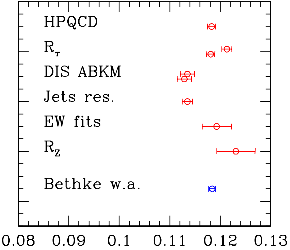
The running coupling of QCD with four flavors (arXiv:1006.0672)

Rainer Sommer, Fatih Tekin & Ulli Wolff

Lattice 2010, June 2010, Villasimius, Sardinia

α_{s}: the fundamental parameter of QCD

Summary of precise determinations 2010



Spread is larger than the estimated precision!

From [S. Bethke, 2009; S. Alekhin et al, 2009, A. Hoang et al., 2009]

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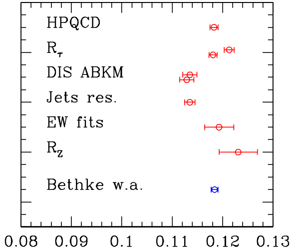

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ALPHA Collaboration : precise and reliable determination

$\leftrightarrow$ precise and non-perturbative computation up to high μ

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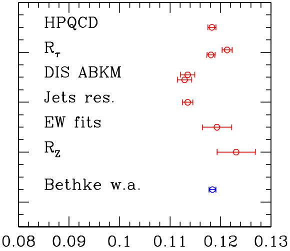

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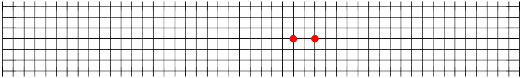
2009: $N_\mathrm{f} = 3$ running [PACS-CS]

One step further: The non-perturbative running for four flavors

Problem in a lattice computation (α_{qq} as an example)



$$\underset{\substack{\uparrow\\ \text{box size}}}{L} \gg \underset{\substack{\uparrow\\ \text{confinement scale, } m_\pi}}{\frac{1}{0.2\mathrm{GeV}}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg \underset{\substack{\uparrow\\ \text{spacing}}}{a}$$

$$\Downarrow$$

$$L/a \gg 50$$

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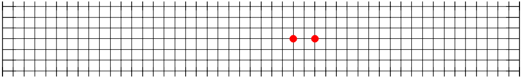


$$L \gg \frac{1}{0.2\mathrm{GeV}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg a$$

box size ↑ confinement scale, m_π ↑ spacing ↑

$\Downarrow$

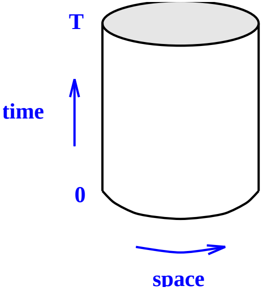
$L/a \gg 50$

Solution: $\boxed{L = 1/\mu}$ ⟶ left with $L/a \gg 1$

[Wilson, ... , Lüscher, Weisz, Wolff]

Finite size effect as a physical observable; finite size scaling!

Definition of $\bar{g}_{\mathrm{SF}}$



$$\begin{aligned} \exp\{-\Gamma\} &= \int \mathrm{D}[U,\overline{\psi},\psi] \exp\{-S[U,\overline{\psi},\psi]\} \\ U(x,k)|_{x_0=0} &= \exp\{aC_k(\eta)\}, \\ U(x,k)|_{x_0=T} &= \exp\{aC'_k(\eta)\} \end{aligned}$$

$$\Gamma' = \left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0} = \frac{k}{\bar{g}^2(L)} \propto \langle\, F_{0k}|_{\mathrm{boundary}} \rangle$$

[ALPHA Collaboration 1991 – 2001]

The step scaling function

- … is a discrete *beta* function:

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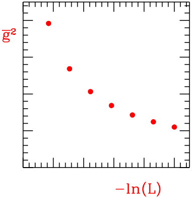
- determines the non-perturbative running:



$$\begin{aligned} u_0 &= \bar{g}^2(L_{\max}) \\ &\downarrow \\ \sigma(2, u_{k+1}) &= u_k \\ &\downarrow \\ u_k &= \bar{g}^2(2^{-k} L_{\max}) \end{aligned}$$

The step scaling function: $\sigma(s,u) = \bar{g}^2(sL)$ with $u = \bar{g}^2(L)$

On the lattice:
additional dependence on the resolution a/L

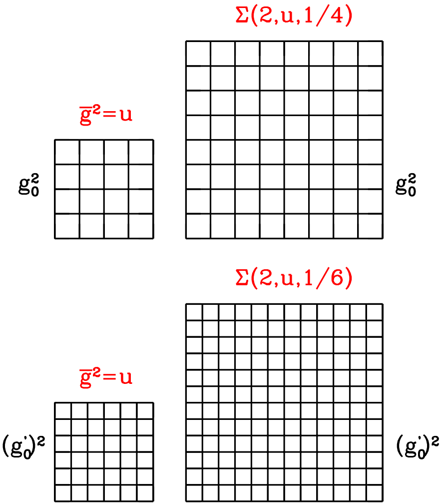
g_0 fixed, L/a fixed:

$$\bar{g}^2(L) = u, \qquad \bar{g}^2(sL) = u'\,,$$
$$\Sigma(s,u,a/L) \quad = \quad u'$$

continuum limit:

$$\Sigma(s,u,a/L) = \sigma(s,u) + \mathrm{O}(a/L)$$

in the following always $s = 2$



everywhere: $m = 0$ (PCAC mass defined in $(L/a)^4$ lattice)

