Classification and Generalization of Minimal-doubling Actions

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M. Creutz(BNL) and T. Misumi, work in progress

(T. Kimura(UT) and T. Misumi, PTP 2010)

Lattice2010@Villasimius in Italy

Introduction

◆ Doubling problem : obstacle to simulations

Several ways to bypass No-Go theorem, but….

- Wilson (broken chiral sym.) ➡ Mass renormalization
- DW or Overlap (Non-locality) ➡ Numerical expense
- Staggered (4 tastes) ➡ Rooting procedure

➢ Another possibility : **Minimally doubled fermion**

i) 2 flavors ← 4 in Staggered

ii) Exact chiral: $U(1)_A \subset SU(2)$ ← Broken in Wilson

iii) Strict locality ← Not strict in DW or Overlap

- Two known classes
  - **Karsten-Wilczek fermion** <u>*CT, P, Cubic*</u>

$$
\begin{aligned}
D(p) \;=\;& \sin p_1\, i\gamma_1 \\
&+ \sin p_2\, i\gamma_2 \\
&+ \sin p_3\, i\gamma_3 \\
&+(\sin p_4 + \cos p_1 + \cos p_2 + \cos p_3 \;-\; 3)\, i\gamma_4
\end{aligned}
$$

  - **Borici-Creutz fermion** <u>*CTP, S*4</u>

$$
\begin{aligned}
D(p) \;=\;& \quad (\sin p_1 + \sin p_2 - \sin p_3 - \sin p_4) i\gamma_1 \\
&+ \;(\sin p_1 - \sin p_2 - \sin p_3 + \sin p_4) i\gamma_2 \\
&+ \;(\sin p_1 - \sin p_2 + \sin p_3 - \sin p_4) i\gamma_3 \\
&+ \;B(4C - \cos p_1 - \cos p_2 - \cos p_3 - \cos p_4) i\gamma_4
\end{aligned}
$$

Lack of discrete symmetry requires fine-tuning of parameters…..

P. F. Bedaque, *et.al.*, (2008)

S. Capitani, *et al.*, (2009)

By classifying possible classes of minimal-doubling actions, we search for possibility of application.

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1. Minimal-doubling actions

A general form of chirally-symmetric *O(a)* Dirac operator.

$$D(p) = i\gamma_\mu R_{\mu\nu} \sin p_\nu + i\gamma_\mu R'_{\mu\nu} \cos p_\nu + \sum_\nu i\gamma_\mu R''_{\mu\nu}$$

Three matrices (*R, R', R"*) characterize the operator.

- Advantage of this form : Easy to see discrete symmetry

Find another class of Minimal-doubling actions in this framework.

- Karsten-Wilczek action *CT, P, Cubic*

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R' = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & \lambda & \lambda & 0 \end{pmatrix}, \quad R'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & \lambda & \lambda & 0 \end{pmatrix}$$

- Creutz action *CPT, S*$_4$

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R' = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B & B & B & B \end{pmatrix}, \quad R'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ BC & BC & BC & BC \end{pmatrix}$$

(Borici action)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R' = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad R'' = -\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Starting with one simple *O(a)* Dirac op

$$\begin{aligned} D(p) \; &= (\sin p_1 + \cos p_1 - 1)\, i\gamma_1 \\ &+(\sin p_2 + \cos p_2 - 1)\, i\gamma_2 \\ &+(\sin p_3 + \cos p_3 - 1)\, i\gamma_3 \\ &+(\sin p_4 + \cos p_4 - 1)\, i\gamma_4 \end{aligned}$$

- Number of zeros (species)

$\tilde{p}_\mu = 0 \;\text{ or }\; \pi/2$ ⇨ 16

Just similar to the naive action except for *O(a)* terms.

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Permuting *O(a)* terms

$$\begin{aligned} D(p) = & (\sin p_1 + \cos p_1 - 1)\, i\gamma_1 \\ & +(\sin p_2 + \cos p_2 - 1)\, i\gamma_2 \\ & +(\sin p_3 + \boxed{\cos p_4} - 1)\, i\gamma_3 \\ & +(\sin p_4 + \boxed{\cos p_3} - 1)\, i\gamma_4 \end{aligned}$$

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Permuting *O(a)* terms

$$D(p) = (\sin p_1 + \cos p_1 - 1)\, i\gamma_1 + (\sin p_2 + \cos p_2 - 1)\, i\gamma_2 + (\sin p_3 + \boxed{\cos p_4} - 1)\, i\gamma_3 + (\sin p_4 + \boxed{\cos p_3} - 1)\, i\gamma_4$$

- Number of zeros (species)

$$\tilde{p}_1 = 0 \text{ or } \pi/2, \quad \tilde{p}_2 = 0 \text{ or } \pi/2, \quad (\tilde{p}_3, \tilde{p}_4) = (0,0) \text{ or } (\pi/2, \pi/2)$$

⇒ 8

Twisted-ordering reduces the number of species !

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Permuting *O(a)* terms twice

$$
\begin{aligned}
D(p) = & (\sin p_1 + \boxed{\cos p_2} - 1)\, i\gamma_1 \\
& + (\sin p_2 + \boxed{\cos p_1} - 1)\, i\gamma_2 \\
& + (\sin p_3 + \boxed{\cos p_4} - 1)\, i\gamma_3 \\
& + (\sin p_4 + \boxed{\cos p_3} - 1)\, i\gamma_4
\end{aligned}
$$

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Permuting *O(a)* terms twice

$$
\begin{aligned}
D(p) = &(\sin p_1 + \boxed{\cos p_2} - 1)\, i\gamma_1 \\
&+(\sin p_2 + \boxed{\cos p_1} - 1)\, i\gamma_2 \\
&+(\sin p_3 + \boxed{\cos p_4} - 1)\, i\gamma_3 \\
&+(\sin p_4 + \boxed{\cos p_3} - 1)\, i\gamma_4
\end{aligned}
$$

◆ Number of zeros (species)

$$(\tilde{p}_1, \tilde{p}_2) = (0,0) \text{ or } (\pi/2, \pi/2) \quad (\tilde{p}_3, \tilde{p}_4) = (0,0) \text{ or } (\pi/2, \pi/2)$$

➡ 4

More twisted-orderings reduce more species !

