Effective Potential and Phase Diagram in the Strong-Coupling Lattice QCD with Next-to-Next-to-Leading Order and Polyakov loop effects

T. Z. Nakano (Kyoto Univ, YITP)

Collaborators:

K. Miura (INFN-LNF), A. Ohnishi (YITP)

Polyakov loop effects

TZN, K. Miura, A. Ohnishi, in preparation

NLO and NNLO effects

TZN, K. Miura, A. Ohnishi, Prog. Theor. Phys. **123** (2010), 825

K. Miura, TZN, A. Ohnishi, Prog. Theor. Phys. **122** (2009) , 1045

K. Miura, TZN, A. Ohnishi, N. Kawamoto, Phys. Rev. D **80** (2009) , 074034.

Strong Coupling Lattice QCD (SC-LQCD)

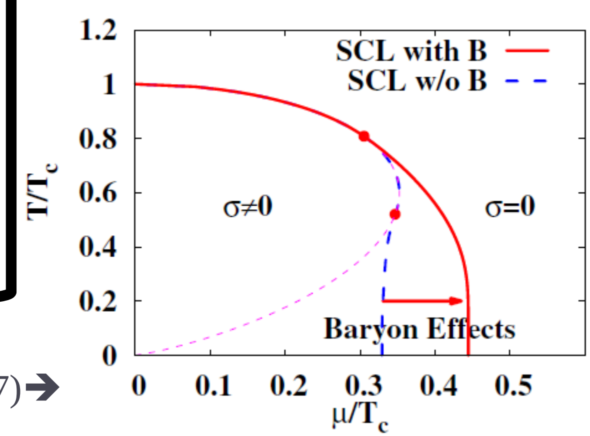
- **Finite μ in Monte Carlo simulation➔ sign problem …**

Strong Coupling Lattice QCD

- **(Semi) Analytic Lattice QCD based on $1/g^2$ expansion**
- **Applicable to the High Density matter**

- **Chiral symmetry in the strong coupling**
  - Spontaneous breaking of the chiral symmetry
    N. Kawamoto and J. Smit ('81),
    H. Kluberg-Stern, A. Morel and B. Petersson ('83)
  - QCD phase diagram
    P. H. Damgaard, N. Kawamoto and K. Shigemoto('84),
    Y. Nishida, K. Fukushima and T. Hatsuda ('04), K. Fukushima ('04)
    A. Ohnishi, N. Kawamoto and K. Miura ('07),
    Ph. de Forcrand and M. Fromm ('09) (Monte Carlo)

Most studies are in $1/g^2=0$.



A. Ohnishi, N. Kawamoto, K. Miura(2007)➔

Purpose

- **Beyond Strong Coupling Limit**
  - Higher order of $1/g^2$ ➔ NLO, **NNLO**

    Faldt, Petersson ('86), N. Bilic, F. Karsch and K. Redlich ('92),
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    A. Ohnishi, K. Miura, TZN, (LAT09)
  - Deconfinement transition ➔ **Polyakov loop**

    Kogut, Snow, Stone ('82), Ilgenfritz, Kripfganz ('85), Gocksch, Ogilvie ('85), Fukushima ('03)

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  - Deconfinement transition ➔ **Polyakov loop**
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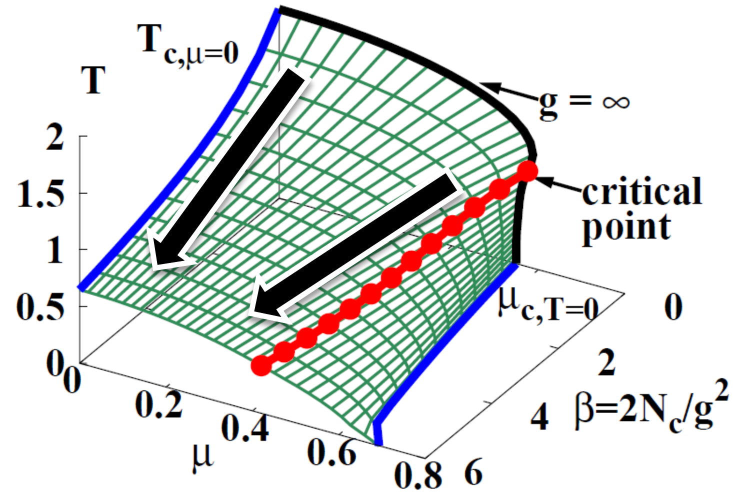
- **NNLO results**
  - $T_{c,\,\mu=0}$ ↓
  - Critical point ➔ small μ
    ➔favorable, however insufficient.
    first order in the staggered fermion (N_f=4)
    D'Elia-Lombardo('03)



in the lattice unit , chiral limit (m_0=0)

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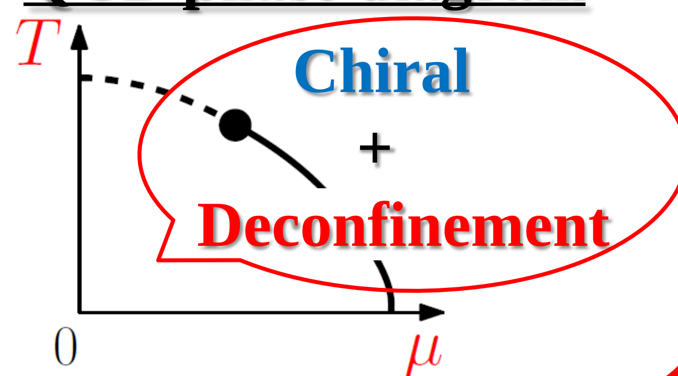
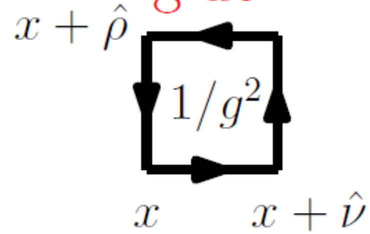
Effective Potential

QCD phase diagram



Lattice QCD Action

$$S_{\mathrm{LQCD}}[\chi, \bar{\chi}, U_\nu]$$

= antiquark quark $\eta_{\nu,x}$  $\bar{\chi}_x \; U_{\nu,x} \; \chi_{x+\hat{\nu}}$ + gluon 

$\mathrm{tr}_c U_{\nu\rho,x}$ + h.c. + mass (m_0) term  $\bar{\chi}_x \chi_x$

staggered factor

$$\eta_{\nu,x} = \begin{cases} e^{\mu} & (\nu = 0) \\ (-1)^{x_0+\cdots+x_{j-1}} & (\nu \neq 0) \end{cases}$$

$(\nu = 0)$ ← **chemical potential**

$(\nu \neq 0)$ ← **γ_5 − related factor**

- **SU(3)$_c$, Staggered fermion (N_f = 4 in the continuum limit)**

Formulation

Partition function

$Z[\chi, \bar{\chi}, U_\nu]$

$= \int \mathcal{D}[\chi, \bar{\chi}, U_\nu] \exp(-S_{\mathrm{LQCD}}[\chi, \bar{\chi}, U_\nu])$

Formulation

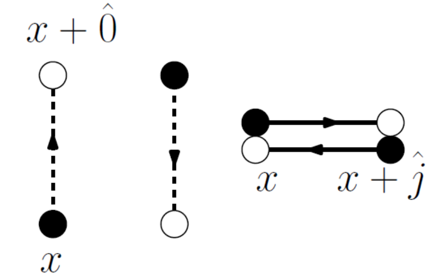
Partition function

$$Z[\chi,\bar{\chi},U_\nu]$$
$$= \int \mathcal{D}[\chi,\bar{\chi},U_\nu]\exp(-S_{\rm LQCD}[\chi,\bar{\chi},U_\nu])$$
$$= \int \mathcal{D}[\chi,\bar{\chi},U_0]\exp(-S_{\rm SCL})\langle\exp(-S_G)\rangle$$
$$= \int \mathcal{D}[\chi,\bar{\chi},U_0]\exp(-S_{\rm eff}[\chi,\bar{\chi},U_0;\mathbf{\Phi}])$$
$$(\mathbf{\Phi}=\sigma,\omega_\tau,\ell,\bar{\ell})$$

1. $\mathcal{D}U_j$

2. bosonization

3. MF, saddle pt. approximation



Formulation

Partition function

$$Z[\chi, \bar{\chi}, U_\nu]$$
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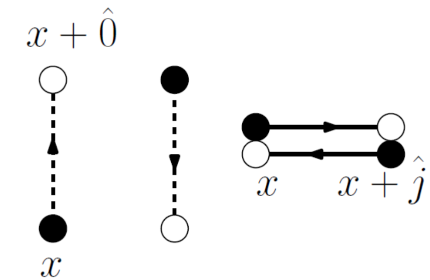
$$= \int \mathcal{D}[\chi, \bar{\chi}, U_0] \exp(-S_{\mathrm{SCL}}) \langle \exp(-S_G) \rangle$$
$$= \int \mathcal{D}[\chi, \bar{\chi}, U_0] \exp(-S_{\mathrm{eff}}[\chi, \bar{\chi}, U_0; \mathbf{\Phi}])$$
$$(\mathbf{\Phi} = \sigma, \omega_\tau, \ell, \bar{\ell})$$