Tuning / Interpolation

PCAC mass in SF for "small" lattice: tuning κ

$$m(L) = 0 \quad \longrightarrow \quad \kappa_c(\beta, a/L)$$

Interpolation in β

$$\bar{g}^2(\beta)_{L/a,\kappa=\kappa_c(\beta,a/L)} = \frac{6}{\beta}\left[\sum_{m=0}^{n} c_{m,L/a}\left(\frac{6}{\beta}\right)^m\right]^{-1}$$

[Appelquist, Fleming & Neil, 2009]

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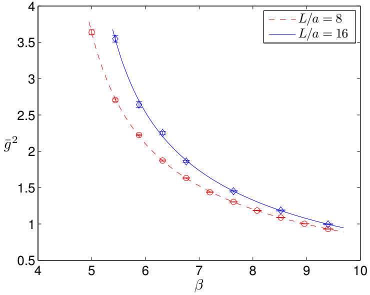

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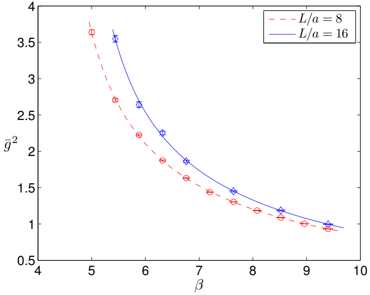


Lattice step scaling function $\Sigma(u, a/L)$

Improvement: very important for $\bar{g}^2_{\mathrm{SF}}$ [ALPHA]

- Standard $\mathrm{O}(a)$ improvement: $c_{\mathrm{sw}}(g_0)$, NP for $\beta \geq 5.0$ [Tekin, S., Wolff, 2009]
- boundary $\mathrm{O}(\mathrm{a})$-terms, e.g. $c_t(g_0)a^4 \sum_{\mathbf{x}} F_{0k}F_{0k}$ at $x_0 = 0$ and $x_0 = T$
  c_{t} to two-loops from [Bode, Weisz & Wolff, 1999]
  check of remaining uncertainty for $N_{\mathrm{f}} = 2$
- remaining cutoff effects of SSF:

$$\begin{aligned}
\delta(u, a/L) &= \frac{\Sigma(u,a/L)-\sigma(u)}{\sigma(u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \ldots \\
\delta_1(a/L) &= \delta_{10}(a/L) + \delta_{11}(a/L)N_{\mathrm{f}} \\
\delta_2(a/L) &= \delta_{20}(a/L) + \delta_{21}(a/L)N_{\mathrm{f}} + \delta_{22}(a/L)N_{\mathrm{f}}^2
\end{aligned}$$

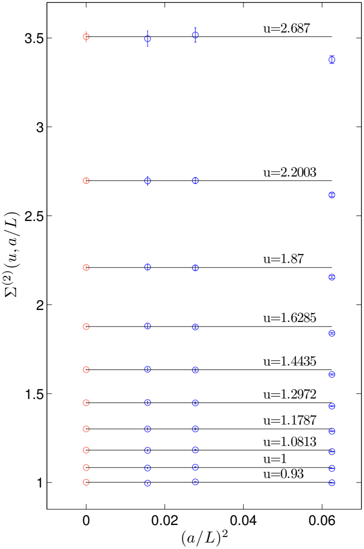
[ALPHA Collaboration, 1993-1997]

[Bode, Weisz & Wolff, 1999]

- Observable improvement [De Divitiis et al., 1993] improved step scaling function:

$$\begin{aligned}
\Sigma^{(2)}(u, a/L) &\equiv \frac{\Sigma(u, a/L)}{1 + \delta_1(a/L)u + \delta_2(a/L)u^2} \\
&= \sigma(u) + \mathrm{O}(u^4\, a/L)
\end{aligned}$$