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Permuting *O(a)* terms in a cyclic way

$$\begin{aligned} D(p) &= (\sin p_1 + \boxed{\cos p_2} - 1)\, i\gamma_1 \\ &+ (\sin p_2 + \cos p_3 - 1)\, i\gamma_2 \\ &+ (\sin p_3 + \cos p_4 - 1)\, i\gamma_3 \\ &+ (\sin p_4 + \cos p_1 - 1)\, i\gamma_4 \end{aligned}$$

2. Twisted-Ordering

M.Creutz and T.Misumi (2010)

Permuting *O(a)* terms in a cyclic way

$$
\begin{aligned}
D(p) = &(\sin p_1 + \boxed{\cos p_2} - 1)\, i\gamma_1 \\
&+(\sin p_2 + \cos p_3 - 1)\, i\gamma_2 \\
&+(\sin p_3 + \cos p_4 - 1)\, i\gamma_3 \\
&+(\sin p_4 + \cos p_1 - 1)\, i\gamma_4
\end{aligned}
$$

- Number of zeros (species)

$$\tilde{p}_\mu = (0,0,0,0),\ \ (\pi/2, \pi/2, \pi/2, \pi/2) \ \Rightarrow\ 2$$

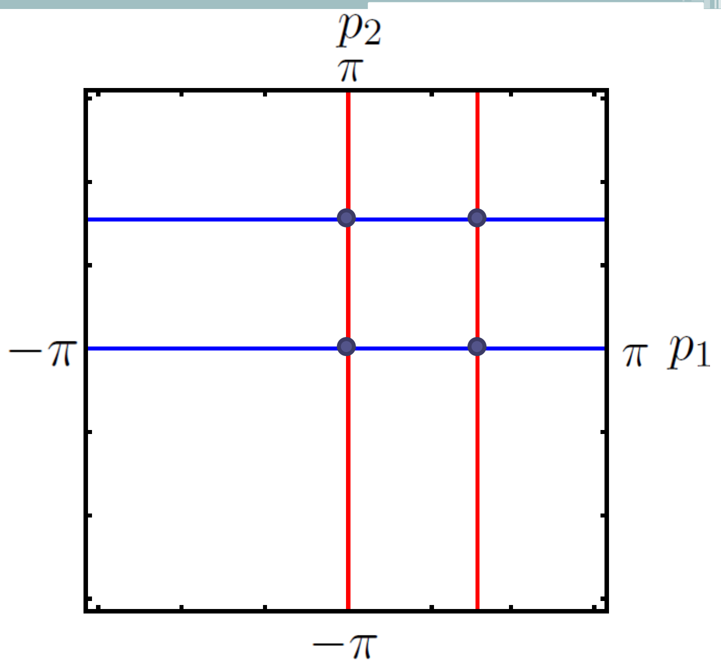
Minimal-doubling ! (On the orthogonal lattice)

◆ 2d

i) Untwisted case

$$D(p) = (\sin p_1 + \cos p_1 - 1)\, i\gamma_1 + (\sin p_2 + \cos p_2 - 1)\, i\gamma_2$$

zeros: $(0,0),\ (0,\pi/2),\ (\pi/2,0),\ (\pi/2,\pi/2)$

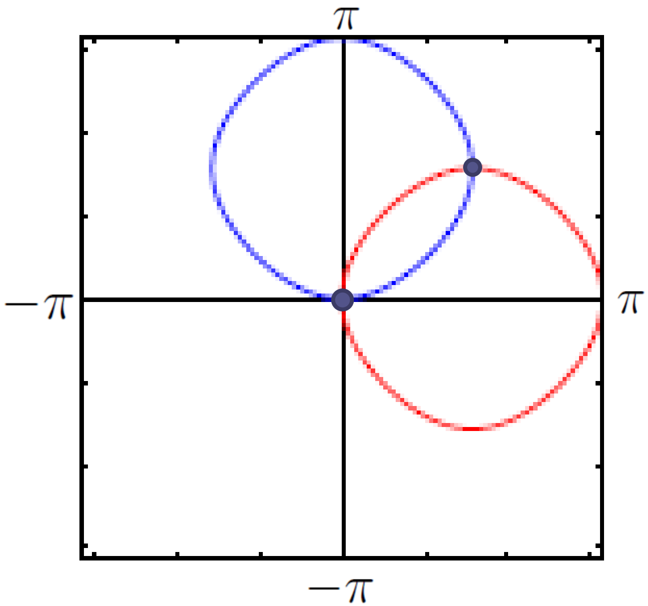


ii) Twisted case

$$D(p) = (\sin p_1 + \cos p_2 - 1)\, i\gamma_1 + (\sin p_2 + \cos p_1 - 1)\, i\gamma_2$$

zeros: $(0,0)\ (\pi/2,\pi/2)$

Minimal number of species!



◆ Two choices to place a parameter

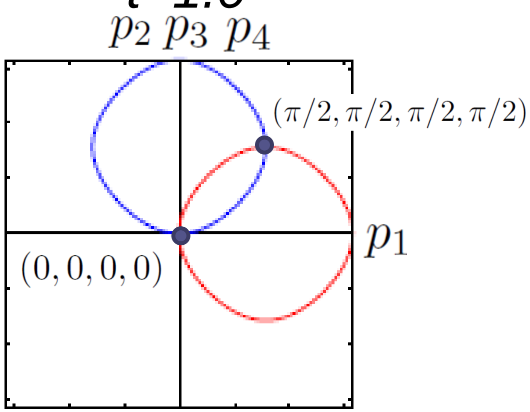
1. *t*-parameter

$$
\begin{aligned}
D(p) = & (\sin p_1 + \cos p_2 - 1)\, i\gamma_1 \\
& +(\sin p_2 + \cos p_3 - 1)\, i\gamma_2 \\
& +(\sin p_3 + \cos p_4 - 1)\, i\gamma_3 \\
& +(\sin p_4 + t(\cos p_1 - 1))\, i\gamma_4
\end{aligned}
$$