1. $\mathcal{D}U_j$

2. bosonization

3. MF, saddle pt. approximation

4. $\mathcal{D}[\chi, \bar{\chi}]$

5. $\mathcal{D}U_0$

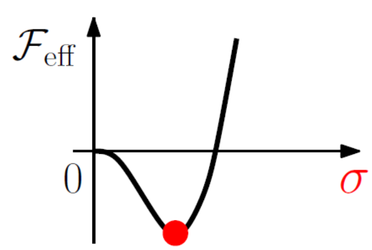
P. H. Damgaard, N. Kawamoto and K. Shigemoto ('84)

Effective potential

$$\mathcal{F}_{\mathrm{eff}}[\mathbf{\Phi}; \mu, T] = -\frac{T}{V} \log Z$$

Stationary condition

$$\frac{\partial \mathcal{F}_{\mathrm{eff}}}{\partial \mathbf{\Phi}} = 0$$



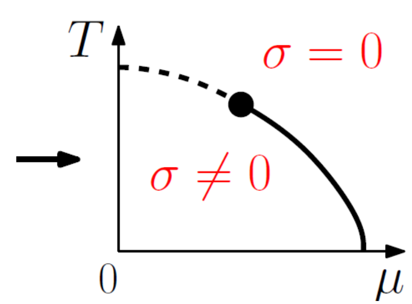


Diagram (NNLO + Polyakov loop)

- **Cluster expansion of exp(-S_G) ➔Connected diagrams**
  - Quark : leading order of 1/d expansion Kluberg-Stern, Morel Petersson ('83)

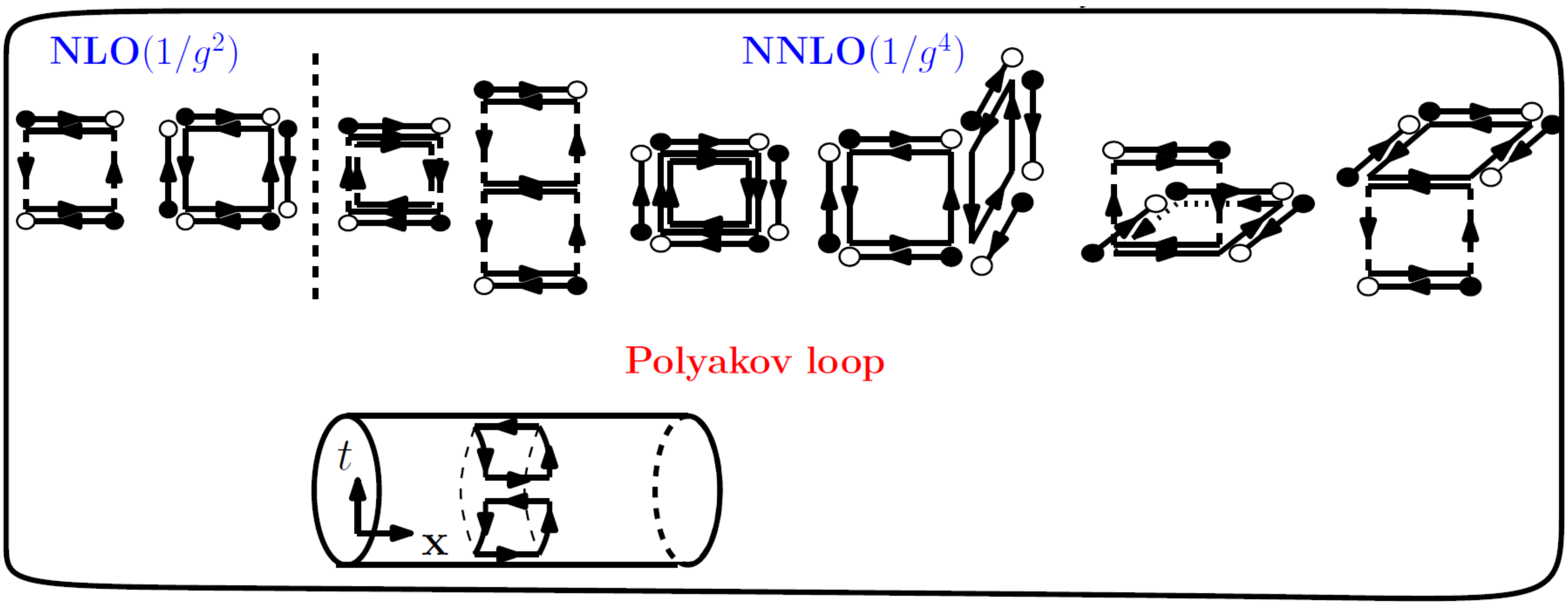
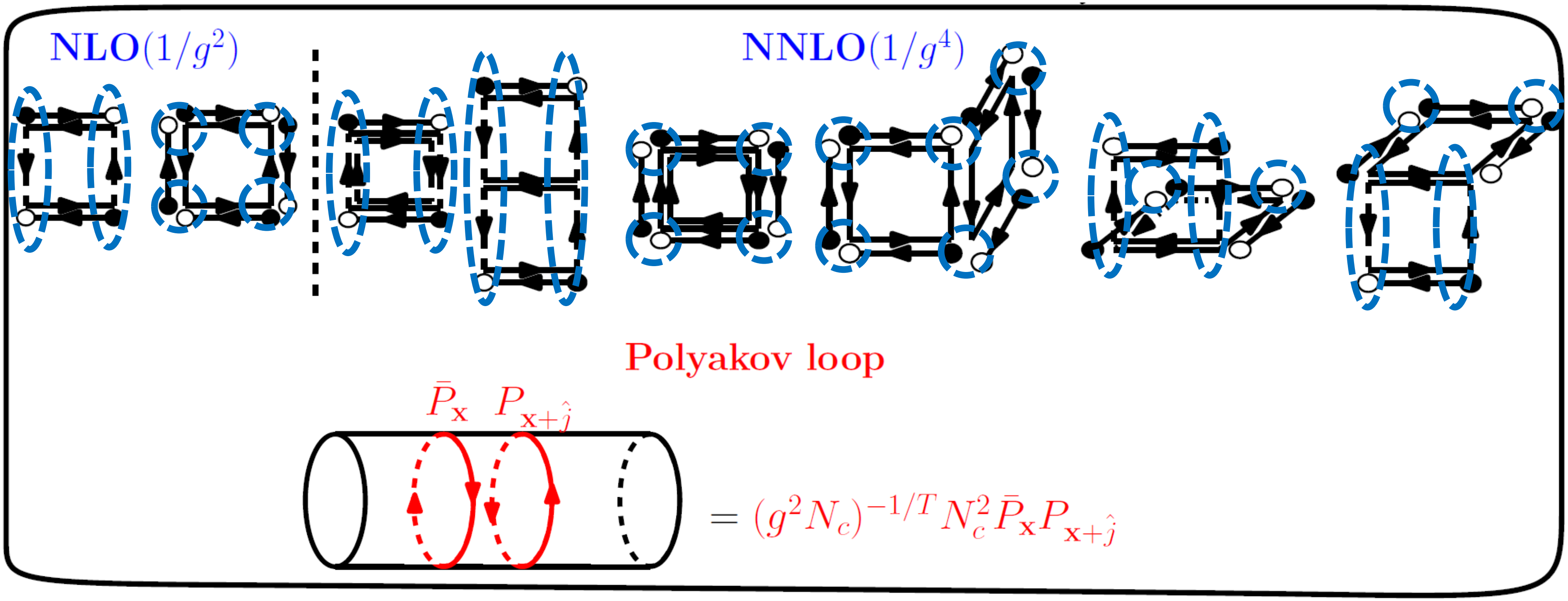


Diagram (NNLO + Polyakov loop)

- **Cluster expansion of $\exp(-S_G)$ ➔ Connected diagrams**
  - Quark : leading order of 1/d expansion Kluberg-Stern, Morel Petersson ('83)
- **Spatial link intgral ➔ Hadron, Polyakov loop**



Bosonization

- **Extended Hubbard-Stratonovich (EHS) transformation**

K. Miura, TZN, A. Ohnishi (2009) , K. Miura, TZN, A. Ohnishi, N. Kawamoto (2009)

$$e^{\alpha AB} = \int d\varphi d\phi e^{\alpha\left\{[\varphi-(A+B)/2]^2+[\phi-i(A-B)/2]^2+\alpha AB\right\}}$$

$$\approx e^{-\alpha\left\{\varphi^2-(A+B)\varphi+\phi^2-i(A-B)\phi\right\}}\Big|_{\text{stationary}}$$

$$\varphi = \langle A+B\rangle/2,\ \phi = i\langle A-B\rangle/2$$

$$\bar{\psi} = \varphi - i\phi,\ \psi = \varphi + i\phi$$

$$\approx e^{-\alpha\left\{\bar{\psi}\psi - A\psi - \bar{\psi}B\right\}}\Big|_{\text{stationary}}$$

$$\bar{\psi} = \langle A\rangle,\ \psi = \langle B\rangle$$

- **Polyakov loop action**

$$\Delta S_p \propto (\bar{P}_{\mathbf{x}} P_{\mathbf{x}+\hat{j}} + h.c.)$$

$$\approx 2(\bar{\ell}\ell - \bar{P}_{\mathbf{x}}\ell - \bar{\ell}P_{\mathbf{x}})$$

$$\begin{cases} \bar{\ell} = \langle \bar{P}_{\mathbf{x}}\rangle \\ \ell = \langle P_{\mathbf{x}}\rangle \end{cases}$$

