# Large N exact beta function and inter-quark potential

Marco Bochicchio

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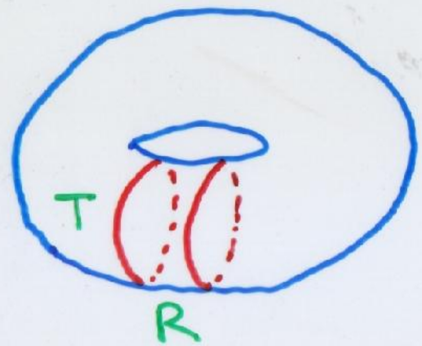
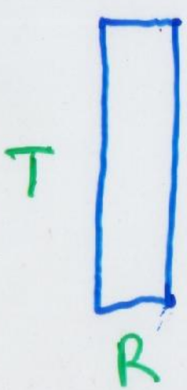
M.B. JHEP 0905(2009) 116 [hep-th/0809.4662]

PoS EPS-HEP(2009) 075 [hep-th/0910.0776]

and to appear




$$\langle P(0)\, P^*(R)\rangle \sim e^{-T V(R)} \quad ; \quad T \gg R$$




$$\langle W(T,R)\rangle \sim e^{-T V(R)}$$

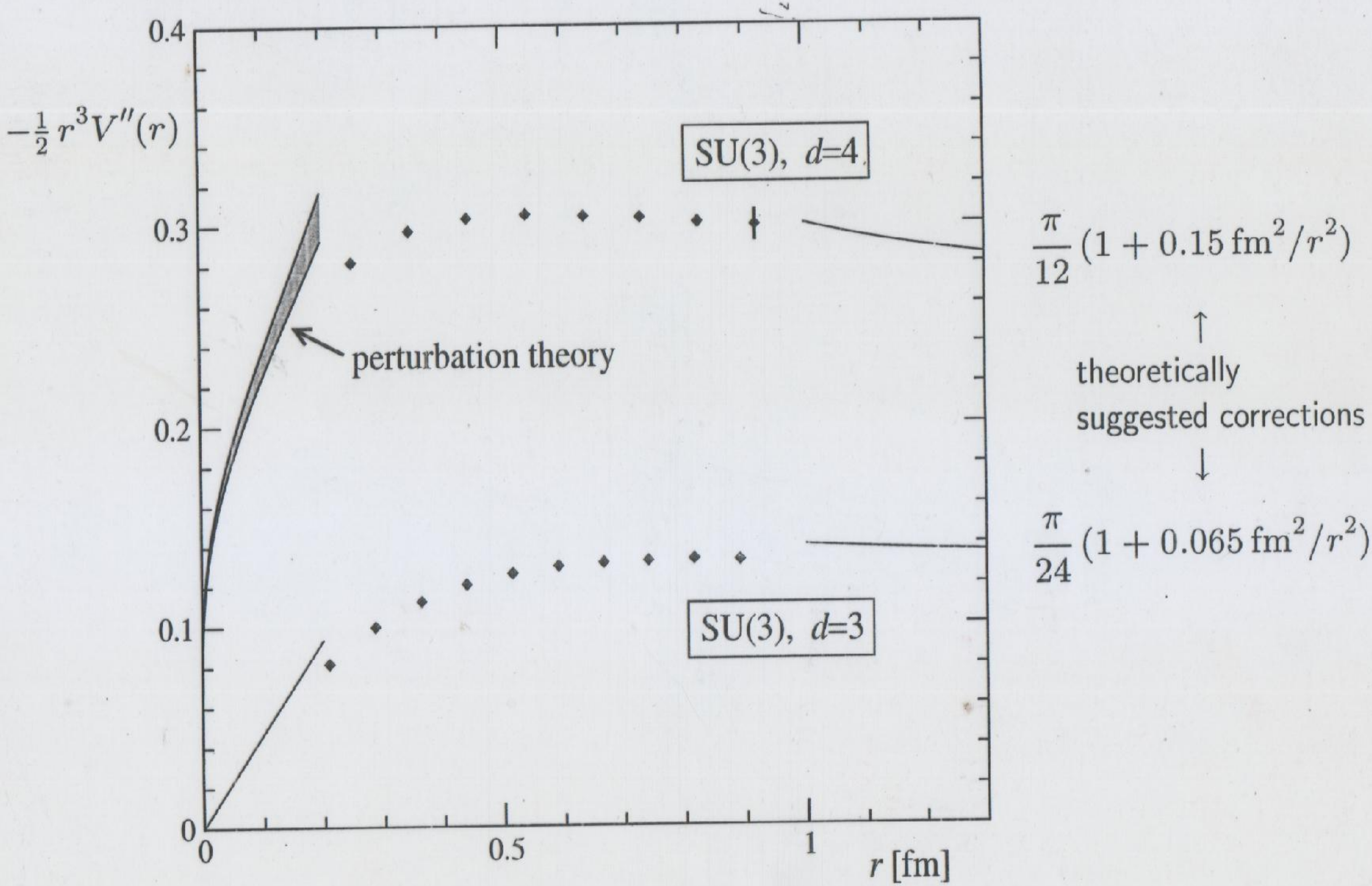
Luscher Weisz (2002)