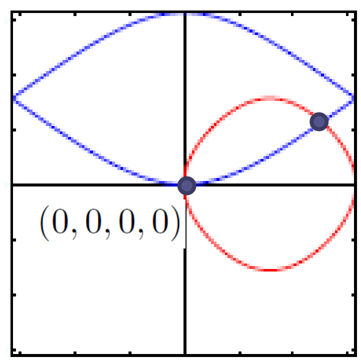
One of zeros shifts with *t*.

$$0 \leq t \leq 1$$

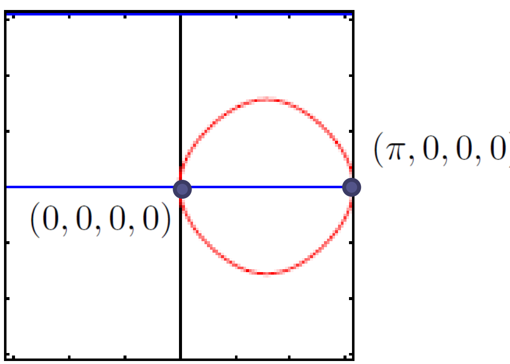
t=1.0



t=0.5



t=0



$(\pi/2, \pi/2, \pi/2, \pi/2)$ ⟶ $(\pi, 0, 0, 0)$

◆ Symmetries

Translation, Gauge, $U(1)_B$, $U(1)_A \subset SU(2)$

One exact chiral symmetry

Common with all Minimal-doubling actions. P.F.Bedaque, *et.al.*, PLB **662**, 449 (2008)

➢ Discrete symmetries

$t=0$: CP T Z_2 First *CP*-invariant MD action

$t=1$: CPT Z_4 Z_2 Hypercubic sym . → Z_4

Fine-tuning to cancel redundant operators is still required.

Note: Flavored *C*,*P* or *T* may exist.

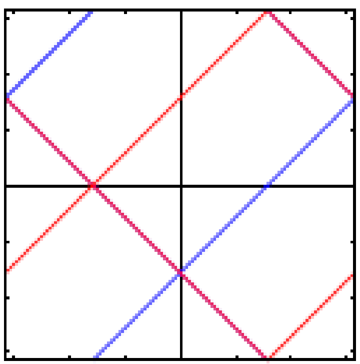
2. α-parameter

$$
\begin{aligned}
D(p) \;=\; & (\sin p_1 + \cos p_2 - \alpha)\, i\gamma_1 \\
& + (\sin p_2 + \cos p_3 - \alpha)\, i\gamma_2 \\
& + (\sin p_3 + \cos p_4 - \alpha)\, i\gamma_3 \\
& + (\sin p_4 + \cos p_1 - \alpha)\, i\gamma_4
\end{aligned}
$$

Minimal-doubling persists within

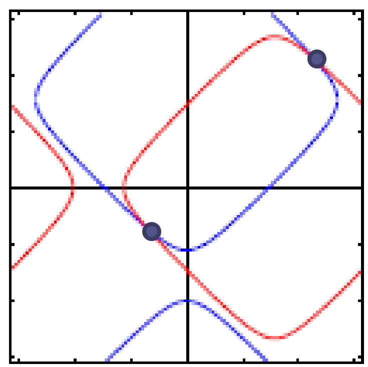
$$0 < \alpha < \sqrt{2}$$

α = 0



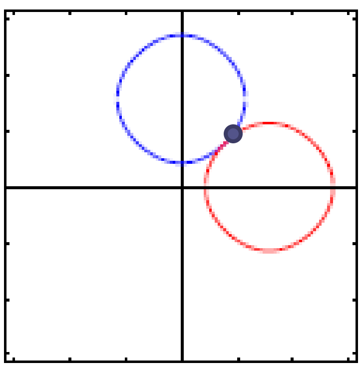
"cut" solution

α = 0.1



p^2 dispersion arises

$\alpha = \sqrt{2}$



One mutilated pole

A small *α* leads to unphysical dispersion relation. (*d=2n*)

◆ Gauged action in position space

$$S = \frac{1}{2}\sum_{n,\mu}\Big[\bar\psi_n\,\gamma_\mu\,(U_{n,\mu}\psi_{n+\mu} - U^\dagger_{n-\mu,\mu}\psi_{n-\mu})$$

$$+\, i\bar\psi_n\,\underline{\gamma_{\mu-1}}\,(U_{n,\mu}\psi_{n+\mu} + U^\dagger_{n-\mu,\mu}\psi_{n-\mu} - \alpha\,\psi_n)\Big]$$

Twisted

We term this mechanism Twisted-Ordering.

➡ Minimal-doubling action with chiral symmetry and strict locality on orthogonal lattices !

Note: Maximally Twisted-Ordering ➔ Minimal-doubling

Partially Twisted-Ordering ➔ 4 or 8 species

3. Higher dimensions

T. Kimura and T. Misumi (2009)

- Clifford algebra in even dimensions

$$\Gamma_\mu^{(2m)} = \tau_1 \otimes \Gamma_\mu^{(2m-2)} = \begin{pmatrix} 0 & \Gamma_\mu^{(2m-2)} \\ \Gamma_\mu^{(2m-2)} & 0 \end{pmatrix} \quad \text{for} \quad \mu = 1, \cdots, 2m-1,$$

$$\Gamma_{2m}^{(2m)} = \tau_2 \otimes \mathbb{1}_{[2^{m-1}]} = \begin{pmatrix} 0 & -i\mathbb{1}_{[2^{m-1}]} \\ i\mathbb{1}_{[2^{m-1}]} & 0 \end{pmatrix}, \qquad \Gamma_{2m+1}^{(2m)} = \tau_3 \otimes \mathbb{1}_{[2^{m-1}]} = \begin{pmatrix} \mathbb{1}_{[2^{m-1}]} & 0 \\ 0 & -\mathbb{1}_{[2^{m-1}]} \end{pmatrix}$$

