# Low-energy constants from Dirac eigenvalue correlators at NNLO in the epsilon expansion

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# Outline

# The $\varepsilon$ expansion

We consider

- Lattice QCD: Euclidean QCD at finite volume $V$
- Small temperatures: Theory dominated by pions

# The $\varepsilon$ expansion

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- Lattice QCD: Euclidean QCD at finite volume $V$
- Small temperatures: Theory dominated by pions
- Small quark masses

Compton wavelength of pions large compared to volume $V$, i.e.,

$$\sqrt{V} m_\pi^2 < 1 \,.$$

QCD described by $\chi$PT in the $\varepsilon$ expansion

# The $\varepsilon$ expansion

The $\varepsilon$ expansion (systematic expansion in $1/F^2\sqrt{V}$)

Leading order:

- Described by random matrix theory (RMT)
- Dirac eigenvalue distributions in RMT known
- Fit Dirac eigenvalue distributions: Extract LECs of $\chi$PT

NLO:

- Finite-volume corrections to LECs

NNLO:

- Systematic deviations from RMT

# JLQCD lattice data

Setup:

- $a = 0.107(3)$ fm, $32 \times 16^3$ lattice points, $V^{1/4} \approx 1.7$ fm
- Overlap fermions: $N_f = 2$, $am_u = am_d = 0.002$, $m_\pi^2 \sqrt{V} \approx 1$
- Sea quarks at zero chemical potential
- Valence quarks at zero and nonzero imaginary chemical potential

Fit to RMT Dirac eigenvalue distributions:

- Extract leading-order LECs $\Sigma$ and $F$ (Normalization: $F_{\text{experiment}} \approx 90$ MeV)

# JLQCD lattice data

JLQCD: Fit to cumulant Dirac eigenvalue distribution:

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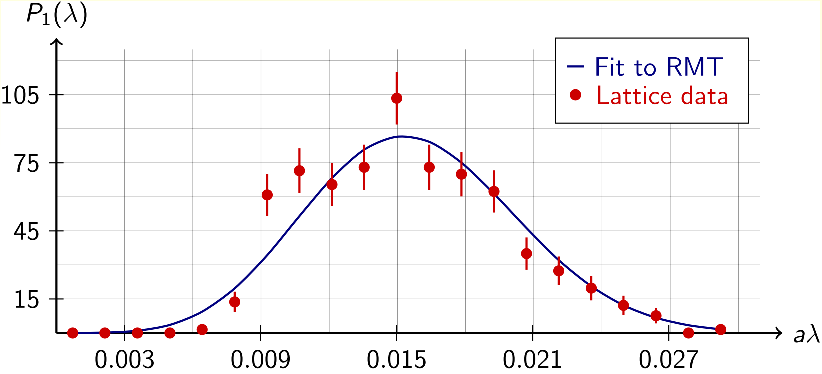
Check: Fit to distribution of lowest Dirac eigenvalue:




$$\chi^2/\mathrm{dof} = 2.9\,, \qquad a^3\Sigma^{\mathrm{RMT}} = 0.00208(2)\,.$$

JLQCD: Meson correlators (Fukaya et. al. 2008):

$$F^{\text{meson}} = 87(6)\ \text{MeV}$$

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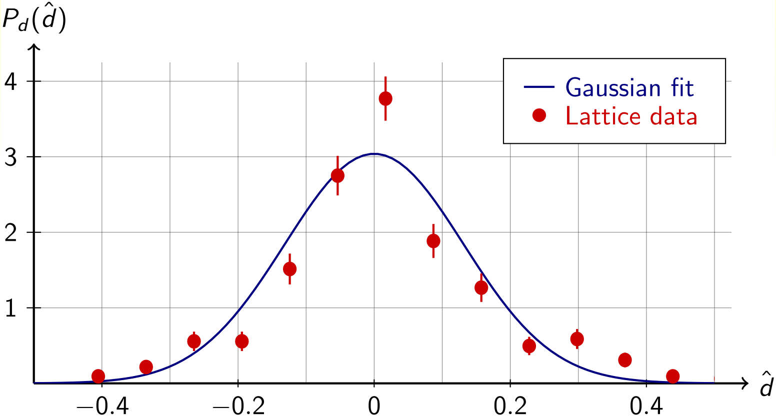
Compare: Dirac eigenvalue shift due to imaginary chemical potential $a\mu = 0.01$ ($\hat{d} = d\Sigma V$):




RMT prediction: Gaussian distribution, $\sigma^2 = \mu^2 F^2 V$

$$\chi^2/\text{dof} = 4.2\,, \qquad F^{\text{RMT}} = 67(5)\ \text{MeV}\,.$$

# JLQCD lattice data

Result from meson correlators

$$F^{\text{meson}} = 87(6)\ \text{MeV}$$

and result from Dirac eigenvalue shift due to imaginary chemical

$$F^{\text{RMT}} = 67(5)\ \text{MeV}$$

do not agree!