Effective Potential (NNLO + Polyakov loop)

$$\mathcal{F}_{\text{eff}}[\sigma, \omega_\tau, \underbrace{\ell, \bar{\ell}}; \mu, T] = \mathcal{F}_q + U_g + \cdots$$

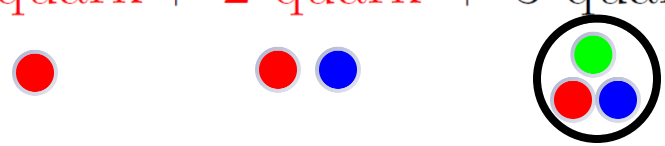
Polyakov loop

Quark number density

Chiral condensate

●Quark

$$\mathcal{F}_q = \text{1-quark} + \text{2-quark} + \text{3-quark} + \cdots$$



●Pure glue

$$U_g = 2T \underbrace{\beta_p}_{(g^2 N_c)^{-1/T} 3N_c^2} \bar{\ell}\ell - T \log \underbrace{D_1}$$

finite μ modifed bessel function

$$\Sigma_n e^{N_c n \log(\bar{\ell}/\ell)/2} [I_n^3 + \cdots]$$

Effective Potential (NNLO + Polyakov loop)

$$\mathcal{F}_{\rm eff}[\sigma, \omega_\tau, \underbrace{\ell, \bar{\ell}}; \mu, T] = \mathcal{F}_q + U_g + \cdots$$

- σ → Chiral condensate
- ω_τ → Quark number density
- $\ell, \bar{\ell}$ → Polyakov loop

● **Quark**

$$\mathcal{F}_q = \text{1-quark} + \text{2-quark} + \text{3-quark} + \cdots$$

Polyakov loop effects
➔ Quark excitation

NNLO effects

$E_q, \tilde{\mu}, Z_\chi$

● **Pure glue**

$$U_g = 2T \underbrace{\beta_p}_{(g^2N_c)^{-1/T}3N_c^2} \bar{\ell}\ell - T \log \underbrace{D_1}$$

D_1 → $\Sigma_n e^{N_c n \log(\bar{\ell}/\ell)/2}[I_n^3 + \cdots]$

finite μ → $e^{N_c n \log(\bar{\ell}/\ell)/2}$; modifed bessel function → I_n^3

Effective Potential (NNLO + Polyakov loop)

$$\mathcal{F}_{\rm eff}[\sigma, \omega_\tau, \underbrace{\ell, \bar{\ell}}; \mu, T] = \mathcal{F}_q + U_g + \cdots$$

Polyakov loop

Quark number density

Chiral condensate

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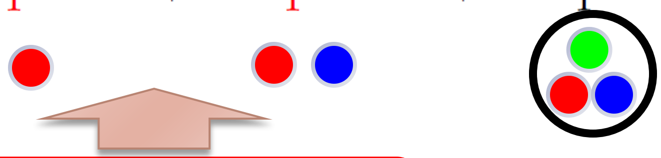


Polyakov loop effects
➔ Quark excitation

NNLO effects

$E_q, \tilde{\mu}, Z_\chi$

●**Pure glue** Kogut, Snow, Stone ('82) : μ=0

finite μ modifed bessel function

$$U_g = 2T \underbrace{\beta_p}_{(g^2N_c)^{-1/T}3N_c^2} \bar{\ell}\ell - T \log \underbrace{D_1}$$

$$\to \Sigma_n e^{N_c n \log(\bar{\ell}/\ell)/2}[I_n^3 + \cdots]$$

Effective Potential (NNLO + Polyakov loop)

- Analytically….

$$\mathcal{F}_q = -T\log(Z_P/D_1) - N_c \log Z_\chi \ ,$$

$$\begin{aligned} Z_P/D_1 =& \frac{\sinh((N_c+1)E_q/T)}{\sinh(E_q/T)} + 2\cosh(N_c\tilde{\mu}/T) \\ &+4\cosh^2(E_q/T)(\widetilde{D}_4 e^{\tilde{\mu}/T} + \widetilde{D}_{-4} e^{-\tilde{\mu}/T}) \\ &+2\cosh(E_q/T)(\widetilde{D}_{-4} e^{2\tilde{\mu}/T} + \widetilde{D}_4 e^{-2\tilde{\mu}/T} + \widetilde{D}_2 + 2) \\ &+\widetilde{D}_6 e^{\tilde{\mu}/T} + \widetilde{D}_{-6} e^{-\tilde{\mu}/T} \ , \end{aligned}$$

Effective Potential (NNLO + Polyakov loop)

- Many modified Bessel functions …

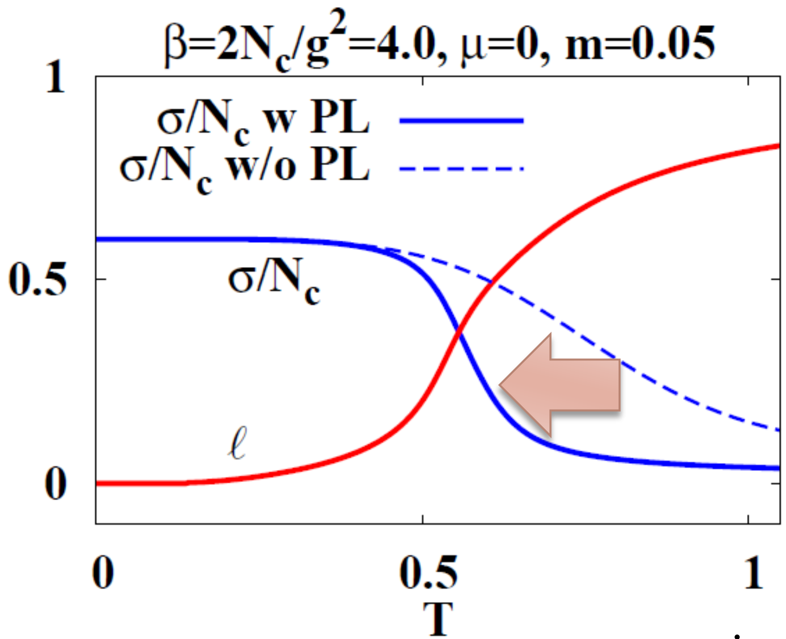
$$D_i(L,\bar{L}) \equiv \sum_{n=-\infty}^{\infty} e^{N_c n\phi} D_{n,i} \ ,$$

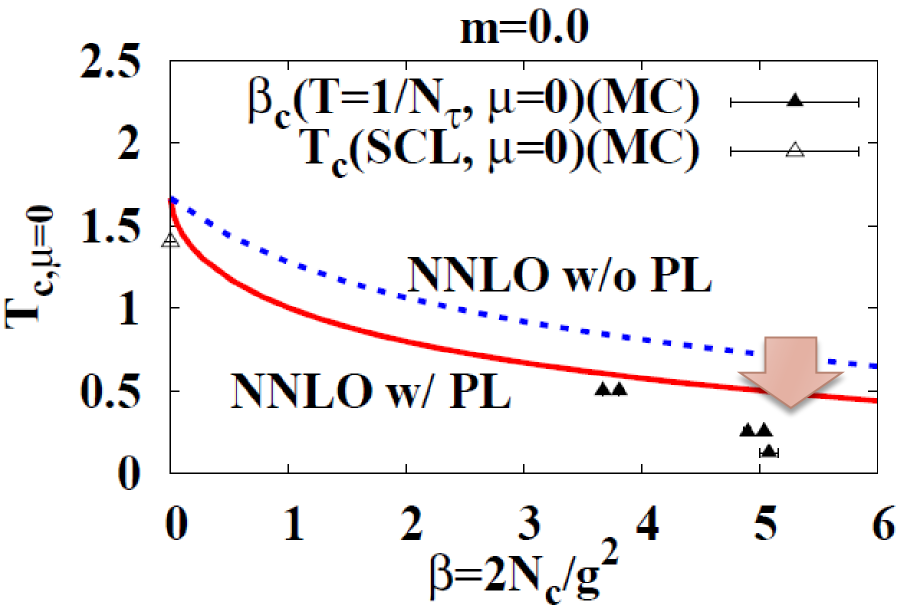
$$\begin{aligned}
D_{n,1} &= I_n^3 - I_{n+2}I_nI_{n-2} - 2I_{n+1}I_nI_{n-1} \\
&+ I_{n+1}^2 I_{n-2} + I_{n-1}^2 I_{n+2} \ , \\
D_{n,2} &= -2(I_n^3 - I_{n+2}I_nI_{n-2}) + 5I_{n+1}I_nI_{n-1} \\
&- 3(I_{n+1}^2 I_{n-2} + I_{n-1}^2 I_{n+2}) - I_{n+3}I_nI_{n-3} \\
&+ I_{n-1}I_{n-2}I_{n+3} + I_{n+1}I_{n+2}I_{n-3} \ , \\
D_{n,\pm 3} &= D_{n\mp 1,1} \ , \\
D_{n,\pm 4} &= I_n^2 I_{n\mp 1} + I_{n\pm 1}^2 I_{n\mp 3} \\
&- I_{n\mp 1}^2 I_{n\pm 1} + I_{n\pm 2}I_{n\mp 1}I_{n\mp 2} - I_{n\pm 2}I_nI_{n\mp 3} \\
&- I_{n\pm 1}I_nI_{n\mp 2} \ , \\
D_{n,\pm 5} &= D_{n\mp 1,\mp 4} \ , \\
D_{n,\pm 6} &= -(I_n^2 I_{n\mp 1} + I_{n\pm 1}^2 I_{n\mp 3}) \\
&+ 2(I_{n\mp 1}^2 I_{n\pm 1} - I_{n\pm 2}I_{n\mp 1}I_{n\mp 2} + I_{n\pm 2}I_nI_{n\mp 3}) \\
&+ I_{n\mp 2}^2 I_{n\pm 3} - I_{n\pm 3}I_{n\mp 1}I_{n\mp 3} \ .
\end{aligned}$$