Cf.) Creutz action ($d=2m$)

$$\begin{aligned} S_{\mathrm{C}} = \frac{1}{2} \sum_x \Big[\sum_{\mu=1}^{2m} \Big(& \bar{\psi}^A_{x-\mathrm{a}_\mu} \left(\Sigma^{(2m)} \cdot \mathrm{e}^\mu\right) \psi^B_x - \bar{\psi}^B_{x+\mathrm{a}_\mu} \left(\Sigma^{(2m)} \cdot \mathrm{e}^\mu\right) \psi^A_x \\ & - \bar{\psi}^A_{x+\mathrm{a}_\mu} \left(\bar{\Sigma}^{(2m)} \cdot \mathrm{e}^\mu\right) \psi^B_x + \bar{\psi}^B_{x-\mathrm{a}_\mu} \left(\bar{\Sigma}^{(2m)} \cdot \mathrm{e}^\mu\right) \psi^A_x \Big) \\ & + \bar{\psi}^A_x \left(\Sigma^{(2m)} \cdot \mathrm{e}^{2m+1}\right) \psi^B_x - \bar{\psi}^B_x \left(\Sigma^{(2m)} \cdot \mathrm{e}^{2m+1}\right) \psi^A_x \\ & - \bar{\psi}^A_x \left(\bar{\Sigma}^{(2m)} \cdot \mathrm{e}^{2m+1}\right) \psi^B_x + \bar{\psi}^B_x \left(\bar{\Sigma}^{(2m)} \cdot \mathrm{e}^{2m+1}\right) \psi^A_x \Big] \end{aligned}$$

➢ Minimal-doubling range of parameters

- Creutz *(d=2m)* $\frac{m-1}{m} < C < 1$: *m=2* ➔ *4d* case

- K-W (extended version) $i\gamma_4 \sum_\mu (r - \cos p_\mu)$

$$\frac{d+\sqrt{2}-3}{d} < r < \frac{d+\sqrt{2}-1}{d}$$

Minimal-doubling range tends to be narrower with *d*.

More species need to be eliminated for higher dim.

- TO (α-parameter) $0 < \alpha < \sqrt{2}$ for any *d=2m*

Special case! ➔ Twisted-ordering itself excludes species.

4. Summary and Discussion

I. Twisted-ordering reduces species in any dimensions.

II. Maximally-Twisted-ordering produces Minimal-doubling actions.

- Three classes

1. Karsten-Wilczek

   CT, P, Cubic, Z_2

2. Creutz-Borici

   CPT, S_4, Z_2

3. Twisted-Ordering

   $t=0$: CP, T, Z_2

   $t=1$: CPT, Z_4, Z_2

Which is best for application?
Minimize fine-tune parameters?
A general form?

S. Capitani, *et al*., (2010)
Talk by S. Capitani,
Poster by J. Weber.

◆ Candidate for a general form

$$D(p) = i\sum_{\mu}[\gamma_\mu \sin(p_\mu + \beta_\mu) - \gamma'_\mu \sin(p_\mu - \beta_\mu)] - i\Gamma$$

$\gamma'_\mu = A_{\mu\nu}\gamma_\nu$ ➡ Another gamma matrix

β_μ ➡ two zeros $\pm\beta_\mu$

$A_{\mu\nu}\sin 2\beta_\nu = \sin 2\beta_\mu$

Borici and some TO actions are unified in this form.

Drawbacks: Including PTO too
Not including *t*-cases of TO and K-W

We need to search more to find a general form…

Appendix 1. General form

- Borici action

$$\beta_\mu = \pi/4 \qquad A = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

- Twisted-Ordering action

$$\beta_\mu = \pi/4 \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

<u>Note there are equivalent actions</u>

$$p_\mu \;\longrightarrow\; p_\mu + \alpha_\mu \qquad \gamma_\mu \;\longrightarrow\; C_{\mu\nu}\gamma_\nu$$

Appendix 2. Redundant operators

KW fermion

Relevant: $\mathcal{O}_3 = \overline{\psi} i \gamma_4 \psi$

Marginal:

$$\mathcal{O}_4^{(1)} = F_{\mu\nu} F_{\mu\nu}$$
$$\mathcal{O}_4^{(2)} = F_{4\mu} F_{4\mu}$$
$$\mathcal{O}_4^{(3)} = \bar{\psi}\, D_4 \gamma_4\, \psi$$
$$\mathcal{O}_4^{(4)} = \bar{\psi} D_\mu \gamma_\mu \psi$$

Ren. for speed of light

BC fermion

Relevant:

$$\mathcal{O}_3^{(1)} = \bar{Q}(\gamma_4 \otimes \tau^3) Q$$
$$\mathcal{O}_3^{(2)} = \bar{Q}(\gamma_4 \gamma_5 \otimes \tau^3) Q$$

Marginal:

$$\mathcal{O}_4^{(1)} = F_{\mu\nu} F_{\mu\nu}$$
$$\mathcal{O}_4^{(2)} = F_{4\mu} F_{4\mu}$$
$$\mathcal{O}_4^{(3)} = F_{\mu\nu} \tilde{F}_{\mu\nu}$$
$$\mathcal{O}_4^{(4)} = F_{4\mu} \tilde{F}_{4\mu}$$
$$\mathcal{O}_4^{(5)} = \bar{Q}(\gamma_\mu \otimes \mathbf{1}) D_\mu Q$$
$$\mathcal{O}_4^{(6)} = \bar{Q}(\gamma_4 \otimes \mathbf{1}) D_4 Q$$
$$\mathcal{O}_4^{(5)} = \bar{Q}(i\gamma_\mu \gamma_5 \otimes \mathbf{1}) D_\mu Q$$
$$\mathcal{O}_4^{(6)} = \bar{Q}(i\gamma_4 \gamma_5 \otimes \mathbf{1}) D_4 Q$$

Twisted-Ordering fermion (t=0)

Relevant: $\mathcal{O}_3 = \bar{\psi} i \gamma_j \psi \qquad (j = 1, 2, 3)$

Marginal:

$$\mathcal{O}_4^{(1)} = \bar{\psi}\gamma_\mu D_\mu \psi$$

$$\mathcal{O}_4^{(2)} = \bar{\psi}\gamma_j D_j \psi$$

$$\mathcal{O}_4^{(3)} = F_{\mu\nu} F_{\mu\nu}$$

$$\mathcal{O}_4^{(4)} = F_{j\nu} F_{j\mu}$$

Renormalization for the speed of light

The number of fine tuning parameters is as few as KW fermion.

