Witten index from lattice simulation

Issaku Kanamori (Universita di Torino)

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Ref.

I.K. arXiv:1006.2468

Introduction

Witten index: useful index to detect spontaneous SUSY breaking

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A method to measure the index

using a traditional monte calro simulation

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Witten index: useful index to detect spontaneous SUSY breaking

$\boxed{\text{A method to measure the index}}$

using a traditional monte calro simulation

plan of the talk

- Introduction: What is Witten index?
- Method: how to determine the normalization?
- Numerical check: supersymmetric quantum mechanics
- Conclusion

Motivation: What is Witten index?

- $\cancel{\text{SUSY}}$: cannot be broken by loop effects

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  Witten index: $w = \mathrm{tr}(-1)^F e^{-\beta H} = (N_B - N_F)\Big|_{E=0}$

  index $\neq 0$: SUSY

  index $= 0$: SUSY or ~~SUSY~~

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other approaches to measure Witten index

talks by Baumgartner and Wenger (the day before yesterday)

talk by Kawai (next)

Method

Normalization...?

Witten index in path integral

$$w = Z_{\mathrm{P}} = \int \mathcal{D}\phi\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi \, \exp(-S_{\mathrm{P}})$$

P: Periodic boundary condition

a proper definition of the measure is needed

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Expectation value

$$\langle A \rangle = \frac{\int \mathcal{D}\phi\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\, A \exp(-S)}{\int \mathcal{D}\phi\, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\, \exp(-S)}$$

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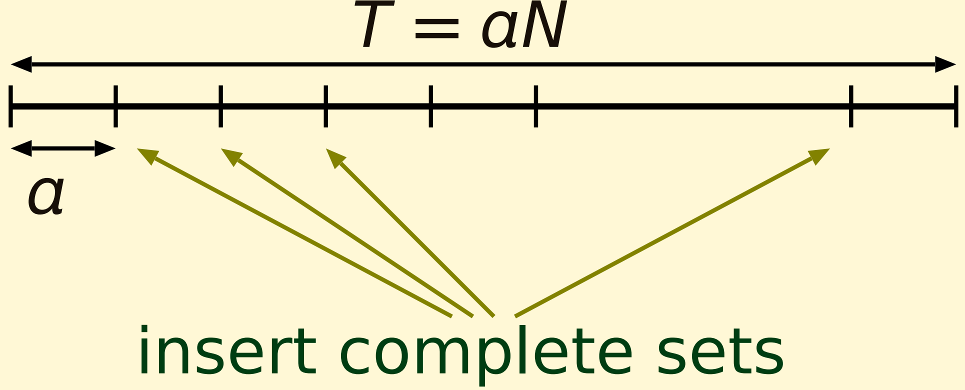
Overall normalization of Z_{P} seems impossible to determine

Measure

Operator formalism to path integral



$T = aN$

a

insert complete sets

$$\langle q_{\mathrm{fin}} | e^{-T\hat{H}} | q_{\mathrm{ini}} \rangle$$

$$\Rightarrow \qquad \int \mathcal{D}\phi = \int_{-\infty}^{\infty} \prod_i \frac{1}{\sqrt{2\pi}} d\phi_i^{(\mathrm{lat})}$$

$$\int \mathcal{D}\overline{\psi}\, \mathcal{D}\psi = \int \prod_i d\overline{\psi}_i^{(\mathrm{lat})}\, d\psi_i^{(\mathrm{lat})}$$

Partition function

Special observable

$$\left\langle e^{+S} \underbrace{\exp\left[-\frac{1}{2}\sum_i \mu^2 (\phi_i^{\mathrm{lat}})^2\right]}_{\text{regularization functional}} \right\rangle \equiv \frac{C}{\int \mathcal{D}\phi e^{-S}},$$

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$$Z = \int \mathcal{D}\phi\, e^{-S} = \frac{C}{\left\langle e^{+S}\exp\left[-\frac{1}{2}\sum_i \mu^2(\phi_i^{\mathrm{lat}})^2\right]\right\rangle}$$