Chiral Phase Transition (μ=0)

- Polyakov loop effects ➔ Chiral condensate decreases.
- Close to the Monte Carlo results at β~4.

▲(MC)➔ From left,
Ph. de Forcrand and M. Fromm ('09),
Ph. de Forcrand, private communication,
S.A.Gottlieb et al. ('87) ,
D'Elia and Lombardo ('03)
Z.Fodor and S. D. Katz ('02),
R.V.Gavai et al. ('90)

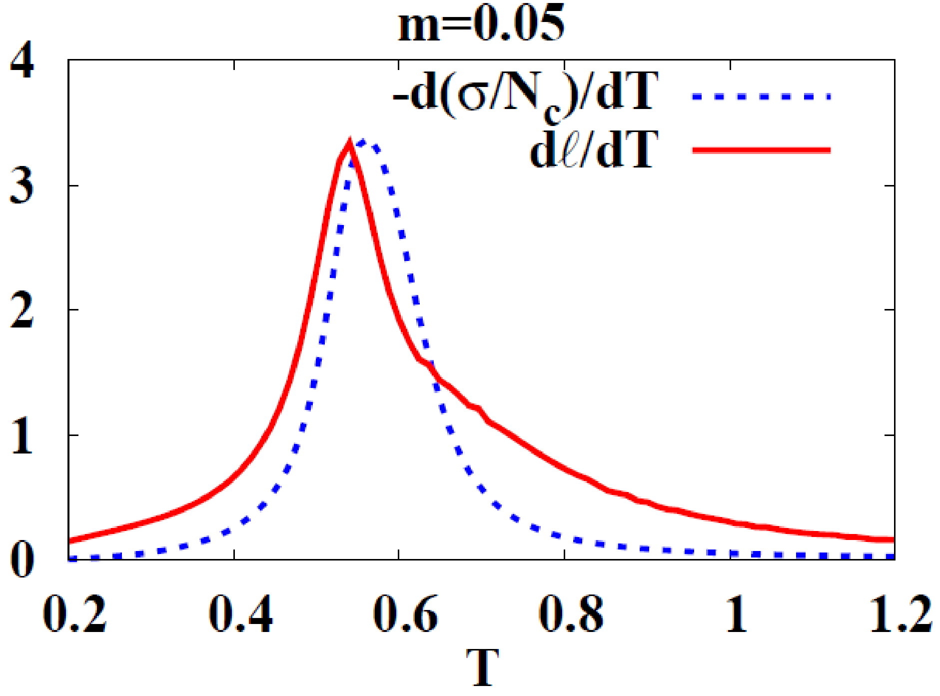




in the lattice unit

Deconfinement Phase Transition (μ=0)

- Simultaneous treatment
  - Chiral phase transition
  - Deconfinement phase transition
- $T_\chi \sim T_d$ cf. Aoki, Fodor, Katz, Szabo ('06)



in the lattice unit

Haar Measure Method

- Polyakov loop (cf. Miura's talk)
  - Background field
  - w/o explicit temporal link integral

$$\Delta S_p \propto -\left(\bar{P}_{\mathbf{x}} P_{\mathbf{x}+\hat{j}} + h.c.\right)$$
$$\approx -2\bar{\ell}\ell$$

$$\begin{cases} \bar{\ell} = \langle \bar{P}_{\mathbf{x}} \rangle \\ \ell = \langle P_{\mathbf{x}} \rangle \end{cases}$$

Haar Measure Method

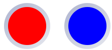
- Polyakov loop (cf. Miura's talk)
  - Background field
  - w/o explicit temporal link integral

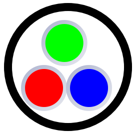
Fukushima ('04), Ratti, Thaler, Weise ('06)

- Polyakov loop effects ➔Haar measure (cf. PNJL model)

$$\mathcal{F}_{\rm eff}[\sigma, \omega_\tau, \ell, \bar{\ell}; \mu, T] = \mathcal{F}_q + U_g + \cdots$$

●**Quark**





$$\mathcal{F}_q = \underbrace{-N_c E_q}_{\text{vacuum}} \underbrace{- T \log[1 + N_c \ell e^{-(E_q - \tilde{\mu})/T} + N_c \bar{\ell} e^{-2(E_q - \tilde{\mu})/T} + e^{-3(E_q - \tilde{\mu})/T}]}_{\text{quarks}} + \text{antiquarks}$$

$$- N_c \log \underbrace{Z_\chi}_{\text{w.f.r.}}$$

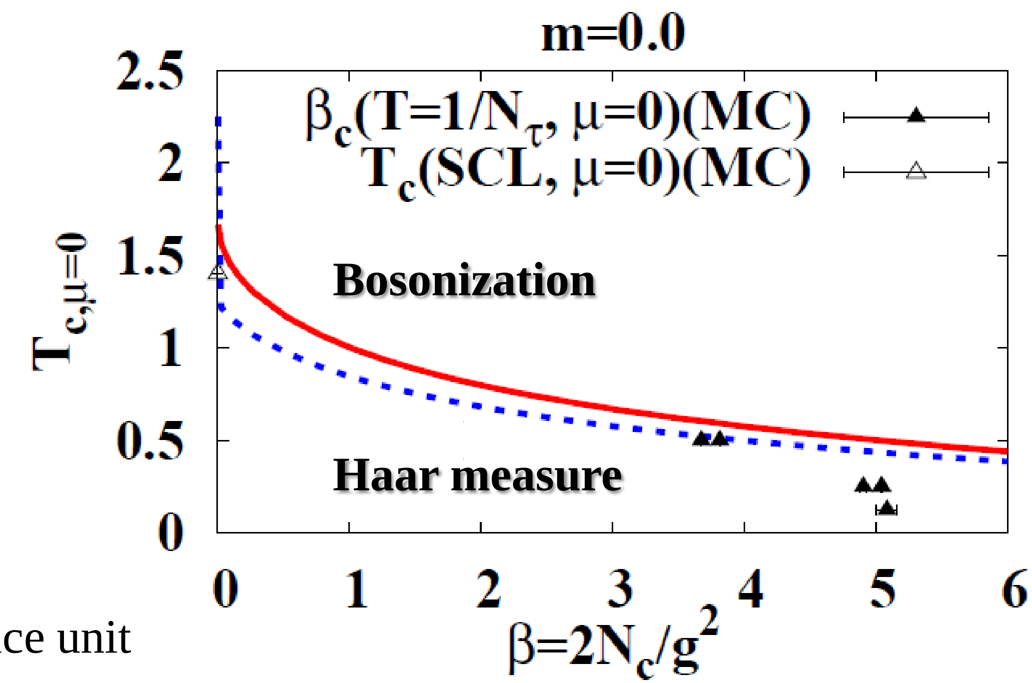
●**Pure glue**

$$U_g = -2T \underbrace{\beta_p}_{(g^2 N_c)^{-1/T} 3N_c^2} \bar{\ell}\ell - T \log \underbrace{[1 - 6\ell\bar{\ell} + 4(\ell^3 + \bar{\ell}^3) - 3(\ell\bar{\ell})^2]}_{\text{Haar measure}}$$

Bosonization v.s. Haar Measure (μ=0)

- Qualitatively same, quantitatively different
  - Bosonization : Weiss MF approx.
  - Haar measure : Background field
- Coupling of the quarks and gluons (Bosonization)
  - Temporal link integral
    ➔Favor color-singlet
    ➔Suppression of
    the quark excitation

    Megias, Arriola, Salcedo ('06)

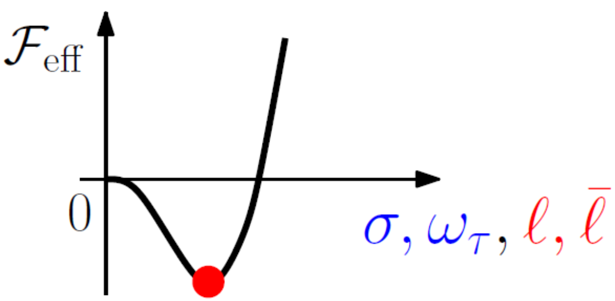


in the lattice unit

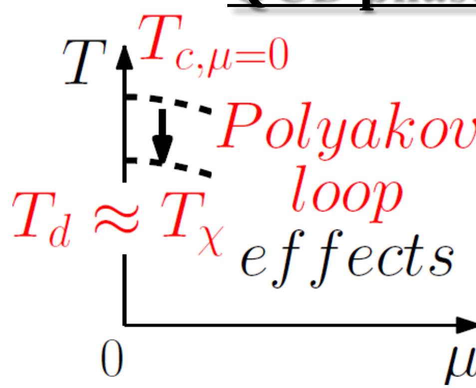
Summary & Future Works

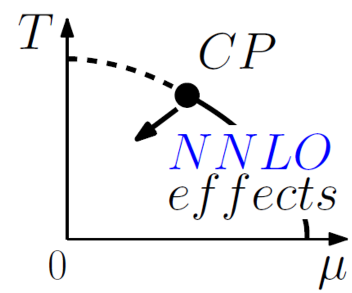
- We have derived the effective potential with **Polyakov loop** and **NNLO** effects in strong coupling lattice QCD.