Appendix 3. KW and BC actions

◆ Karsten-Wilczek fermion

$$D(p) = i\sum_{\mu=1}^{4}\gamma^{\mu}\sin p_{\mu} + i\lambda\gamma^{4}\sum_{j=1}^{3}(1-\cos p_{j})$$

⇒ Two zeros : (0,0,0,0)
(0,0,0,π)

One direction specified

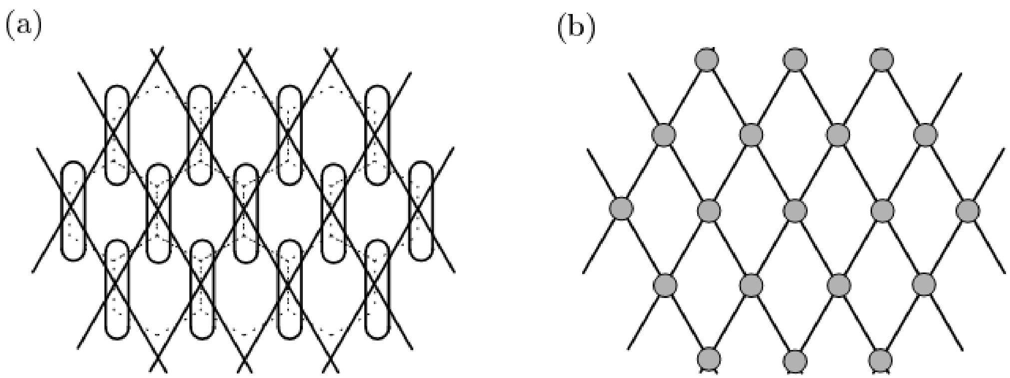
- ➢Hypercubic × →cubic
- ➢C,T ×

⇒ Redundant operators（ψγψ, ψγDψ, etc)

Fine tuning required for a continuum limit!

◆ Creutz action M. Creutz (2007)

$$\begin{aligned} D(p) \;=\; & (\sin p_1 + \sin p_2 - \sin p_3 - \sin p_4) i\gamma_1 \\ + \; & (\sin p_1 - \sin p_2 - \sin p_3 + \sin p_4) i\gamma_2 \\ + \; & (\sin p_1 - \sin p_2 + \sin p_3 - \sin p_4) i\gamma_3 \\ + \; & B(4C - \cos p_1 - \cos p_2 - \cos p_3 - \cos p_4) i\gamma_4 \end{aligned}$$



➤ Karsten-Smit theorem (No-go theorem on cubic lattices)

Conditions of the no-go theorem

+

Permutation sym.of 4 axes: S_4

(Hypercubic Symmetry)

⇒ 16 doublers

(2^d in general dimensions)

cf.) Staggered fermion 16 ⇨ 4

Hypercubic Symmetry × ➔ Twisted HS ○

◆ Generalized Creutz action

Kimura and Misumi (2009)

$$S = \frac{1}{2}\sum_x \left[\sum_{\mu=1}^{4} (\bar{\psi}_x \Gamma \cdot \boldsymbol{e}^\mu \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} \bar{\Gamma} \cdot \boldsymbol{e}^\mu \psi_x) + 2it\bar{\psi}_x \gamma^4 \psi_x\right]$$

i) 4 spinor vectors   ii) On-site term ∝ any of gamma matrices

- Dirac operator

$$D(p) = i\sum_{\mu=1}^{3}\gamma^\mu \left(\sum_{\nu=1}^{4} (\boldsymbol{e}^\nu)_\mu \sin p_\nu\right) + i\gamma^4 \left(t - \sum_{\nu=1}^{4} (\boldsymbol{e}^\nu)_4 \cos p_\nu\right)$$

Minimal-doubling Condition: $\sum_\nu (\boldsymbol{e}^\nu)_\mu = \begin{cases} 0 & \mu = 1,2,3 \\ t/\tilde{C} & \mu = 4 \end{cases}$

⇨ $p = \pm(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p}), \qquad \cos\tilde{p} = \tilde{C}$

A larger class of minimal-doubling fermions

◆ Expansion around zero points

ex.) Creutz action

$$\begin{aligned} D(p) \;=\; & \;\; C(q_1+q_2-q_3-q_4)i\gamma_1 \\ & + \; C(q_1-q_2-q_3+q_4)i\gamma_2 \\ & + \; C(q_1-q_2+q_3-q_4)i\gamma_3 \\ & + \; BS(q_1+q_2+q_3+q_4)i\gamma_4 + \mathcal{O}(q^2) \end{aligned}$$

Coefficients of gamma matrices stand for orthogonal coordinates of momentum. $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$

➢ Momentum basis

$$\boldsymbol{b}^1 = (C, C, C, BS), \quad \boldsymbol{b}^2 = (C, -C, -C, BS),$$
$$\boldsymbol{b}^3 = (-C, -C, C, BS), \quad \boldsymbol{b}^4 = (-C, C, -C, BS)$$