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$$\boxed{\text{We can measure the partition function!}}$$

Sign of the Det(D) (or Pf(D))

Fermions

$$\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\mathcal{D}\phi e^{-S_{\mathrm{B}}-S_{\mathrm{F}}} = \int \mathcal{D}\phi\,\sigma[D] e^{-S'}, \quad S' = S_{\mathrm{B}} - \ln|\mathrm{Det}(D)|$$

Reweighting the sign of Det(D): $\sigma[D]$

$$\langle A \rangle_0 \equiv \frac{\int \mathcal{D}\phi\, A e^{-S'}}{\int \mathcal{D}\phi\, e^{-S'}}, \qquad \langle A \rangle = \frac{\int \mathcal{D}\phi\, A\sigma[D] e^{-S'}}{\int \mathcal{D}\phi\, \sigma[D] e^{-S'}} = \frac{\langle A\sigma[D]\rangle_0}{\langle \sigma[D]\rangle_0}$$

$\langle \sigma[D] \rangle_0$: almost the partition func.

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Normalized partition func.

$\langle\sigma[D]^{-1}e^{+S'}e^{-\frac{1}{2}\sum_i\mu^2\phi_i^2}\rangle$

$$\Rightarrow \boxed{w = Z_\mathrm{P} = C\frac{\langle\sigma[D_\mathrm{P}]\rangle_{0,\mathrm{P}}}{\langle e^{S'_\mathrm{P}-\frac{1}{2}\sum_i\mu^2\phi_i^2}\rangle_{0,\mathrm{P}}}}$$

P: Periodic boundary cond.

Numerical test: SQM

Model

Lattice action with $S = Q\Lambda$, $Q^2 = 0$:
$|\lambda\rangle$ and $Q|\lambda\rangle(\neq 0)$ make a pair as in the continuum

$\Rightarrow$index is well defined

(Giedt-Koniuk-Poppitz-Yavin)

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$$S = \sum_{k=0}^{N-1} \Big[\frac{1}{2}(\phi_{k+1} - \phi_k)^2 + \frac{1}{2} W'(\phi_k)^2 + (\phi_{k+1} - \phi_k) W'(\phi_k) - \frac{1}{2} F_k^2$$

$$+ \overline{\psi}_k (\psi_{k+1} - \psi_k) + W''(\phi_k) \overline{\psi}_k \psi_k \Big]$$

Catterall,...

ϕ: scalar, F: aux. field, $\psi, \overline{\psi}$: fermions
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- $n = 4$: $W = \lambda_4 \phi^4 + \lambda_2 \phi^2$

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- $n = 4$: $W = \lambda_4 \phi^4 + \lambda_2 \phi^2$ SUSY, $w = 1$
- $n = 3$: $W = \lambda_3 \phi^3 + \lambda_2 \phi^2$

Model

Lattice action with $S = Q\Lambda$, $Q^2 = 0$:
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⇒index is well defined

(Giedt-Koniuk-Poppitz-Yavin)