Continuum limit



- *Constant fit:*

  $$\Sigma^{(2)}(u, a/L) = \sigma(u)$$

  for $L/a = 6, 8$

- *Global fit:*

  $$\Sigma^{(2)}(u, a/L) = \sigma(u) + \rho\, u^4\, (a/L)^2$$

  for $L/a = 6, 8$
  $\longrightarrow \rho = 0.007(85)$

- $L/a = 8$ *data:*

  $$\sigma(u) = \Sigma^{(2)}(u, 1/8)$$

Continuum SSF

u		$\sigma(u)$	
	constant fit	global fit	$L/a = 8$ data
0.9300	1.002 (3)	1.002 (3)	0.997 (5)
1.0000	1.084 (3)	1.084 (3)	1.081 (4)
1.0813	1.182 (3)	1.182 (4)	1.181 (5)
1.1787	1.301 (4)	1.301 (5)	1.301 (6)
1.2972	1.448 (5)	1.448 (7)	1.450 (7)
1.4435	1.634 (5)	1.634(10)	1.637 (8)
1.6285	1.877 (7)	1.877(16)	1.880(11)
1.8700	2.209(10)	2.207(27)	2.212(17)
2.2003	2.698(14)	2.694(49)	2.697(24)
2.6870	3.507(30)	3.50 (10)	3.496(44)

↑ **result**

Recursive reconstruction of $\bar{g}_{\mathrm{SF}}(L)$

$$\begin{aligned} u_i &\equiv \bar{g}^2\left(L_{\max}/2^i\right) \\ u_i &= \sigma(u_{i+1}), \quad i = 0, \ldots, n, \quad u_0 = u_{\max} = \bar{g}^2\left(L_{\max}\right), \end{aligned}$$

solve for u_{i+1}, $i = 0 \ldots n = 10$

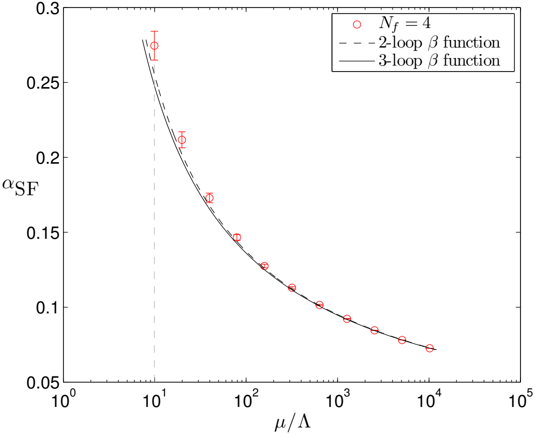
Lambda parameter

for large i, small $u_i = \bar{g}^2(L_i)$:

$$
\begin{aligned}
L_i\Lambda &= \left[b_0\bar{g}^2(L_i)\right]^{-\frac{b_1}{2b_0^2}} \exp\left\{-\frac{1}{2b_0\bar{g}^2(L_i)}\right\} \times \\
&\quad \exp\left\{-\int_0^{\bar{g}(L_i)} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2x}\right]\right\}
\end{aligned}
$$

	constant fit		global fit		$L/a = 8$ data	
i	u_i	$\ln(\Lambda L_{\max})$	u_i	$\ln(\Lambda L_{\max})$	u_i	$\ln(\Lambda L_{\max})$
0	3.45	−2.028	3.45	−2.028	3.45	−2.028
1	2.660(14)	−2.074(17)	2.666(46)	−2.066(56)	2.660 (21)	−2.073(26)
2	2.173(13)	−2.117(24)	2.179(45)	−2.105(83)	2.173 (20)	−2.116(37)
3	1.842(11)	−2.155(28)	1.847(37)	−2.141(97)	1.842 (17)	−2.153(44)
4	1.6013(90)	−2.188(32)	1.606(30)	−2.17 (10)	1.602 (14)	−2.185(50)
5	1.4187(78)	−2.217(35)	1.422(25)	−2.20 (11)	1.419 (13)	−2.213(56)
6	1.2748(70)	−2.241(39)	1.278(20)	−2.23 (11)	1.275 (11)	−2.238(63)
7	1.1583(63)	−2.263(43)	1.161(17)	−2.25 (12)	1.159 (10)	−2.259(70)
8	1.0620(58)	−2.282(47)	1.064(15)	−2.27 (12)	1.0626(95)	−2.278(76)
9	0.9809(53)	−2.299(50)	0.982(13)	−2.29 (12)	**0.9815(87)**	**−2.294(83)**
10	0.9117(49)	−2.315(54)	0.913(11)	−2.30 (12)	0.9122(81)	−2.309(89)

Running α, comparison to PT



10% (3 sigma) difference to PT (3-loop β) at $\bar{g}^2(L) = 3.5$
more than for $N_{\mathrm{f}} = 2$

Summary, outlook

- a good step closer to Λ for $N_\mathrm{f} = 4$
- $N_\mathrm{f} = 4 \rightarrow N_\mathrm{f} = 5$ by perturbation theory
- relation of L_max to F_K or so remains to be done
  - 2+1+1 simulations with a massive charm quark
    - for large volume
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- tests of universality / increased precision with different discretizations?
  - staggered
  - chirally twisted boundary conditions