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do not agree!

Include leading-order $1/F^2\sqrt{V}$ corrections, i.e., calculation at NLO in $\varepsilon$ (Damgaard et. al. 2007, Akemann et. al. 2008, Lehner et. al. 2009):

$$F^{\text{NLO}} = 51(4)\ \text{MeV}$$

Even worse agreement

# JLQCD lattice data

Problem:

- $\chi^2/\text{dof}$ of fits is large
- Systematic deviations from RMT render this method to determine $F$ useless

This talk:

- Understand the systematic errors
- Solve the discrepancy regarding $F$

# The $\varepsilon$ expansion at NNLO (Lehner et. al. 2010)

- Power counting:

$$V \sim \varepsilon^{-4}\,, \qquad \partial_\rho \sim \varepsilon\,, \qquad \pi(x) \sim \varepsilon\,,$$
$$m_\pi \sim \varepsilon^2\,, \qquad \mu \sim \varepsilon^2$$

  with pion fields $\pi(x)$ and mass $m_\pi$, and chemical potential $\mu$.
- Calculate finite-volume effective action $S_{\text{eff}}$ in terms of zero-momentum mode of Nambu-Goldstone manifold $U_0$ (diagrams generated by computer algebra, calculate special version of two-loop finite-volume sunset diagram).
- Compare resulting $S_{\text{eff}}$ to RMT action

$$S_{\text{eff}}^{\text{LO/RMT}} = -\frac{V\Sigma}{2}\,\text{Tr}(M^\dagger U_0 + U_0^{-1} M) - \frac{VF^2}{2}\,\text{Tr}(C U_0^{-1} C U_0)$$

  with quark mass matrix $M$ and quark chemical potential matrix $C$.

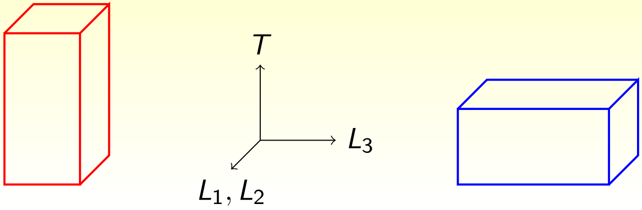
# The $\varepsilon$ expansion at NNLO (Lehner et. al. 2010)

- Action at NNLO:

$$S_{\text{eff}}^{\text{NNLO}} = -\frac{V\Sigma_{\text{eff}}^{\text{NNLO}}}{2}\,\text{Tr}(M^\dagger U_0 + U_0^{-1}M) - \frac{V(F_{\text{eff}}^{\text{NNLO}})^2}{2}\,\text{Tr}(CU_0^{-1}CU_0)$$
$$+ \Upsilon_1 \Sigma(VF)^2\,\text{Tr}(C)[\text{Tr}(U_0\{M^\dagger, C\}) + \text{Tr}(U_0^{-1}\{C, M\})]$$
$$\dots$$
$$+ \Upsilon_8 (V\Sigma)^2[\text{Tr}(MU_0^{-1}MU_0^{-1}) + \text{Tr}(M^\dagger U_0 M^\dagger U_0)]$$
$$+ \mathcal{H}_2 (V\Sigma)^2\,\text{Tr}(M^\dagger M)\,.$$

- Finite-volume corrections to $\Sigma$ and $F$
- Non-RMT terms proportional to $\Upsilon_i$ and $\mathcal{H}_2$
- $\Upsilon_i$, $\mathcal{H}_2$, $\Sigma_{\text{eff}}^{\text{NNLO}}$, and $F_{\text{eff}}^{\text{NNLO}}$ depend on:
  - NLO LEC of $\chi$PT
  - Geometry of spacetime box through finite-volume propagators