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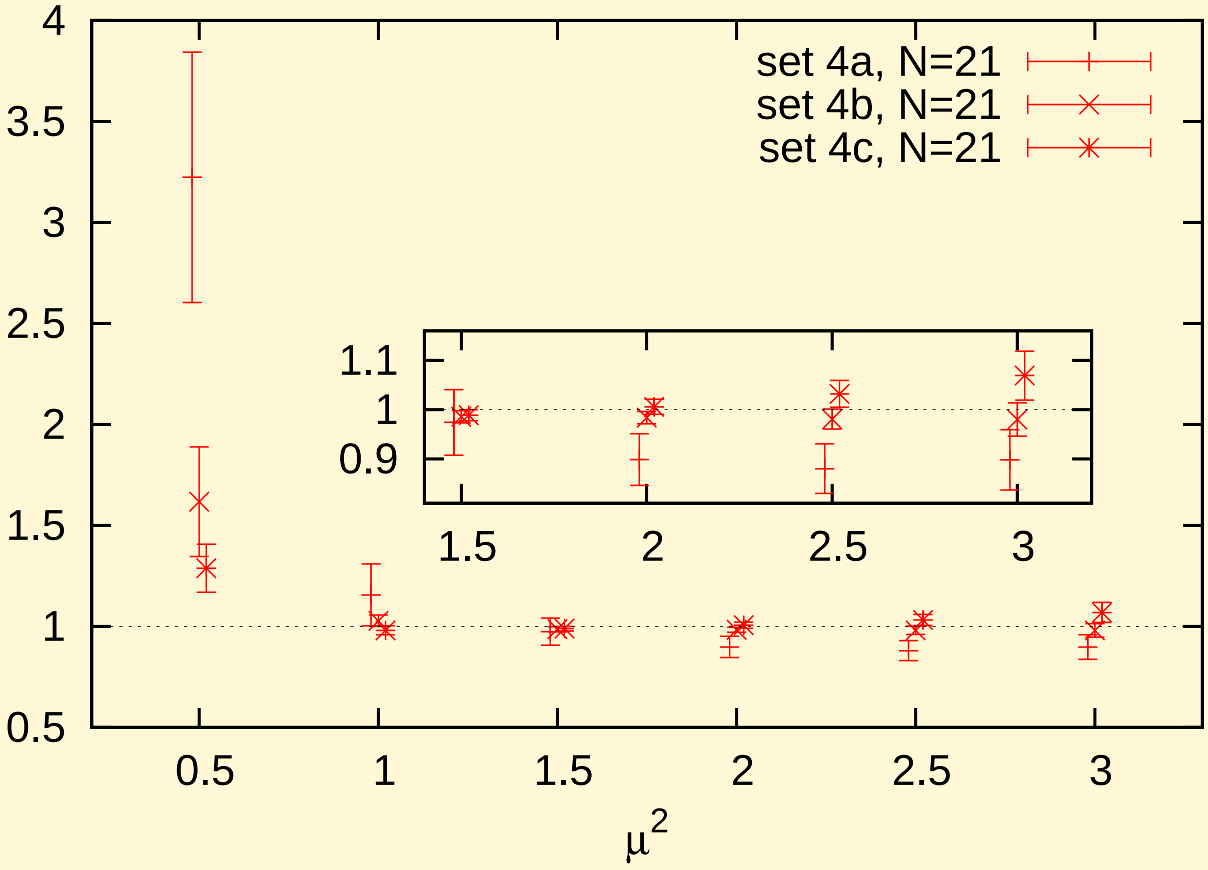
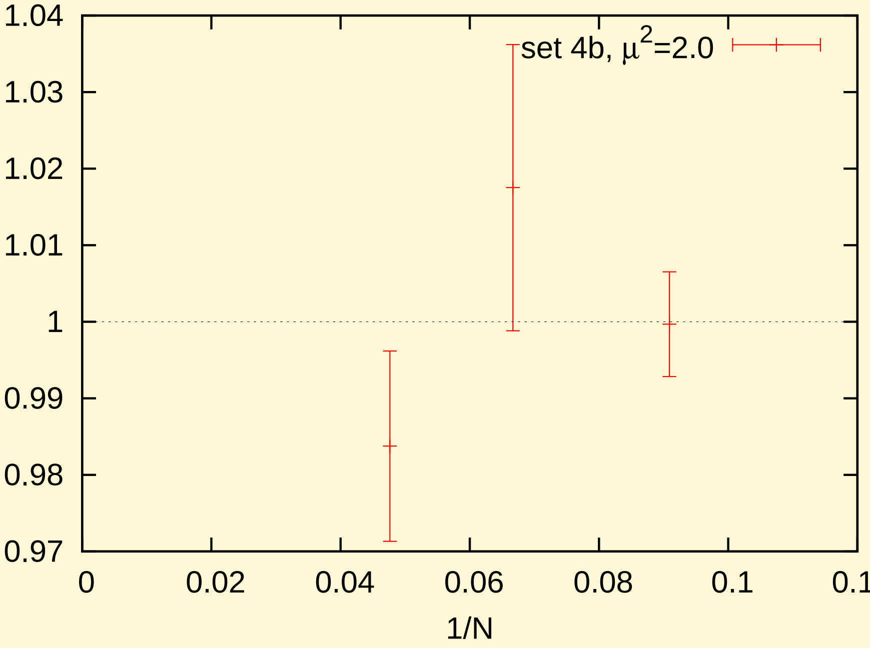
- $n = 4$: $W = \lambda_4 \phi^4 + \lambda_2 \phi^2$ SUSY, $w = 1$
- $n = 3$: $W = \lambda_3 \phi^3 + \lambda_2 \phi^2$ ~~SUSY~~, $w = 0$