Effective Potential



QCD phase diagram





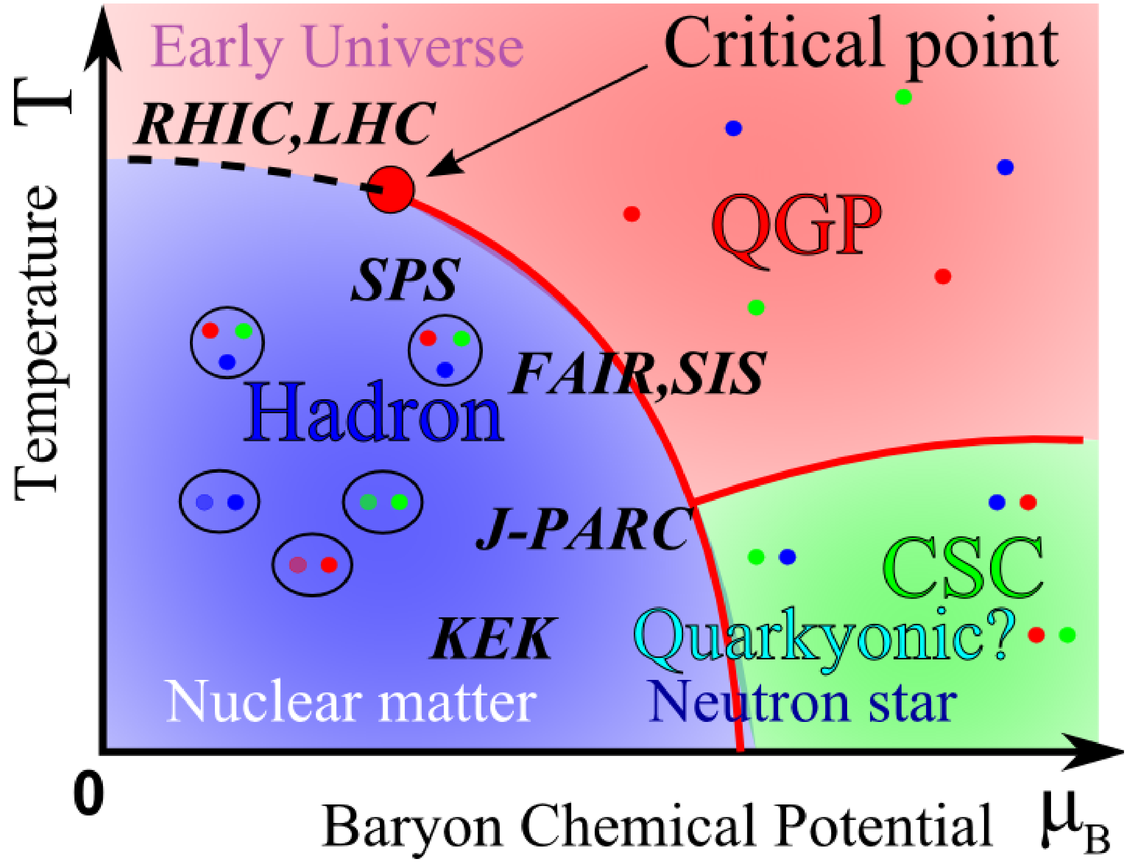
Polyakov loop : quark excitation
NNLO : modification of μ

Bosonization v.s. Haar measure
➔Qualitatively same, quantitatively different

- Future works
  - Phase diagram with the deconfinement transition
  - Higher Order of Polyakov loop Langelage, Münster, Philipsen ('08), Langelage, Philipsen ('10)

Back Up

QCD Phase Diagram



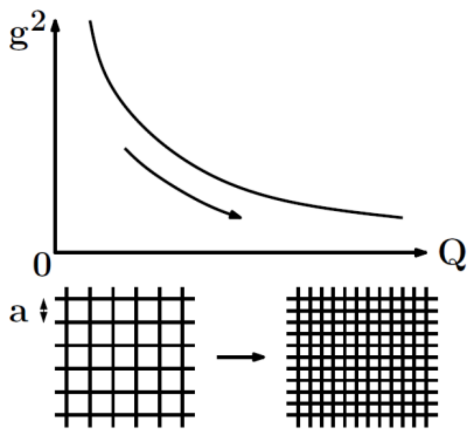
- **Chiral** and **Deconfinement** phase transition
  - **Where is the critical point?**
  - **What phase can appear in the the high density region?**

Strong Coupling Lattice QCD (SC-LQCD)

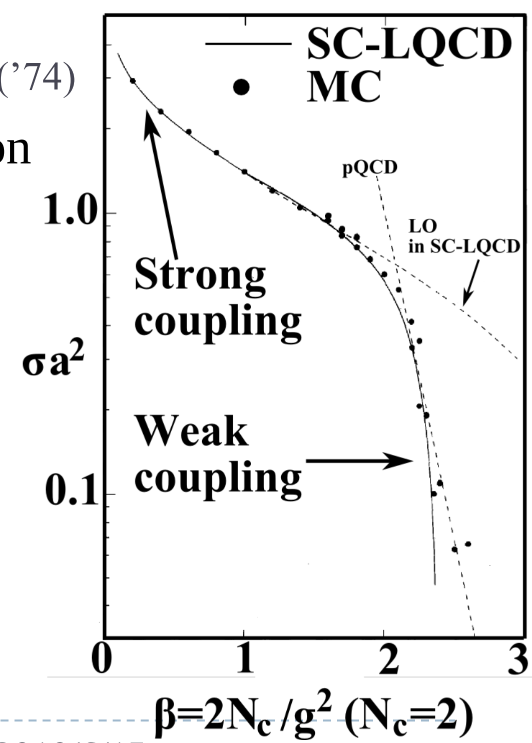
- (Semi) **Analytic** Lattice QCD based on $1/g^2$ expansion
- Applicable to **the High Density** matter
  (Sign problem can be weakened or avoided)

- **Pure glue**
  - Area law of the Wilson loop at strong coupling K.G. Wilson ('74)
  - Continuity between the strong and the weak coupling region

SC-LQCD : G. Münster ('81), MC : M. Creutz ('80, '82)



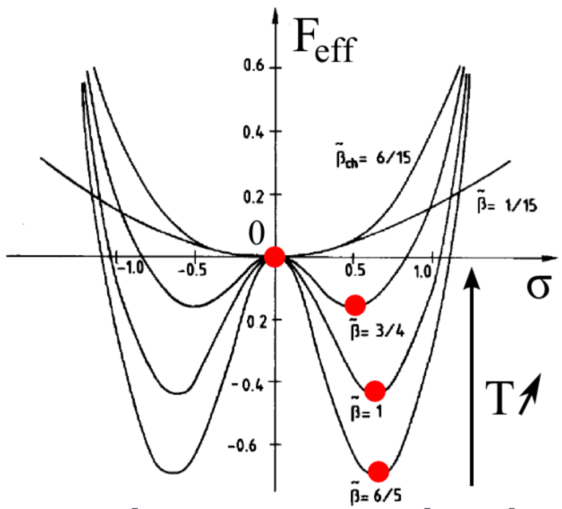
↓G. Münster ('81)



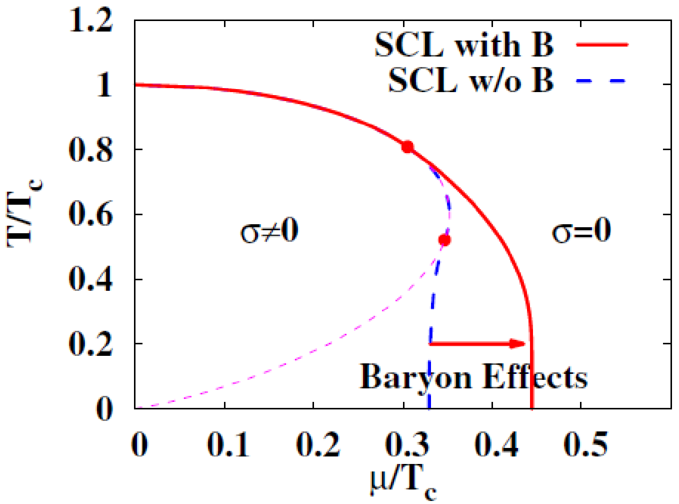
Chiral Symmetry in $1/g^2=0$

- **Many studies…**
  - Spontaneous breaking of the chiral symmetry
    N. Kawamoto and J. Smit ('81),
    H. Kluberg-Stern, A. Morel and B. Petersson ('83)
  - QCD phase diagram
    P. H. Damgaard, N. Kawamoto and K. Shigemoto('84),
    Y. Nishida, K. Fukushima and T. Hatsuda ('04), K. Fukushima ('04)
    A. Ohnishi, N. Kawamoto and K. Miura ('07),
    Ph. de Forcrand and M. Fromm ('09) (Monte Carlo)

Most studis are in $1/g^2=0$.