$$V(R) = \sigma R - \frac{g^2_{phys}(R)}{4\pi R}$$

$$F(R) = V'(R)$$

$$C(R) = \frac{1}{2} R^3 V''(R) \equiv$$

$$\equiv - \frac{g^2_{eff}(R)}{4\pi}$$




M.L. & P. Weisz, JHEP 07 (2002) 049 [hep-lat/0207003]

# Exact beta function

## $N=1$ SUSY YM

$$\frac{\partial g_W}{\partial \log\Lambda} = -\frac{3}{(4\pi)^2}\, g_W^3$$

$$\frac{\partial g}{\partial \log\Lambda} = \frac{-\frac{3}{(4\pi)^2}\, g^3}{1-\frac{2}{(4\pi)^2}\, g^2}$$

gluino condensate $= \mathrm{Tr}\,\lambda^2$

localized on instantons

$$F^-_{\alpha\beta} = 0 \;;\; F^-_{\alpha\beta} = F_{\alpha\beta} - \tilde{F}_{\alpha\beta}$$

Localization = one-loop is exact

Surface operators $\rightarrow$ $Z_N$ vortices

$$P e^{i\int A} = e^{\frac{2\pi i k}{N}}$$

d=2

d=3

d=4

## SU(N) pure YM $N=\infty$

$$\frac{\partial g_W}{\partial \log\Lambda} = -\beta_0 g_W^3 \;;\; \beta_0 = \frac{11}{3}\frac{1}{(4\pi)^2}$$

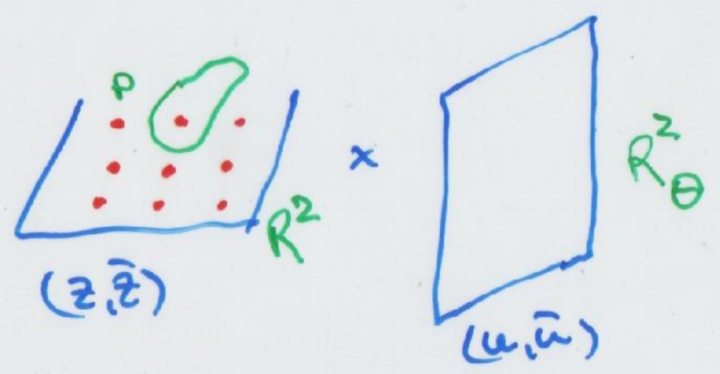
$$\frac{\partial g}{\partial \log\Lambda} = \frac{-\beta_0 g^3 + \frac{g^3}{(4\pi)^2}\frac{\partial \log Z}{\partial \log\Lambda}}{1-\frac{4}{(4\pi)^2}\, g^2}$$

$$\frac{\partial \log Z}{\partial \log\Lambda} = \frac{\gamma g_W^2}{1+c\, g_W^2} \;;\; \gamma = \frac{10}{3}\frac{1}{(4\pi)^2}$$

special Wilson loop $= P\exp i\int (A_z + D_u)\, dz + c.c.$

localized on surface operators

$$F^-_{\alpha\beta} = \sum_p \mu^-_{\alpha\beta}(p)\, \delta^{(2)}(z - z_p(u,\bar{u})) + \theta^{-1}_{\alpha\beta}\mathbb{1}$$