Result: $n = 4$

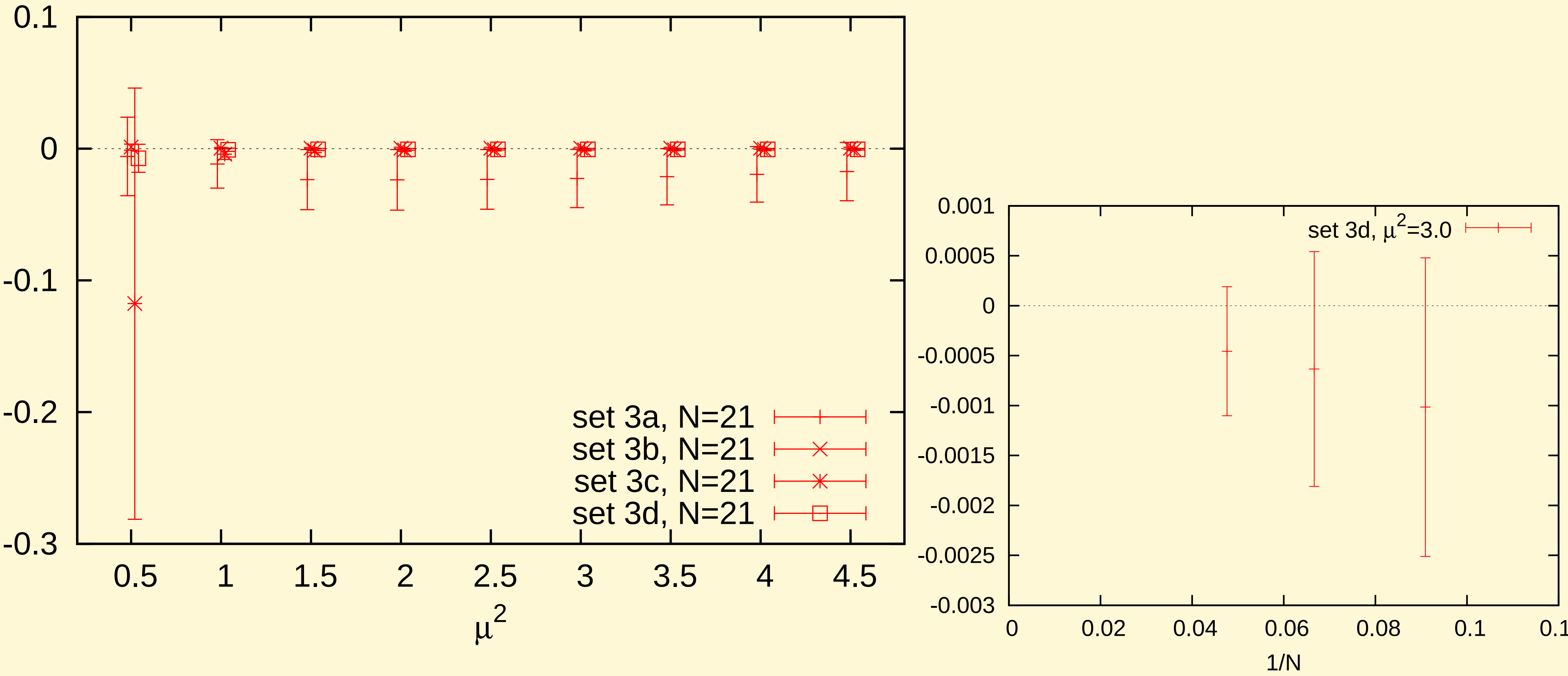


$$w = C \frac{\langle \sigma[D_{\mathrm{P}}]\rangle_{0,\mathrm{P}}}{\langle e^{S'_{\mathrm{P}} - \frac{1}{2}\sum_i \mu^2 \phi_i^2}\rangle_{0,\mathrm{P}}}$$



set 4a ($L\lambda_2 = 1, L^2\lambda_4 = 1$) : $\mu^2 = 2.5, 0.88(5)$
set 4b ($L\lambda_2 = 4, L^2\lambda_4 = 1$) : $\mu^2 = 2.0, 0.984(12)$
set 4c ($L\lambda_2 = 4, L^2\lambda_4 = 4$) : $\mu^2 = 1.5, 0.989(11)$

Result: $n = 3$



set 3a ($L\lambda_2 = 4, L^{3/2}\lambda_4 = 4$) : $\mu^2 = 1.5, -0.024(23)$
set 3b ($L\lambda_2 = 4, L^{3/2}\lambda_4 = 16$) : $\mu^2 = 2.0, 0.0004(7)$
set 3c ($L\lambda_2 = 4, L^{3/2}\lambda_4 = 32$) : $\mu^2 = 1.5, -0.0009(8)$
set 3d ($L\lambda_2 = 2, L^{3/2}\lambda_4 = 16$) : $\mu^2 = 1.5, -0.0005(6)$

Further reweighting

demerit: $w = Z_{\mathrm{P}} = C \frac{\langle \sigma[D_{\mathrm{P}}] \rangle_{0,\mathrm{P}}}{\langle e^{S'_{\mathrm{P}} - \frac{1}{2}\sum_i \mu^2 \phi_i^2} \rangle_{0,\mathrm{P}}}$