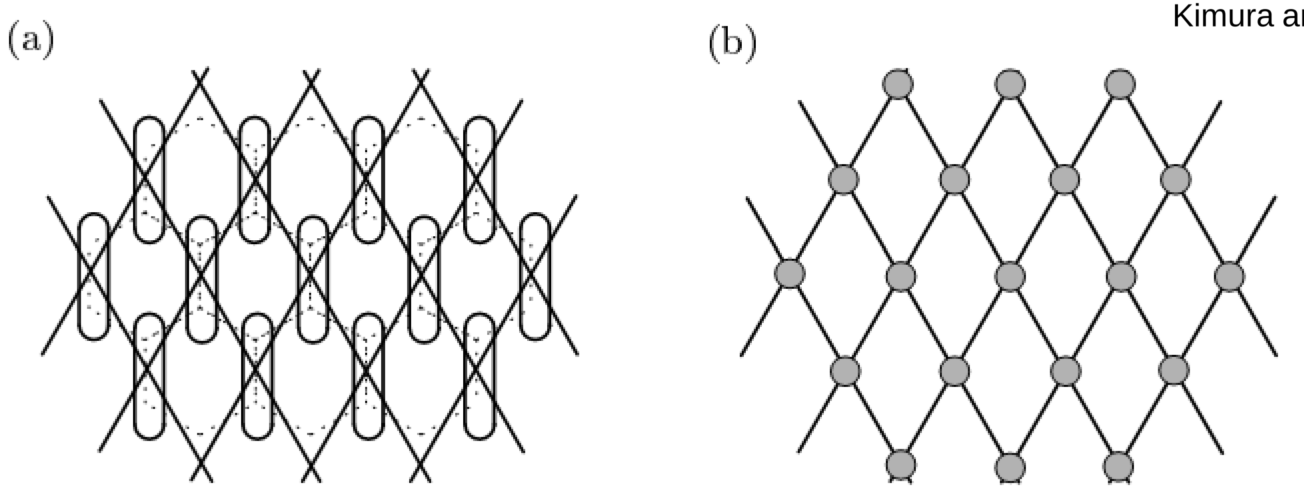
$\boldsymbol{a}_\mu \cdot \boldsymbol{b}_\nu = \delta_{\mu\nu}$ ➡ Spatial basis is found.

◆ Spatial basis (hopping vectors)

$$\boldsymbol{a}^1 = \frac{1}{4C}(1,1,1,\frac{C}{BS}), \quad \boldsymbol{a}^2 = \frac{1}{4C}(1,-1,-1,\frac{C}{BS}),$$
$$\boldsymbol{a}^3 = \frac{1}{4C}(-1,-1,1,\frac{C}{BS}), \quad \boldsymbol{a}^4 = \frac{1}{4C}(-1,1,-1,\frac{C}{BS})$$

These mean <u>deformed-hypercubic (rhombus) lattices</u> except for *C=BS.*

Kimura and Misumi (2009)



(b) is more natural since there is no hopping within one unit.

Creutz-type actions are defined on rhombus lattices.
(It is clear that hypercubic symmetry is broken.)

◆ Generalized Karsten-Wilczek action

$$S = \frac{1}{2}\sum_x \sum_{\mu=1}^{4} \left[\bar{\psi}_x \gamma^\mu \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} \gamma^\mu \psi_x\right] + S_c$$

$$S_c = \frac{i}{2} r \sum_x \left[2(3 + \underline{t)\bar{\psi}_x \gamma^4 \psi_x} - \sum_{\mu=1}^{4} \underline{\left(\bar{\psi}_x \gamma^4 \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} \gamma^4 \psi_x\right)} \right]$$

A new parameter introduced

Sum over μ = 1,2,3,4

- Dirac operator:

$$D(p) = i\sum_{\mu=1}^{4}\gamma^\mu \sin p_\mu + ir\gamma^4\left[\sum_{\mu=1}^{3}(1-\cos p_\mu) + (t-\cos p_4)\right]$$

- Minimal-doubling condition:

$$\left|\frac{r}{\sqrt{1+r^2}}\right| t < 1, \qquad \left|\frac{r}{\sqrt{1+r^2}}\right| (t+2) > 1$$

- Linearized Dirac operator with $p = q + p_0'$:

$$D(p) = i\sum_{\mu=1}^{3}\gamma^\mu q_\mu + i\sqrt{1+r^2(1-t^2)}\gamma^4 q_4 + \mathcal{O}(q^2)$$

- Reciprocal/primitive vecs

$$(\boldsymbol{b}^\nu)_\mu = \begin{cases} \delta^\nu_\mu \\ \delta^\nu_\mu\sqrt{1+r^2(1-t^2)} \end{cases}, \quad (\boldsymbol{a}^\nu)_\mu = \begin{cases} \delta^\nu_\mu & (\mu = 1,2,3) \\ \delta^\nu_\mu/\sqrt{1+r^2(1-t^2)} & (\mu = 4) \end{cases}$$

◆ Comparison with Wilson action

$$\int \frac{d^4k}{(2\pi)^4}\bar{\psi}(-k)[i\frac{\gamma_\mu}{a}\sin(ak_\mu)+m+\frac{r}{a}\sum_\mu(1-\cos(ak_\mu))]\psi(k$$

Wilson term : $$r\sum_{\mu=1}^{4}(1-\cos p_\mu)$$

K-M term : $$ir\gamma^4\sum_{\mu=1}^{4}(1-\cos p_\mu)$$ for *t=1*

You can obtain minimal-doubling actions just by multiplying Wilson term by one of gamma matrices !

Hypercubic and C,T are broken instead of chiral symmetry.

Appendix 4. Details of Higher dims

D=2m : you can generalize straightforward

D=2m+1: extra hoppings are necessary

◆ Spinor vectors in higher dimensions

Isotropic vectors

$$\mathrm{e}^{\mu} \cdot \mathrm{e}^{\nu} = \begin{cases} 1 & \text{for} \quad \mu = \nu \\ \cos\theta_d & \text{for} \quad \mu \neq \nu \end{cases}$$

$$\cos\theta_d = -1/d$$

$$\sum_{\mu=1}^{d+1} \mathrm{e}^{\mu} = 0$$

◆ Spinor vectors (d=2m)