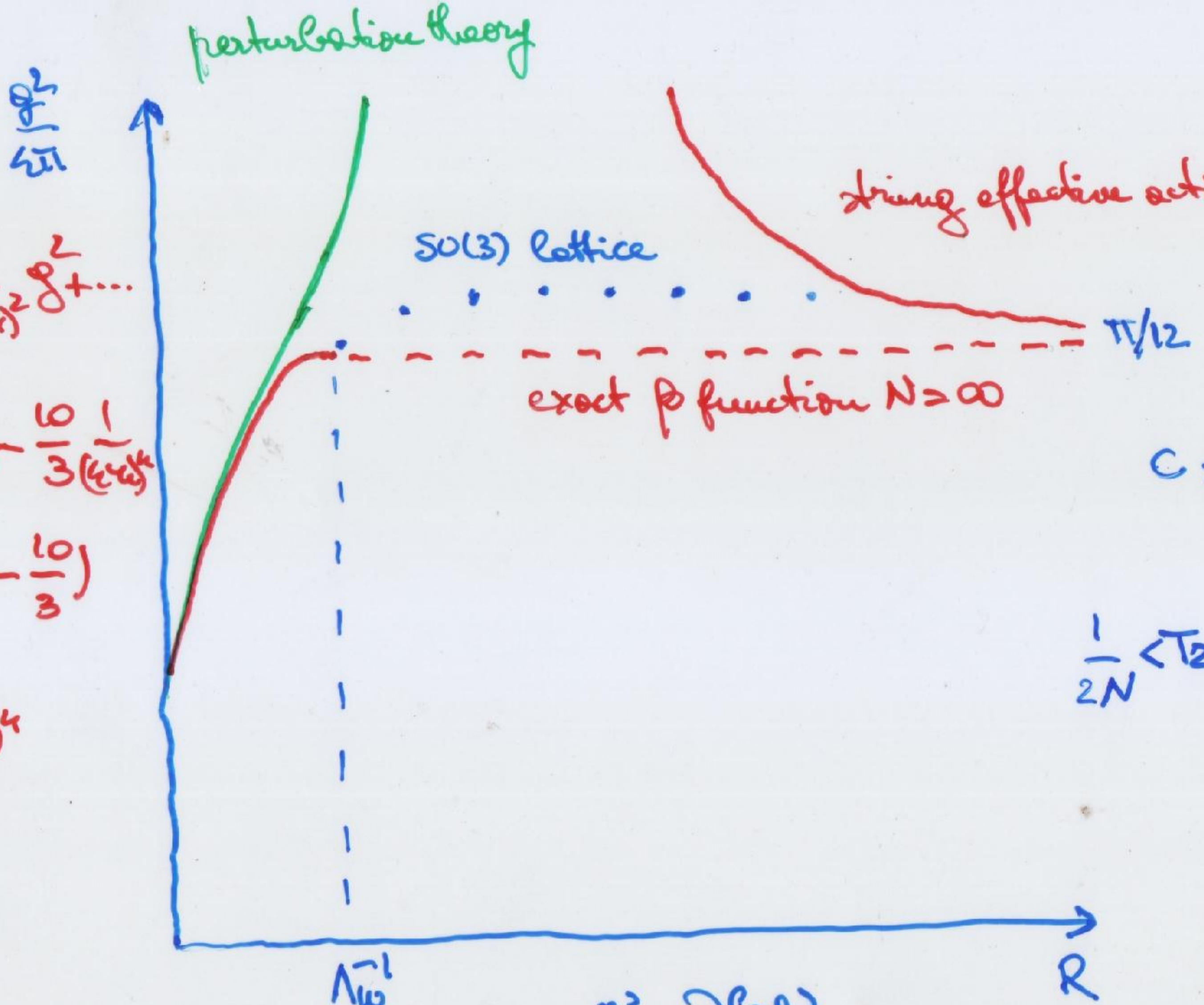


$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{10}{3}\frac{1}{(4\pi)^2} g^2 + \dots$$

$$\beta_1 = \frac{4}{(4\pi)^2}\beta_0 - \frac{10}{3}\frac{1}{(4\pi)^4}$$

$$= \frac{1}{(4\pi)^4}\left(\frac{44}{3} - \frac{10}{3}\right)$$

$$= \frac{34}{3}\frac{1}{(4\pi)^4}$$




$$c = \frac{1}{(4\pi)^2}\frac{\gamma}{\beta_0}$$

$$\frac{1}{2N}\langle \mathrm{Tr}\,\lambda^2\rangle = \frac{(2\pi)^2}{12} = g_*^2$$