↑P. H. Damgaard, N. Kawamoto and K. Shigemoto('84),
in the lattice unit



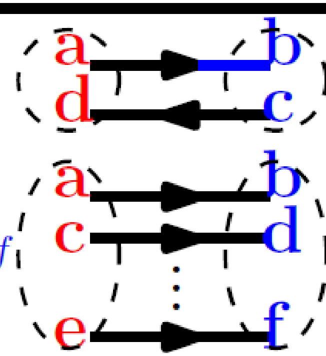
A. Ohnishi, N. Kawamoto, K. Miura(2007)↑

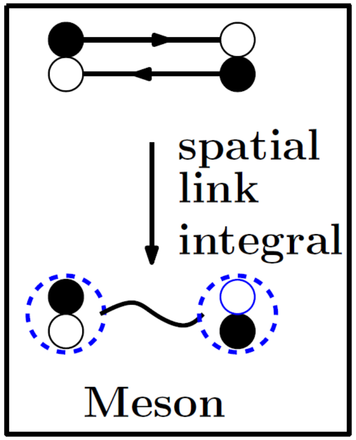
Spatial Link Integral

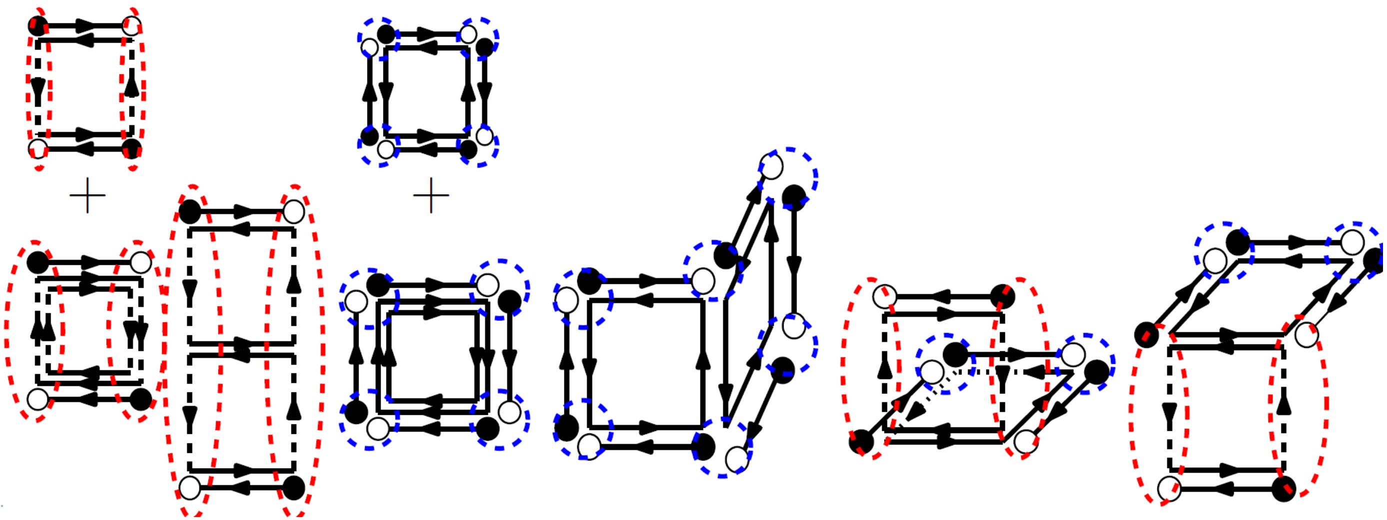
- **Quark composites→ Hadron**

$$\int dU U_{ab} U^{\dagger}_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc}$$

$$\int dU U_{ab} U_{cd} \cdots U_{ef} = \frac{1}{N_c!}\epsilon_{ac\cdots e}\epsilon_{bd\cdots f}$$







NNLO Effective Action

- **Factor ➔ systematic evaluation**

$$S_{\rm NNLO}$$

SCL

$$= \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0} [MM]_{j,x}$$

NLO + NNLO

$$+\frac{1}{4N_c^2 g^2}\left(1+\frac{1}{2g^2}\right)\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x}$$

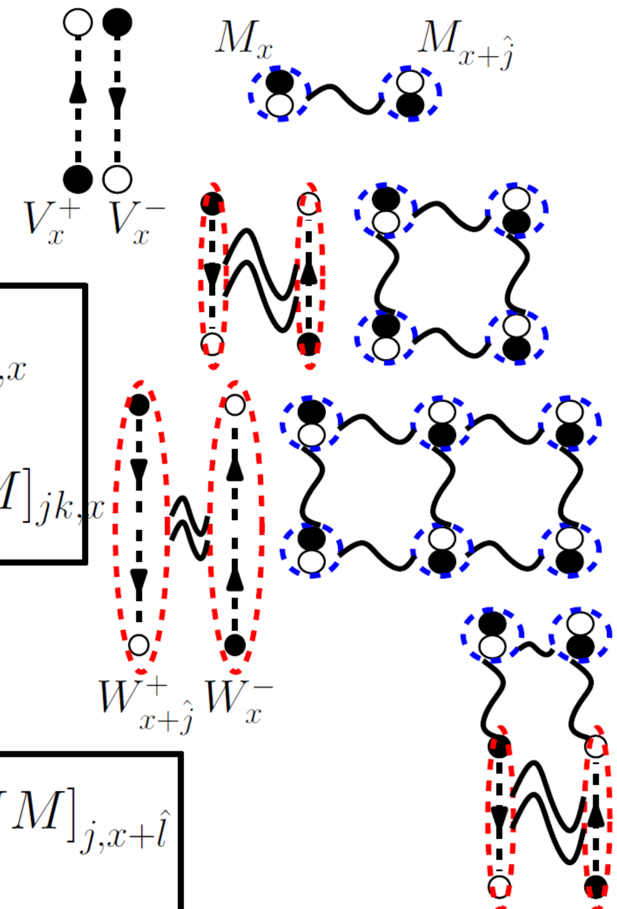
$$-\frac{1}{16N_c^4 g^2}\left(1+\frac{1}{2g^2}\right)\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$

NNLO

$$-\frac{1}{4N_c^3 g^4}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x}$$

$$-\frac{1}{64N_c^7 g^4}\sum_{\substack{x,\, j>0,\, |k|>0,\, |l|>0\\ |k|\neq j,\ |l|\neq j,\ |l|\neq |k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

$$+\frac{1}{16N_c^5 g^4}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$

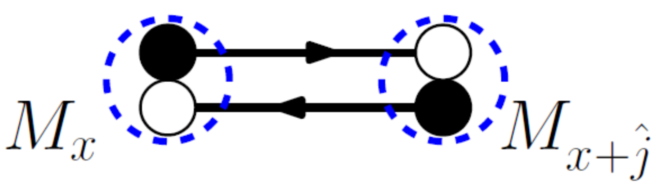


Bosonization

- **Bosonization➔evaluation of the Hadron term**
  - Chiral condensate (σ)
  - Vector potential ($\omega_\tau \approx -\partial F_{eff} / \partial \mu \equiv \rho_q$)
- **Hubbard Stratonovich transformation (cf.Ising model)**

$$e^{\alpha M_x M_{x+\hat{j}}} = \int \mathcal{D}\sigma \exp\left[-\alpha(\sigma + M)_x(\sigma + M)_{x+\hat{j}} + \alpha M_x M_{x+\hat{j}}\right]$$

$$\approx \exp\left(-\alpha\sigma^2 - 2\alpha M_x \sigma\right) \qquad \sigma = \langle -M_x \rangle$$



$$M_x = \bar{\chi}_x \chi_x$$

Bosonization

- **Extended Hubbard-Stratonovich (EHS) transformation**

$$e^{\alpha\left(-V_x^+ V_{x+\hat{j}}^-\right)} = \int \mathcal{D}[\varphi,\phi] e^{-\alpha\left\{\varphi_x - \left(-V_x^+ + V_{x+\hat{j}}^-\right)/2\right\}^2 + \left\{\phi_x - i\left(-V_x^+ - V_{x+\hat{j}}^-\right)/2\right\}^2 + \alpha\left(-V_x^+ V_{x+\hat{j}}^-\right)}$$

