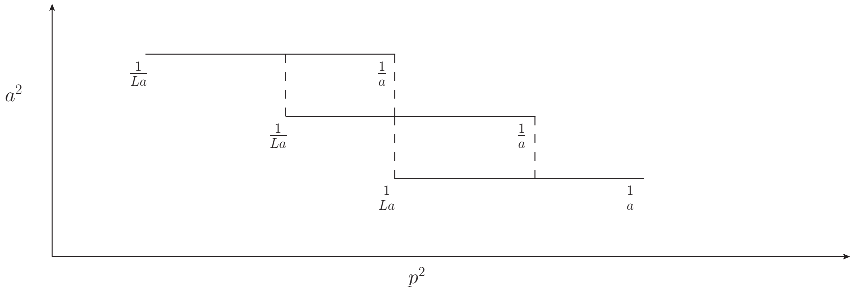
Why Step Scaling ?

- Now a controlled and precise continuum extrapolation of $Z_O \langle O \rangle$ possible
- Formerly, required $\Lambda^2_{QCD} << p^2 << \left(\frac{\pi}{a}\right)^2$ on each lattice
- Now perturbative and discretization error can be disentangled.
- Use continuum extrapolation to control a^2 errors.
- Look at perturbative error in continuum.
- Raise the scale without brute force → Step Scaling.

Step Scaling

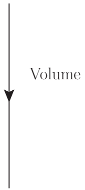


- Sufficient virtuality to not see finite volume.
- $\left(\frac{\pi}{La}\right)^2 << p^2 << \left(\frac{\pi}{a}\right)^2$
- A series of overlapping scaling windows.
- Only need convergence of perturbation theory after scale evolution.

Step Scaling

- $R_O(p, a) = \frac{\Lambda_A}{\Lambda_O} = \frac{Z_O(p,a)}{Z_A(a)}$
- $\Sigma_O(p, sp, a) = \frac{R_O(sp,a)}{R_O(p,a)} = \frac{Z_O(sp,a)}{Z_O(p,a)}$
- s is a scale factor $\simeq 1.5$
- $\lim_{a\rightarrow 0} \Sigma_O(p, sp, a) = \sigma(p, sp) = \frac{Z_O(sp)}{Z_O(p)}$
- $\frac{Z_O(sp)}{Z_O(p)} = exp\left(\int_{\alpha(p)}^{\alpha(sp)} dx \frac{\gamma_O(x)}{\beta(x)}\right)$

Step Scaling



$$
\begin{aligned}
\langle O^{\overline{MS}}(\mu)\rangle &= \langle O^{SMOM}(p)\rangle \sigma(p, sp) \\
&\times \sigma(sp, s^2 p) \\
&\times \dots \\
&\times \sigma(s^{n-1}p, s^n p) \\
&\times C_O^{SMOM \to \overline{MS}}(\mu)_{\{\mu = s^n p\}}
\end{aligned}
$$

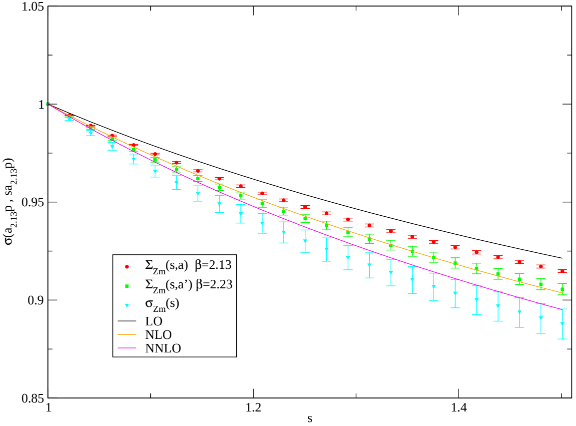
Step Scaling

- Multiple simulations and high energy required
  $\rightarrow$ small volumes.
- Small volume need to use non-hadronic quantity to set scale.
  eg. Schrodinger functional scheme uses $g(\frac{1}{L})$.
- Coupling shrinks as volume reduced.
  Always a finite volume safe observable.
- Trajectory of Σ to continuum has $g(\frac{1}{L})$ fixed
  $\rightarrow$ Match β to ensure quantities join in continuum limit.
- Renormalization scale coupled to volume.