$e^{S'_{\mathrm{P}}}$ spoils the important sampling method

$\Rightarrow$ large fluctuations, poor efficiency

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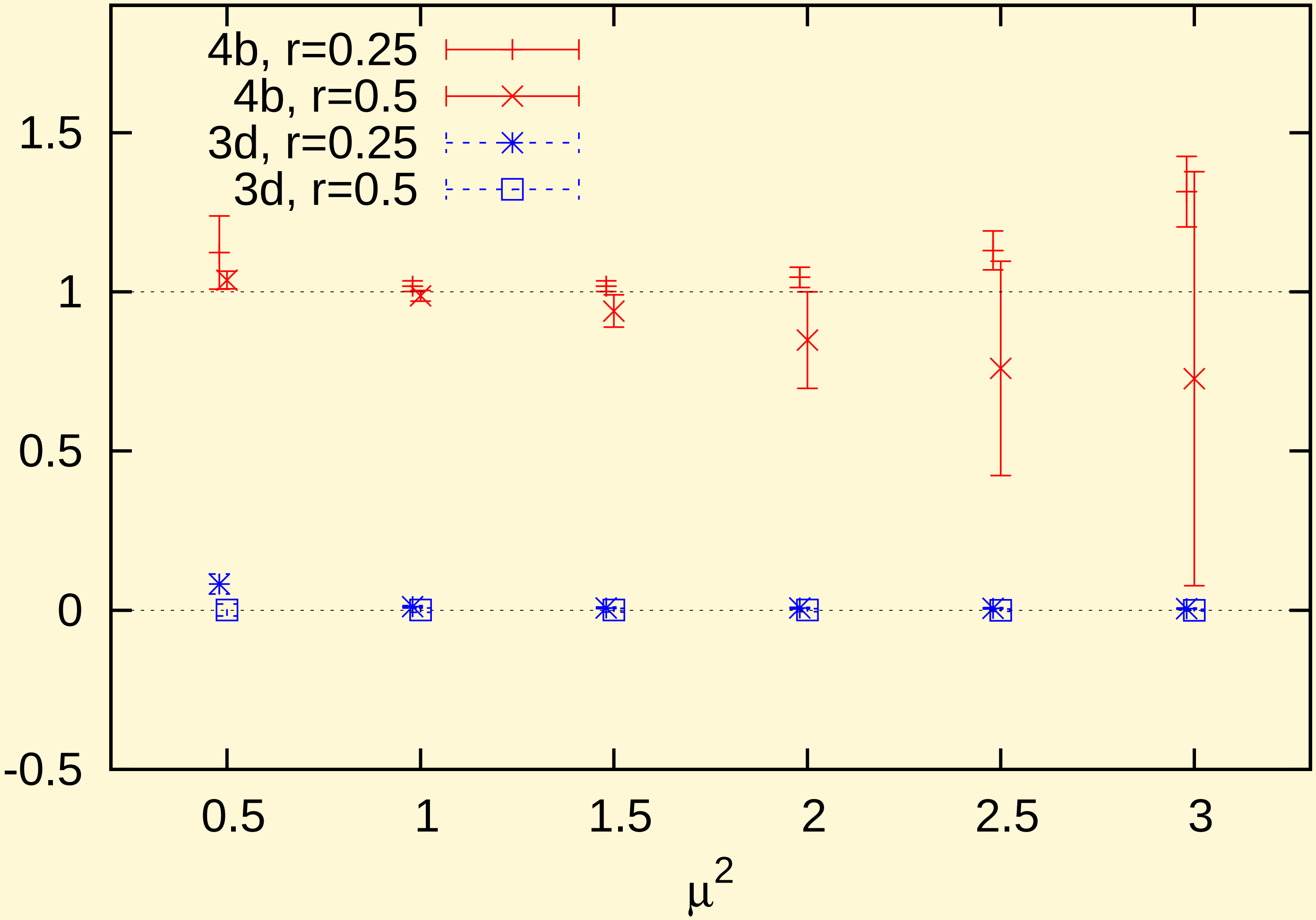
Less important sampling $\quad e^{-S} = e^{-rS} e^{-(1-r)S}$

$$\langle A \rangle = \frac{\int \mathcal{D}\phi\, A e^{-rS} e^{-(1-r)S}}{\int \mathcal{D}\phi\, e^{-rS} e^{-(1-r)S}} = \frac{\langle A e^{-rS} \rangle_r}{\langle e^{-rS} \rangle_r}$$