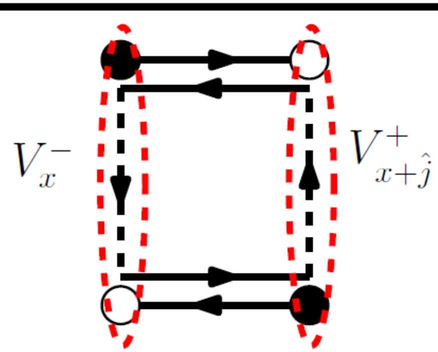
$$\approx e^{-\alpha\left[\varphi^2 - \varphi\left(-V_x^+ + V_x^-\right)/2 + \phi^2 - i\phi\left(-V_x^+ - V_x^-\right)/2\right]}$$

$$\varphi = -\langle V_x^+ - V_x^- \rangle/2$$
$$\phi = i\langle V_x^+ + V_x^- \rangle/2$$

chiral invariant

$$\phi = i\omega$$
$$\omega = -\langle V_x^+ + V_x^- \rangle/2 = \rho_q$$

$$-\omega^2 + \omega\left(-V_x^+ - V_x^-\right)/2$$



$$V_x^+ = \bar{\chi}_x e^{\mu} U_{0,x} \chi_{x+\hat{0}}$$
$$V_x^- = \bar{\chi}_{x+\hat{0}} e^{-\mu} U_{0,x}^{\dagger} \chi_x$$

Bosonization

$$e^{\alpha AB} = \int d\varphi\, d\phi\, e^{-\alpha\left\{[\varphi-(A+B)/2]^2+[\phi-i(A-B)/2]^2\right\}+\alpha AB}$$

$$\approx e^{-\alpha\left\{\varphi^2-(A+B)\varphi+\phi^2-i(A-B)\phi\right\}}\Big|_{\text{stationary}}$$

$$\approx e^{-\alpha\left\{\bar{\psi}\psi-A\psi-\bar{\psi}B\right\}}\Big|_{\text{stationary}} ,$$

$$e^{\alpha AB} \approx e^{-\alpha\left\{\varphi^2-(A+B)\varphi-\omega^2+(A-B)\omega\right\}}\Big|_{\text{stationary}} .$$

Cluster Expansion

- n=1 ➔NLO, n=2 ➔NNLO

$$e^{-S_{\rm eff}(\chi,\bar{\chi},U_0)} = \int \mathcal{D}U_j e^{-S_{\rm LQCD}} = e^{-S_F^{(\tau)}} \int \mathcal{D}U_j e^{-S_F^{(s)}-S_G}$$

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}U_j \mathcal{O}[U_j] e^{S_F^{(s)}}}{\int \mathcal{D}U_j e^{S_F^{(s)}}}$$

$$\int \mathcal{D}U_j e^{S_F^{(s)}} = e^{-S_{\rm SCL}^{(s)}}$$

$$= e^{-S_F^{(\tau)}-S_{\rm SCL}^{(s)}} \langle e^{-S_G} \rangle$$

$$\langle e^{-S_G} \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_G^n \rangle = \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle S_G^n \rangle_c\right]$$

cumulant (↑ $\langle S_G^n \rangle_c$)

1/d Expansion (N_c=3)

- H.Kluberg-Stern, A.Morel, B.Petersson (1983)
- K.Miura, TZN, A.Ohnishi and N.Kawamoto (2009)

$$Z^{(s)}_{(\mathrm{SCL})} = \int \mathcal{D}U_j e^{-S_F^{(s)}} = \prod_{j,x}\left[\int \mathcal{D}U_j e^{-s_{j,x}}\right]$$

$$\int \mathcal{D}U_j e^{-s_{j,x}} = \exp\left(-s^{(\mathrm{eff})}_{j,x}\right)$$

$$M_x = \bar{\chi}_x \chi_x,\ \ B_x = \frac{1}{N_c!}\epsilon^{ab\cdots c}\left(\chi^a_x \chi^b_x \cdots \chi^c_x\right)$$

$$s^{(\mathrm{eff})}_{j,x} = \sum_{n=1}^{N_c} A_n \left(M_x M_{x+\hat{j}}\right)^n + A_{j,x}\left(\bar{B}_x B_{x+\hat{j}} + (-1)^{N_c}(h.c.)\right)$$

$$\sum_j A_1 M_x M_{x+\hat{j}} \Longrightarrow \mathrm{O}(1/d^0)$$

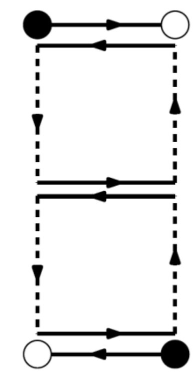
$$\chi_x \Longrightarrow \mathrm{O}(1/d^{1/4})$$

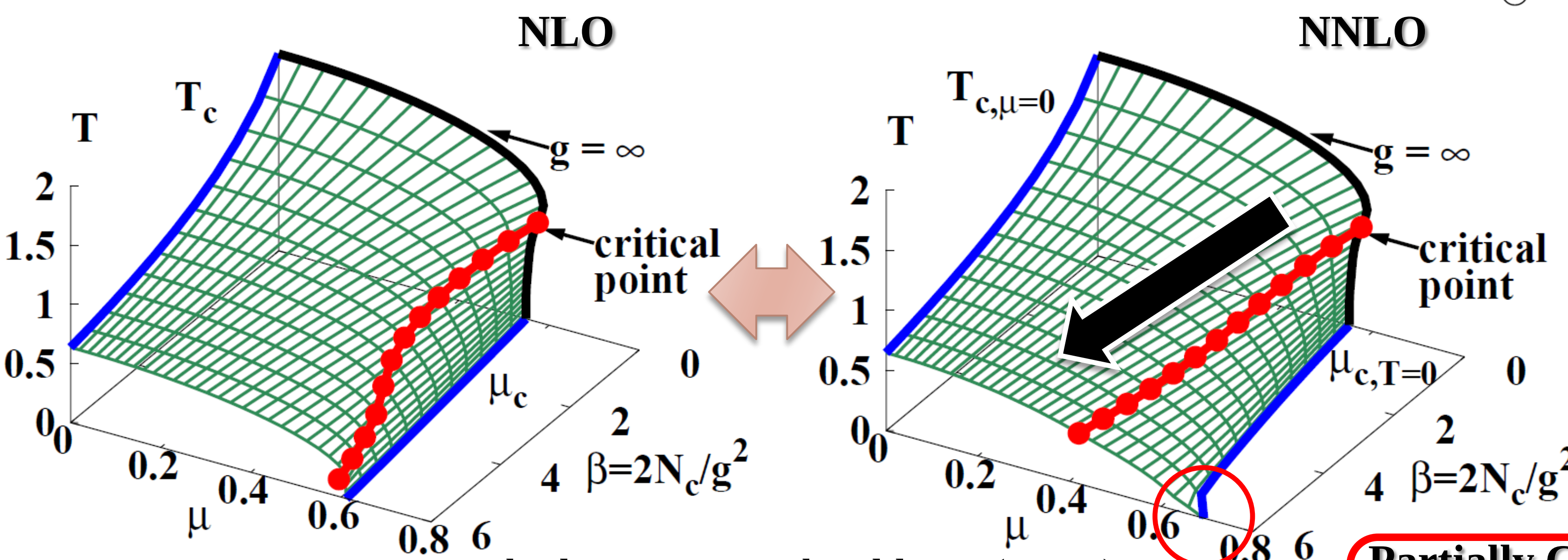
$$n = 0 \Longrightarrow \mathrm{O}(1/d^0)$$
$$n = 1 \Longrightarrow \mathrm{O}(1/d^1)$$
$$n = 2 \Longrightarrow \mathrm{O}(1/d^2)$$

Baryon term: $\mathrm{O}(1/d^{1/2})$

QCD Phase Diagram (NNLO)

- **Critical Point(CP)**
  - NNLO effects ➔ **smaller μ**
    ➔ favorable, however insufficient.
    first order in the staggered fermion (N_f=4) D'Elia-Lombardo('03)
  - Origin : next-to-nearest neighbor interaction