$$\frac{\partial g_\omega}{\partial \log \Lambda} = -\beta_0 g_\omega^3 \;;\; \frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{g^3}{(4\pi)^2}\frac{\partial \log Z}{\partial \log \Lambda}}{1 - \frac{4}{(4\pi)^2} g^2} \;;\; \frac{\partial \log Z}{\partial \log \Lambda} = \frac{\gamma g_\omega^2}{1 + c g_\omega^2} \;;\; \gamma = \frac{10}{3}\frac{1}{(4\pi)^2}$$

# Effective charge in the glueball propagator

$$Z = \int e^{-\frac{N}{2g^2}\int \sum_{\alpha\beta} \mathrm{Tr} F_{\alpha\beta}^2 d^4x} DA \, ; \quad g^2 = g_{YM}^2 N = \text{const} \, ; \; N \to \infty$$

$$\left\langle \frac{1}{N}\mathrm{Tr} F_{\alpha\beta}^2(x_1) \ldots \frac{1}{N}\mathrm{Tr} F_{\alpha\beta}^2(x_k) \right\rangle \overset{N\to\infty}{=} \left\langle \frac{1}{N}\mathrm{Tr} F_{\alpha\beta}^2(x_1) \right\rangle \ldots \left\langle \frac{1}{N}\mathrm{Tr} F_{\alpha\beta}^2(x_k) \right\rangle$$

$$\int \left\langle \frac{1}{N}\mathrm{Tr} F_{\alpha\beta}^2(x) \, \frac{1}{N}\mathrm{Tr} F_{\alpha\beta}^2(0) \right\rangle_{conn} e^{ip\cdot x} d^4x = \sum_r \frac{Z_r}{p^2 + m_r^2} \sim g^4(p^2) p^4 \log\left(\frac{p^2}{\mu^2}\right)$$

two loops pert. theory.

Localization on surface operators

$\mu = F_{01}^- + i F_{02}^-$

$$\int \left\langle \frac{1}{N}\mathrm{Tr}(\mu^2) \, \frac{1}{N}\mathrm{Tr}(\bar{\mu}^2) \right\rangle_{conn} e^{ip_+x_- + ip_-x_+} dx_+ dx_- \sim g^4(p^2) \sum_{k=1}^{\infty} \frac{k^2 \Lambda_\omega^6}{\alpha p_+ p_- + (k\delta - \gamma)\Lambda_\omega^2}$$

exact $\beta$ function

$k$ = magnetic charge $Z_N$ vortices

$$\Lambda_\omega^4 k^2 = \frac{1}{\delta^2}\left[(k\delta-\gamma)\Lambda_\omega^2 + \alpha p^2\right]\left[(k\delta-\gamma)\Lambda_\omega^2 - \alpha p^2\right] + \left(\frac{\alpha}{\delta}\right)^2 p^4 + \ldots$$

$$\sim \left(\frac{\alpha}{\delta}\right)^2 \sum_k \frac{p^4 \Lambda_\omega^2}{\alpha p^2 + (k\delta-\gamma)\Lambda_\omega^2} \sim p^4 \log\frac{p^2}{\Lambda_\omega^2}$$

$$+ \frac{1}{\delta^2}\sum_k \left[(k\delta-\gamma)\Lambda_\omega^2 - \alpha p^2\right] \sim$$

$\sim$ contact terms

# Localization

Finite dimension

$$Z(t)=\int e^{-\omega+t\,d\alpha}$$

$$\to\int e^{-\omega}$$

$$d\omega=0$$

$$\left.\frac{dZ}{dt}\right|_{t=0}=\int d\alpha\, e^{-\omega}$$

$$=\int d(\alpha e^{-\omega})$$

$$=0$$

$$\int d\alpha=0$$

$$d^2=0$$

$\mathcal{N}=2$ SUSY YM

$$\{Q^{\alpha i},\bar{Q}^{\dot\beta j}\}=\sigma^{\mu}_{\alpha\dot\beta}P^{\mu}\delta_{ij}$$