$$
\begin{array}{lllllll}
\mathrm{e}^1 & = (& c_1 s_2 \cdots s_{d-1} s_d, & c_2 s_3 \cdots s_d, & \cdots, & c_{d-2} s_{d-1} s_d, & c_{d-1} s_d, & B c_d \ ) \\
\mathrm{e}^2 & = (& s_2 \cdots s_{d-1} s_d, & c_2 s_3 \cdots s_d, & \cdots, & c_{d-2} s_{d-1} s_d, & c_{d-1} s_d, & B c_d \ ) \\
\mathrm{e}^3 & = (& 0, & s_3 \cdots s_d, & \cdots, & c_{d-2} s_{d-1} s_d, & c_{d-1} s_d, & B c_d \ ) \\
 & \vdots & & & & & & \\
\mathrm{e}^{d-1} & = (& 0, & 0, & \cdots, & s_{d-1} s_d, & c_{d-1} s_d, & B c_d \ ) \\
\mathrm{e}^{d} & = (& 0, & 0, & \cdots, & 0, & s_d, & B c_d \ ) \\
\mathrm{e}^{d+1} & = (& 0, & 0, & \cdots, & 0, & 0, & BC \ )
\end{array}
$$

$$
c_\mu \equiv \cos\theta_\mu = -1/\mu, \ s_\mu \equiv \sin\theta_\mu = \sqrt{\mu^2 - 1}/\mu
$$

➢ Clifford algebra in even dimensions

$$\Gamma_\mu^{(2m)} = \tau_1 \otimes \Gamma_\mu^{(2m-2)} = \begin{pmatrix} 0 & \Gamma_\mu^{(2m-2)} \\ \Gamma_\mu^{(2m-2)} & 0 \end{pmatrix} \quad \text{for} \quad \mu = 1, \cdots, 2m-1,$$

$$\Gamma_{2m}^{(2m)} = \tau_2 \otimes \mathbb{1}_{[2^{m-1}]} = \begin{pmatrix} 0 & -i\mathbb{1}_{[2^{m-1}]} \\ i\mathbb{1}_{[2^{m-1}]} & 0 \end{pmatrix}, \qquad \Gamma_{2m+1}^{(2m)} = \tau_3 \otimes \mathbb{1}_{[2^{m-1}]} = \begin{pmatrix} \mathbb{1}_{[2^{m-1}]} & 0 \\ 0 & -\mathbb{1}_{[2^{m-1}]} \end{pmatrix}$$

defined from (2m-2)D algebra

We define the following vectors.

$$\Gamma^{(2m)} = \begin{pmatrix} 0 & \bar{\gamma}^{(2m)} \\ \gamma^{(2m)} & 0 \end{pmatrix}$$

$$\gamma^{(2m)} = \left(\Gamma_1^{(2m-2)}, \cdots, \Gamma_{2m-1}^{(2m-2)}, i\mathbb{1}_{[2^{m-1}]}\right), \quad \bar{\gamma}^{(2m)} = \left(\Gamma_1^{(2m-2)}, \cdots, \Gamma_{2m-1}^{(2m-2)}, -i\mathbb{1}_{[2^{m-1}]}\right)$$

◆ Creutz action (*d*=2*m*)

$$
\begin{aligned}
S_{\mathrm{C}} = \frac{1}{2}\sum_x \Big[\sum_{\mu=1}^{2m} & \Big(\bar{\psi}^A_{x-\mathrm{a}_\mu}\left(\Sigma^{(2m)}\cdot \mathrm{e}^\mu\right)\psi^B_x - \bar{\psi}^B_{x+\mathrm{a}_\mu}\left(\Sigma^{(2m)}\cdot \mathrm{e}^\mu\right)\psi^A_x \\
& -\bar{\psi}^A_{x+\mathrm{a}_\mu}\left(\bar{\Sigma}^{(2m)}\cdot \mathrm{e}^\mu\right)\psi^B_x + \bar{\psi}^B_{x-\mathrm{a}_\mu}\left(\bar{\Sigma}^{(2m)}\cdot \mathrm{e}^\mu\right)\psi^A_x\Big) \\
& +\bar{\psi}^A_x\left(\Sigma^{(2m)}\cdot \mathrm{e}^{2m+1}\right)\psi^B_x - \bar{\psi}^B_x\left(\Sigma^{(2m)}\cdot \mathrm{e}^{2m+1}\right)\psi^A_x \\
& -\bar{\psi}^A_x\left(\bar{\Sigma}^{(2m)}\cdot \mathrm{e}^{2m+1}\right)\psi^B_x + \bar{\psi}^B_x\left(\bar{\Sigma}^{(2m)}\cdot \mathrm{e}^{2m+1}\right)\psi^A_x\Big]
\end{aligned}
$$

➢ Momentum basis vectors

$$
\begin{aligned}
\mathrm{b}_1 &= (\ Cc_1s_2\cdots s_{d-1}s_d,\ Cc_2s_3\cdots s_d,\ \cdots,\ Cc_{d-2}s_{d-1}s_d,\ Cc_{d-1}s_d,\ -BSc_d\ ) \\
\mathrm{b}_2 &= (\ Cs_2\cdots s_{d-1}s_d,\ Cc_2s_3\cdots s_d,\ \cdots,\ Cc_{d-2}s_{d-1}s_d,\ Cc_{d-1}s_d,\ -BSc_d\ ) \\
\mathrm{b}_3 &= (\ 0,\ Cs_3\cdots s_d,\ \cdots,\ Cc_{d-2}s_{d-1}s_d,\ Cc_{d-1}s_d,\ -BSc_d\ ) \\
&\ \vdots \\
\mathrm{b}_{d-1} &= (\ 0,\ 0,\ \cdots,\ Cs_{d-1}s_d,\ Cc_{d-1}s_d,\ -BSc_d\ ) \\
\mathrm{b}_d &= (\ 0,\ 0,\ \cdots,\ 0,\ Cs_d,\ -BSc_d\ )
\end{aligned}
$$

➢ Internal multiply

$$
\cos\eta = \frac{\mathrm{b}_\mu\cdot\mathrm{b}_\nu}{|\mathrm{b}_\mu|\,|\mathrm{b}_\nu|} = \frac{B^2S^2c_d^2 + C^2s_d^2c_{d-1}}{B^2S^2c_d^2 + C^2s_d^2}
$$

➢ Minimal-doubling condition

Exclude redundant zeros

$p^{(\pm)} = \pm(\tilde{p}_C, \cdots, \tilde{p}_C)$ ○ $p = (\tilde{p}_C, \cdots, \tilde{p}_C, \pi - \tilde{p}_C)$ ×

$$\frac{m-1}{m} < C < 1$$ ➔ *m*=*2*:reduce to 4D case

Minimal-doubling region gets narrower with D

➢ Creutz condition

$$\tilde{p}_C = \frac{\pi}{d+1}, \qquad C = \cos\left(\frac{\pi}{d+1}\right), \quad B = (d+1)\cot\left(\frac{\pi}{d+1}\right)$$

➢ Borici condition

$$C = \cos\tilde{p}_C, \quad B = \sqrt{d+1}\cot\tilde{p}_C$$

◆Odd dimensions

T.Kimura and T.M.