prepare the configurations using $(1-r)S$

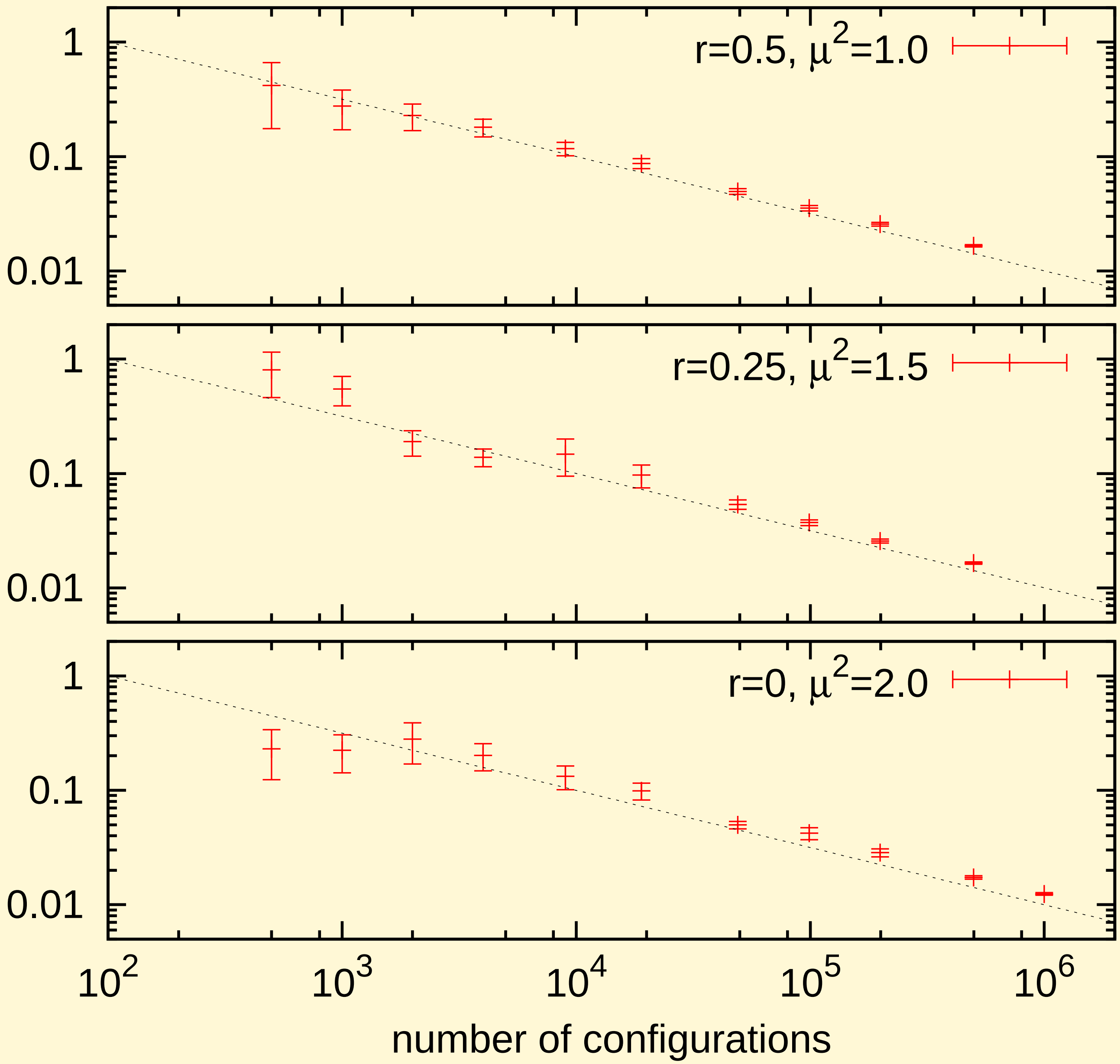
$$\Rightarrow \qquad w = C \frac{\langle \sigma[D_{\mathrm{P}}] e^{-rS'_{\mathrm{P}}} \rangle_{r,\mathrm{P}}}{\langle e^{(1-r)S'_{\mathrm{P}} - \frac{1}{2}\sum_i \mu^2 \phi_i^2} \rangle_{r,\mathrm{P}}}$$

Result



Behavior of errors

(LI sampling)



(Normal)