$$\mathcal{Q}=\varepsilon_{\alpha i}Q^{\alpha i}=\text{twisted s.c.}$$

$$\mathcal{Q}^2=0$$

$$(d,\mathcal{Q},\int)\quad \int\mathcal{Q}\alpha=0$$

$$Z(t)=\int e^{-S_{SUSY}-t\mathcal{Q}\alpha}$$

$$=Z(0)$$

$t\to\infty$

$Z(t)$ localizes on the critical points of $\mathcal{Q}\alpha$

$\mathcal{N}=1$ SUSY YM

No twist exists

$\mathcal{Q}$ generated by tautological Parisi-Sourlas SUSY associated to the Nicolai map:

$$A_\alpha\to F^-_{\alpha\dot\beta}+\text{gauge fixing}$$

$$Z=\int e^{-\frac{1}{4g^2}\int \mathrm{Tr}\,F^2_{\alpha\beta}}\,\mathrm{Det}\,\not{D}\;DA$$

$$=\int e^{-\frac{16\bar{\alpha}^2\mathcal{Q}}{2g^2}-\frac{1}{8g^2}\int\mathrm{Tr}\,F^{-2}_{\alpha\beta}}\,\underbrace{\mathrm{Det}\,\not{D}\,\frac{DA}{DF^-}}_{1}\,DF^-$$

$A_+=0$

Localization in pure YM? No $\mathcal{Q}$ but Nicolai map still exists as a change of variables

# Cohomology

$$[\omega] = \int \omega = \int \omega + d\alpha$$

$$Q S_{SUSY} = 0 \sim d\omega = 0$$

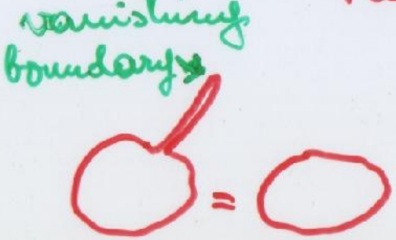
the co-boundary $Q$ is the generator of the (super) symmetry

$$Z(t) = \int e^{-S_{SUSY} + tQ\alpha}$$

$t \to \infty$

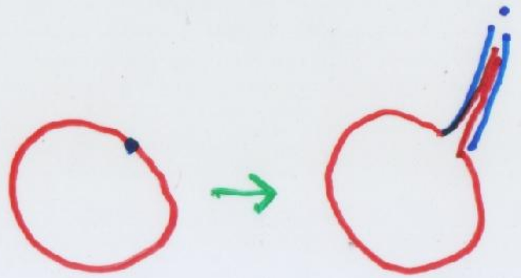
localization on critical points of the co-boundary $Q\alpha$

# Homology

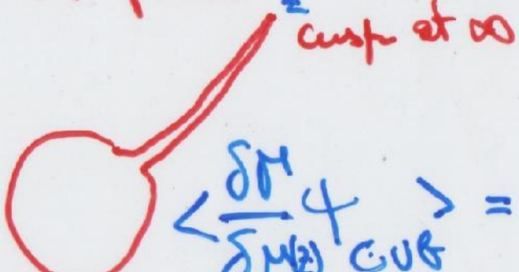



$$\langle W \rangle \sim e^{p\Lambda + \gamma \log \Lambda}$$

$$\langle \psi_c \rangle = \langle \psi_{CUG} \rangle$$



the vanishing boundary is generated by a conformal transformation = symmetry of the RG flow




$$\left\langle \frac{\delta \Gamma}{\delta \mu(z)} \psi_{CUG} \right\rangle = \frac{1}{2\pi} \int_{CUG} \frac{dw}{z-w} \langle \psi_{CUG} \rangle = 0$$

# Holomorphic loop equation in the Nicolai variables

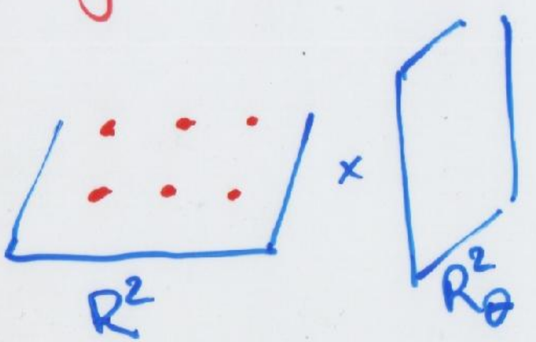
$$Z = \int e^{-\frac{N}{2g^2}\int \mathrm{Tr}\, F_{\alpha\beta}^2} DA = \int e^{-\frac{8\pi^2 QN}{g^2} - \frac{N}{4g^2}\int \mathrm{Tr}\, F^{-2}_{\alpha\beta}} DA$$