- Spinor structure on Hyperdiamond lattice

$$\Gamma = \underbrace{\tau}_{\text{sublattice}} \otimes \underbrace{\gamma}_{\text{subspinor}}$$

A→B, B→A hoppings are enough for Even dimensions

A→A, B→B hoppings are necessary for Odd dimensions

Because we need diagonal hoppings in any representation in odd dimensions.

◆Clifford algebra in $d = 2m + 1$

(d=2m) spinor ➔ Parity rep.

$$\Gamma_\mu^{(2m)} = \tau_1 \otimes \Gamma_\mu^{(2m-2)} = \begin{pmatrix} 0 & \Gamma_\mu^{(2m-2)} \\ \Gamma_\mu^{(2m-2)} & 0 \end{pmatrix} \quad \text{for} \quad \mu = 1, \cdots, 2m-1$$

$$\Gamma_{2m}^{(2m)} = \tau_2 \otimes \mathbb{1} = \begin{pmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}$$

$$\Gamma_{2m+1}^{(2m)} = \tau_3 \otimes \mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\gamma = \left(\Gamma_1^{(2m-2)}, \cdots, \Gamma_{2m-1}^{(2m-2)}, i\mathbb{1}, 0\right), \quad \bar{\gamma} = \left(\Gamma_1^{(2m-2)}, \cdots, \Gamma_{2m-1}^{(2m-2)}, -i\mathbb{1}, 0\right)$$

◆ Creutz action (d=2m+1)

$$S = \frac{1}{2}\sum_x \sum_{\mu=1}^{2m+1} \underline{\left(\bar{\psi}^A_{x-a_\mu}(\gamma\cdot e^\mu)\,\psi^B_x - \bar{\psi}^A_{x+a_\mu}(\gamma\cdot e^\mu)\,\psi^B_x + \bar{\psi}^B_{x-a_\mu}(\bar{\gamma}\cdot e^\mu)\,\psi^A_x - \bar{\psi}^B_{x+a_\mu}(\bar{\gamma}\cdot e^\mu)\,\psi^A_x\right)}$$

$$+ \frac{i}{2}\sum_x \left[\sum_{\mu=1}^{2m+1} \underline{\left(\bar{\psi}^A_{x+a_\mu}\psi^A_x + \bar{\psi}^A_{x-a_\mu}\psi^A_x + \bar{\psi}^B_{x+a_\mu}\psi^B_x + \bar{\psi}^B_{x-a_\mu}\psi^B_x\right)} - \underline{2t\left(\bar{\psi}^A_x\psi^A - \bar{\psi}^B_x\psi^B\right)}\right]$$

Nearest-unit AA BB hopping

Mass-like term

Common hopping with even

◆Momentum space

$$S = \int dp \sum_{\mu=1}^{2m+1} \left(i \sin p_\mu \bar{\psi}^A(-p)\,(\gamma \cdot e^\mu)\,\psi^B(p) + i \sin p_\mu \bar{\psi}^B(-p)\,(\bar{\gamma} \cdot e^\mu)\,\psi^A(p) \right)$$
$$+ \int dp \left[i \left(-t + \sum_{\mu=1}^{2m+1} \cos p_\mu \right) \left(\bar{\psi}^A(-p)\psi^A(p) - \bar{\psi}^B(-p)\psi^B(p) \right) \right]$$
$$= \int dp \; \bar{\Psi}(-p) D(p) \Psi(p)$$

◆Dirac operator

$$D(p) = i \sum_{\mu=1}^{2m+1} x_\mu(p) \Gamma_\mu^{(2m)}$$

$$x_\mu(p) = \left(\prod_{\nu=\mu+1}^{2m+1} s_\nu \right) \left(c_\mu \sum_{\nu=1}^{\mu} \sin p_\nu + \sin p_{\mu+1} \right) \quad \text{for} \quad \mu = 1, \cdots, 2m$$

$$x_{2m+1}(p) = -t + \sum_{\mu=1}^{2m+1} \cos p_\mu$$

Minimal-doubling condition

$$2m - 1 < t < 2m + 1 \qquad p = \pm(\tilde{p}, \cdots, \tilde{p}) \quad \text{with} \quad \cos \tilde{p} = \frac{t}{2m+1} \equiv \tilde{C}$$

◆Excitations

$$x_\mu(p) = \tilde{C}\left(\prod_{\nu=\mu+1}^{2m+1} s_\nu\right)\left(c_\mu \sum_{\nu=1}^{\mu} q_\nu + q_{\mu+1}\right) + \mathcal{O}(q^2) \quad \text{for} \quad \mu = 1, \cdots, 2m$$

$$x_{2m+1}(p) = -\tilde{S}\sum_{\mu=1}^{2m+1} q_\mu + \mathcal{O}(q^2)$$

Choosing $t = 2m+1$ ➔ Two zeros reduce to one !

Excitation in (2m+1)th direction disappears: $x_{2m+1} = 0$

$$D(p) = i\sum_{\mu=1}^{2m} x_\mu(p)\Gamma_\mu^{(2m)}$$

:Mutilated pole

One massless fermion appears in terms of 4D.

Chiral symmetry is broken in O(a^2).

5d minimal-doubling action ➔ 4d Wilson-like action