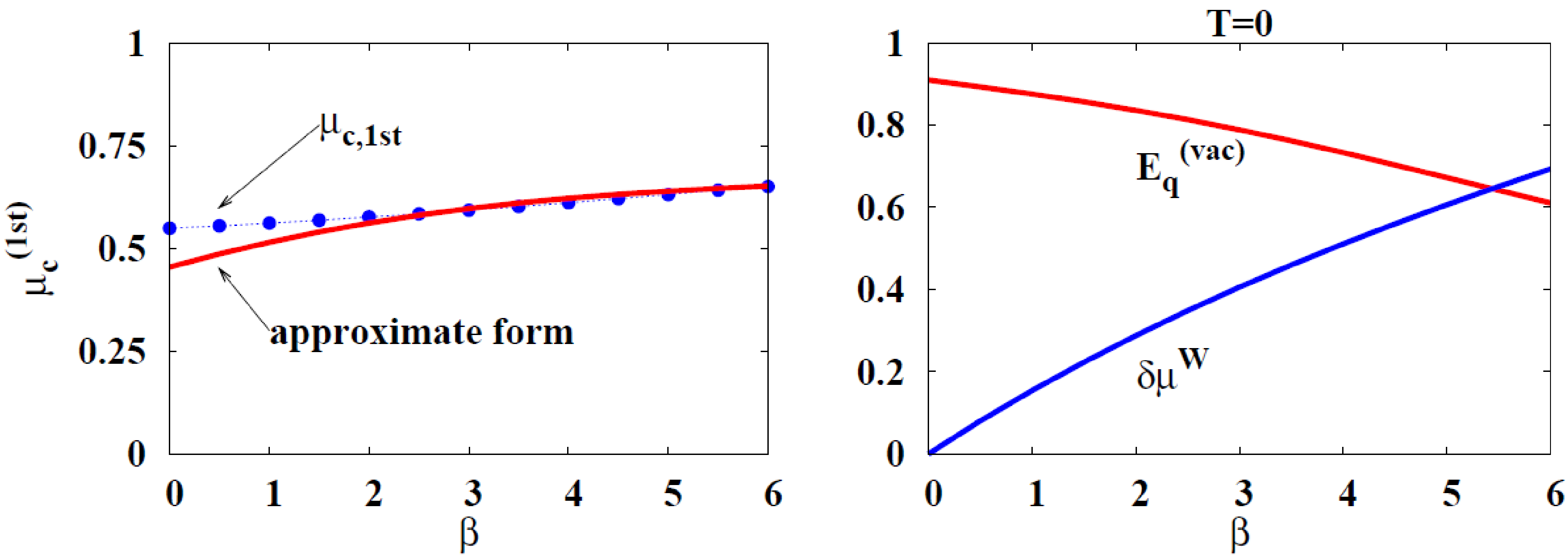


in the lattice unit , chiral limit (m_0=0)

Partially Chiral Restored Phase ($\sigma \ll \sigma_{vac}$, ρ_q>0)

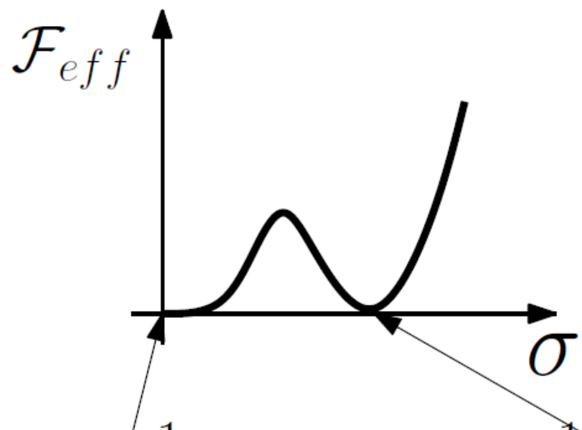
$\mu_{c,T=0}^{(1st)}$

- "Approximate expression" of $\mu_{c,T=0}$(1st)
- Evolution of $E_q^{(vac)}$ and $\delta\mu^{Wigner}$ with β



"Approximate Form" of $\mu_{c,T=0}^{(1st)}$

- Two body interaction dominance
- As β increases, E_q decreases and μ shift increases.



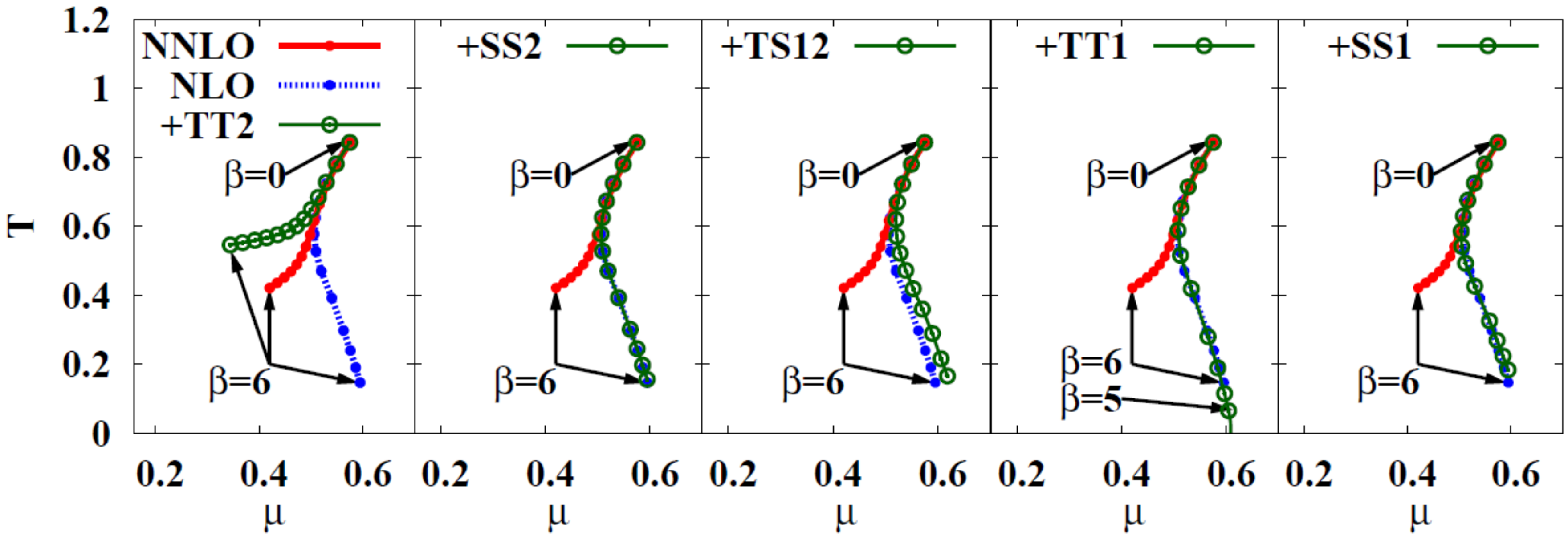
$$-N_c\mu + \frac{1}{2}N_c\delta\mu^{Wigner} \simeq -\frac{1}{2}N_c E_q^{NG}$$

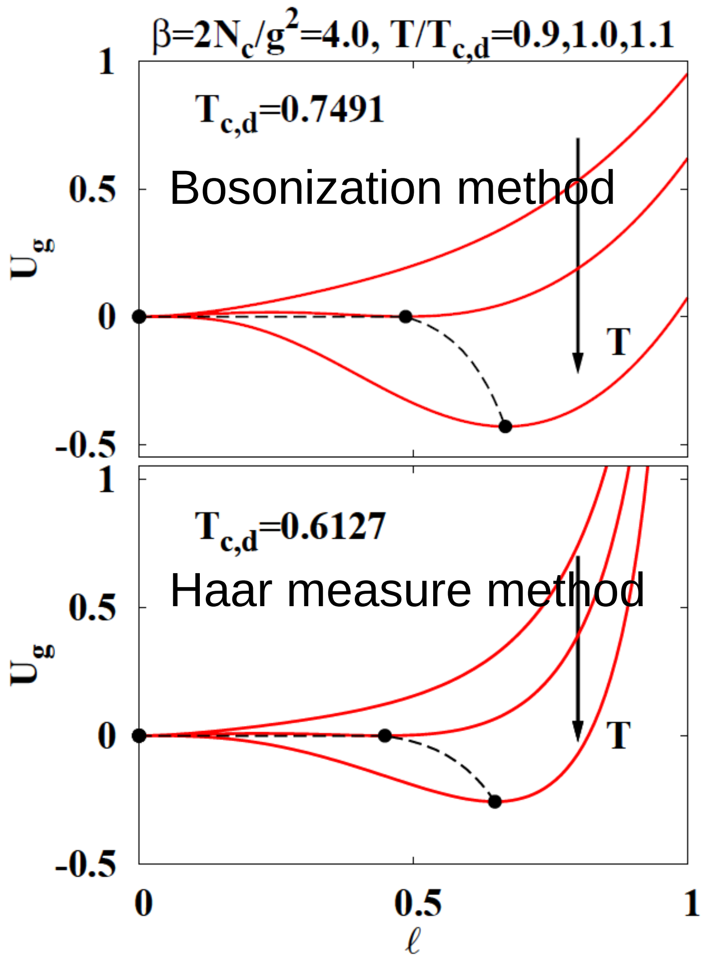
$$\rightarrow \boxed{\mu_{c,T=0}^{(1st,App.)} = \frac{1}{2}\left[E_q^{(vac)} + \delta\mu^{Wigner}\right]}$$

$$\beta \nearrow \quad \Longrightarrow \quad \searrow \qquad\qquad \nearrow$$

Truncation of NNLO Diagrams

- Temporal-temporal, spatial-spatial, temporal-spatial




β=2N_c/g^2=4.0, T/T_c,d=0.9,1.0,1.1
T_c,d=0.7491
Bosonization method
U_g
T
1
0.5
0
-0.5
T_c,d=0.6127
Haar measure method
U_g
T
1
0.5
0
-0.5
0
0.5
1
ℓ