Behavior of errors

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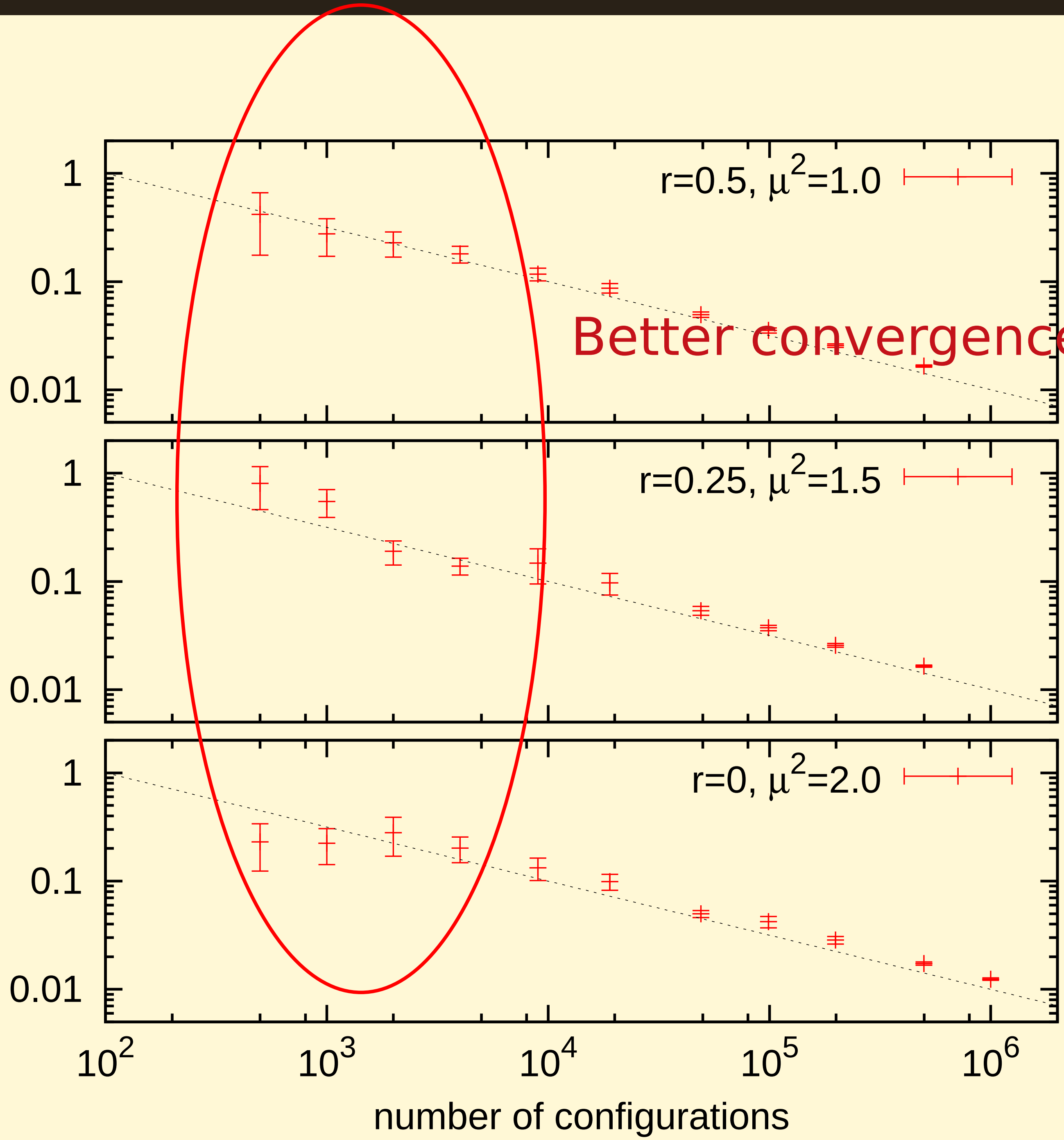

Better convergence to $N_{\text{conf.}}^{-1/2}$

(Normal)

Applications

(may or may not be practical)

- Higher dimensions
- 1-dim supersymmetric Yang-Mills
  M(atrix)-theory [BFSS] : assumes (predicts) index=1

Conclusion

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$$\boxed{\text{Method for measuring the Witten index}}$$

Normalization

- Path integral measure
- $\langle \exp(+S - \frac{1}{2}\mu^2 \sum_i \phi_i^2) \rangle$

Numerical check: SQM (exact Q-symmetry with $Q^2 = 0$)

- correct index
- efficiency: rather poor (less important sampling may help?)

Thank you.

(un negozio ho trovato a Torino)