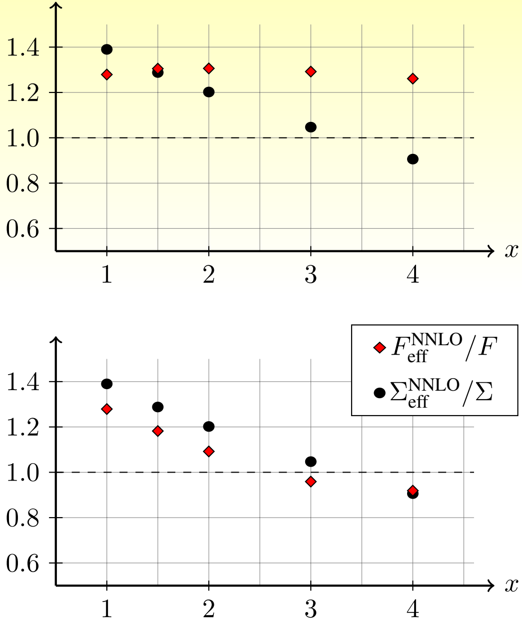
# Different geometries




$$
\begin{array}{lll}
(a_x) & T = xL\,, & L_1 = L_2 = L_3 = L\,, \\
(b_x) & L_3 = xT\,, & T = L_1 = L_2\,.
\end{array}
$$

Reminder: JLQCD uses $(a_2)$

# Finite-volume corrections to $\Sigma$ and $F$




$(a_x)$

$(b_x)$

Parameters:

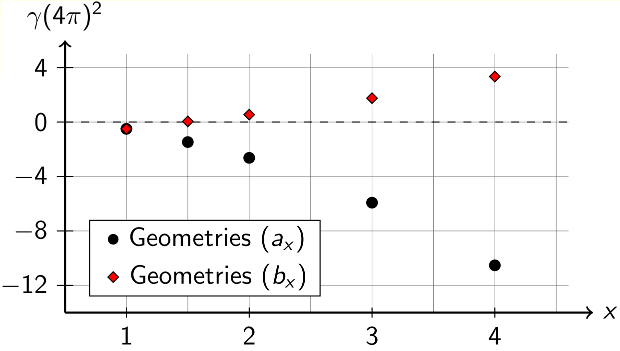
- $m_\pi^2\sqrt{V} = 1$
- $F = 90$ MeV
- $L = 1.71$ fm

# Systematic deviations from RMT

- $\Upsilon_1, \Upsilon_2, \Upsilon_3$ depend on geometry ($a_x$) or ($b_x$) for the same $x$:

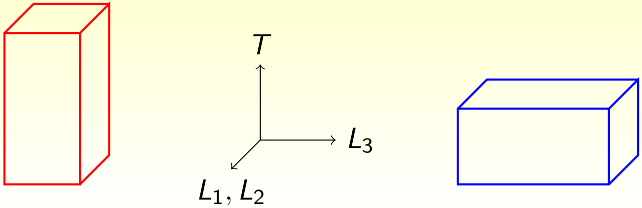
$$\Upsilon_1, \Upsilon_2, \Upsilon_3 \propto \gamma\,,$$

where $\gamma$ contains the dependence on the geometry.




- $\Upsilon_4, \ldots, \Upsilon_8, \mathcal{H}_2$ do not depend on geometry ($a_x$) or ($b_x$) for the same $x$.

# Implications for JLQCD simulation

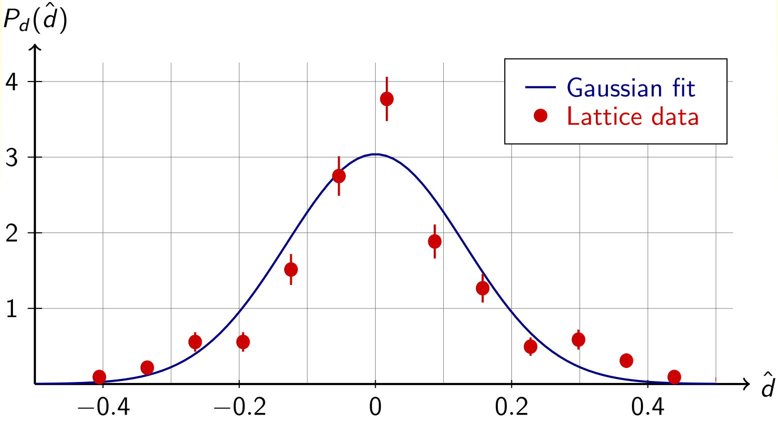



$$(a_x) \qquad T = xL\,, \qquad L_1 = L_2 = L_3 = L\,,$$
$$(b_x) \qquad L_3 = xT\,, \qquad T = L_1 = L_2\,.$$