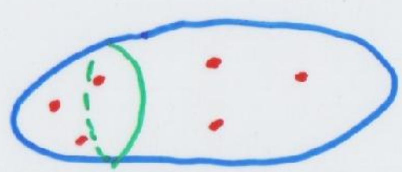
$$1 = \int \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta})\, D\mu^-_{\alpha\beta} \qquad \text{(Nicolai map)}$$

$$Z = \int e^{-\frac{8\pi^2 QN}{g^2} - \frac{N}{4g^2}\int \mathrm{Tr}(\mu^-_{\alpha\beta})^2} \mathrm{Det}^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + i\, \mathrm{ad} F^-_{\alpha\beta})\, D\mu^-_{\alpha\beta} = \int e^{-\Gamma} D\mu$$

## Nicolai map on a lattice of surface operators

$$Z = \int e^{-\frac{8\pi^2 QN}{g^2} - \frac{N}{4g^2}\int \mathrm{Tr}\, F^{-2}_{\alpha\beta}} \delta\Big(F^-_{\alpha\beta} - \sum_p \mu^-_{\alpha\beta}(p)\, \delta^{(2)}(z - z_p(u,\bar u))\Big) \prod_p \mu^-_{\alpha\beta}(p)$$

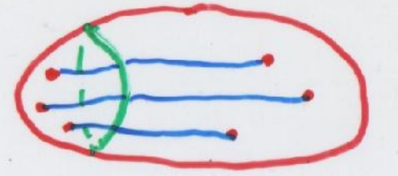


$$\left\langle \frac{\delta\Gamma}{\delta\mu_p} \Psi_{z_p z_p} \right\rangle = \frac{1}{2\pi}\int \frac{dz'}{z_p - z'} \langle \Psi_{z_p z'} \rangle \langle \Psi_{z' z_p} \rangle$$

$$\Psi_{zw}(B) = e^{i\int_{C_{zw}} (A_z + D_u)dz + (A_{\bar z} + D_{\bar u})d\bar z} \quad ; D_u = \partial_u + i A_u$$

$$\left\langle \frac{\delta\Gamma}{\delta\mu_p} \Psi_{z_p z_p}(C \cup L) \right\rangle = 0$$



# Beta function from localization

## $\mathcal{N}=1$ SUSY YM

$$Z_{\text{instantons}} = \int e^{-\frac{(4\pi)^2 Q}{2 g_W^2}} \Lambda^{n_B - \frac{n_F}{2}} \frac{\operatorname{Det}\omega_B}{\operatorname{Det}\omega_F} \;;$$

$$\frac{(4\pi)^2 Q}{2 g_W^2(\mu)} = \frac{(4\pi)^2 Q}{2 g_W^2(\Lambda)} - \left(n_B - \frac{n_F}{2}\right) \log\left(\frac{\Lambda}{\mu}\right) \;;$$

$$A \to g A_c \qquad \frac{\operatorname{Det}\omega_B}{\operatorname{Det}\omega_F} \to g^{n_B - n_F} \;;$$

$$-\frac{1}{2g^2} = -\frac{1}{2 g_W^2} + \frac{n_B - n_F}{(4\pi)^2 Q} \log g \;;$$

## SU(N) YM $N=\infty$

$$Z_{\text{vertices}} = \int e^{-\frac{N}{2 g_W^2} S_{YM}(\text{vortices})} \Delta_{FP} \operatorname{Det}^{-\frac{1}{2}}(\text{non-zero modes}) \times \Lambda^{n_B} \operatorname{Det}\omega_B$$

$$\frac{S_{YM}}{2 g_W^2(\mu)} = \frac{Z^{-1} S_{YM}}{2 g_W^2(\Lambda)} - n_B \log\left(\frac{\Lambda}{\mu}\right)$$

$$A \to g Z^{\frac{1}{2}} A_c$$

$$-\frac{1}{2g^2} = -\frac{1}{2 g_W^2} + \frac{n_B}{S_{YM}} \log g + \frac{n_B}{4 S_{YM}} \log Z$$

gluing