Step Scaling

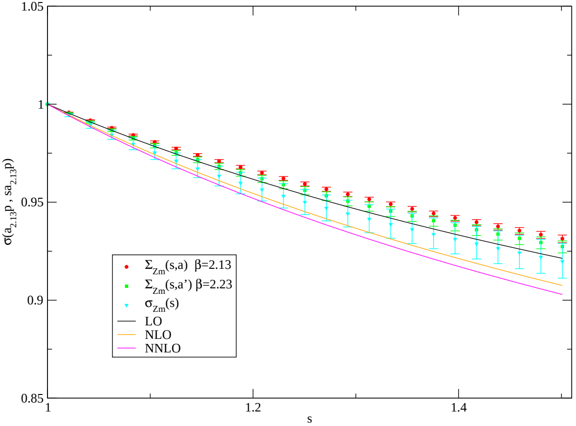
- Consider a series of Sommer scales $r_0, r_1, \ldots$
- $r_n = \frac{r_0}{s^n}$
- $r_n^2 F(r_n) = C_n$ determines r_n
- To ensure scales join up measure C_{n+1} in the continuum limit with scales set by C_n.
- $\lim_{a \to 0} \frac{r_n^2}{s^2} F(\frac{r_n}{s}) = C_{n+1}$
- Scale set using C_{n+1} agrees in continuum with scale set via C_n
- Momentum continuously tunable
  $\longrightarrow$ no need to match β exactly.
- Renormalization scale decoupled from volume.

First Step - $\sigma_{Z_m}(p, sp)$



Perturbation theory from Phys. Rev. D **80** (2009) Aoki et al.$SMOM_{\gamma_\mu}$ scheme

First Step - $\sigma_{Z_m}(p, sp)$



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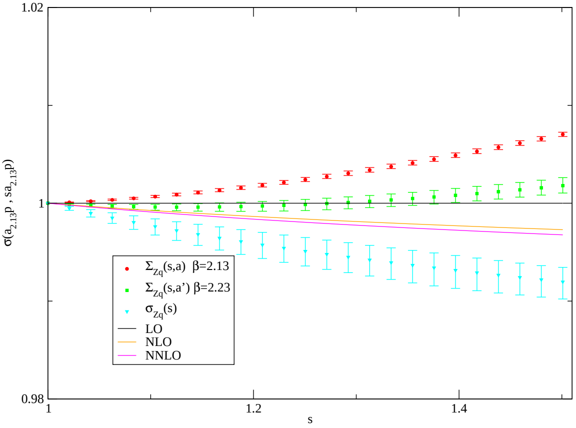
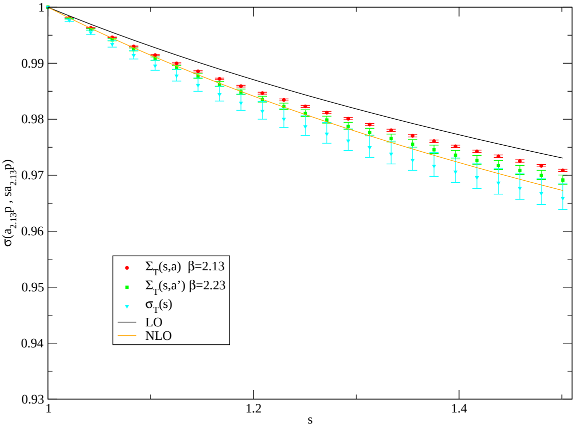
SMOM scheme

First Step - $\sigma_{Z_q}(p, sp)$



$SMOM_{\gamma_\mu}$ scheme

First Step - $\sigma_{Z_T}(p, sp)$



$SMOM_{\gamma_\mu}$ scheme

Summary

- Our combination of volume sources, non-exceptional momenta and twisted boundary conditions enable precise continuum extrapolations.
- Uncertainty from perturbation theory remains.
- A step scaling scheme based on NPR is feasible and can be implemented relatively cheaply.
- The first step has been taken and the results are promising.
- Insofar as the RI/MOM method is general so is our step scaling method. It works for any operator.

Outlook

Compute scale evolution functions for a complete set of fermion operators in steps

$$2 Gev \rightarrow 3 Gev \rightarrow 4.5 GeV$$