**Rotate lattice and use $(b_2)$ instead of $(a_2)$**

# Fit for $F$

Geometry ($a_2$): Dirac eigenvalue shift due to imaginary chemical potential $a\mu = 0.01$ ($\hat{d} = d\Sigma V$):

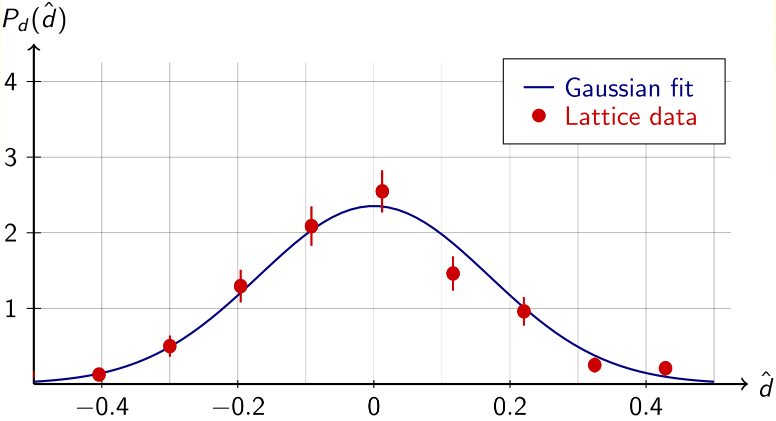



RMT prediction: Gaussian distribution, $\sigma^2 = \mu^2 F^2 V$

$$\chi^2/\text{dof} = 4.2\,, \quad F^{\text{NLO}} = 51(4)\ \text{MeV}\,, \quad F^{\text{meson}} = 87(6)\ \text{MeV}\,.$$

# Fit for $F$

Geometry ($b_2$): Dirac eigenvalue shift due to imaginary chemical potential $a\mu = 0.01$ ($\hat{d} = d\Sigma V$):




RMT prediction: Gaussian distribution, $\sigma^2 = \mu^2 F^2 V$

$$\chi^2/\text{dof} = 0.91\,, \quad F^{\text{NLO}} = 80(5)\ \text{MeV}\,, \quad F^{\text{meson}} = 87(6)\ \text{MeV}\,.$$

**Conclusions (JHEP06(2010)028):**

- The NNLO corrections are responsible for the bad fits to Dirac eigenvalue correlators
- They can be minimized by a suitable choice of lattice geometry
- LECs from JLQCD lattice configurations:

$$\Sigma^{\mathrm{NNLO}} = (221(6)\ \mathrm{MeV})^3\,,$$
$$F^{\mathrm{NNLO}} = 80(5)\ \mathrm{MeV}\,,$$
$$F^{\mathrm{meson}} = 87(6)\ \mathrm{MeV}\,.$$

**Outlook (in preparation):**

- Calculation of spectral density in epsilon expansion beyond RMT
- A more precise statement of the Thouless energy

Backup slides

# Finite-volume corrections to $\Sigma$ and $F$

Same setup, detailed numbers at NLO and NNLO:

| | $(a_1)$ | $(a_{3/2})$ | $(a_2)$ | $(a_3)$ | $(a_4)$ |
|---|---|---|---|---|---|
| $\Sigma_{\text{eff}}^{\text{NLO}}/\Sigma$ | 1.3455 | 1.2477 | 1.1454 | 0.9404 | 0.7355 |
| $\Sigma_{\text{eff}}^{\text{NNLO}}/\Sigma$ | 1.39(1) | 1.288(7) | 1.202(5) | 1.047(3) | 0.906(3) |
| $F_{\text{eff}}^{\text{NLO}}/F$ | 1.3004 | 1.3182 | 1.3192 | 1.3193 | 1.3193 |
| $F_{\text{eff}}^{\text{NNLO}}/F$ | 1.279(9) | 1.305(4) | 1.306(2) | 1.292(1) | 1.261(2) |
| | | $(b_{3/2})$ | $(b_2)$ | $(b_3)$ | $(b_4)$ |
| $F_{\text{eff}}^{\text{NLO}}/F$ | | 1.1894 | 1.06816 | 0.7710 | 0.2186 |
| $F_{\text{eff}}^{\text{NNLO}}/F$ | | 1.182(8) | 1.092(7) | 0.959(6) | 0.919(5) |

Convergence of $\varepsilon$ expansion for JLQCD setup is OK