Species doublers as super multiplets in lattice supersymmetry :

Exact supersymmetry with interactions for D=1 N=2

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-Basic ideas and coordinate representation-

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★ Lattice SUSY transformation

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half lattice shift translation (1dim.)

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alternating sign → species doublers as supermultiplets

★ Exact SUSY in momentum space with

$$\delta\left(\sin\frac{ap_1}{2}+\sin\frac{ap_2}{2}+\cdots+\sin\frac{ap_n}{2}\right)$$

★ New (commutative) star product in coordinate space

$$\hat{\partial}(F(x)*G(x)) = (\hat{\partial}F(x))*G(x) + F(x)*(\hat{\partial}G(x))$$

Lebniz rule

↑ difference operator

Link approach

The simplest example (continuum)

D=1 N=1

super field $$\Phi(x,\theta)=\varphi(x)+i\theta\psi(x),$$

supercharge $$Q=\frac{\partial}{\partial\theta}+i\theta\frac{\partial}{\partial x}$$

SUSY algebra $$Q^2=i\frac{\partial}{\partial x}.$$

$$\{\frac{\partial}{\partial\theta}\ ,\ \theta\}\ =\ 1 \qquad \theta=\begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad \frac{\partial}{\partial\theta}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$[\theta,\varphi]=\left[\frac{\partial}{\partial\theta},\varphi\right]=0 \qquad \{\theta,\psi\}=\left\{\frac{\partial}{\partial\theta},\psi\right\}=0$$

$$\varphi(x)=\begin{pmatrix}\varphi(x)&0\\0&\varphi(x)\end{pmatrix} \qquad \psi(x)=\begin{pmatrix}\psi(x)&0\\0&-\psi(x)\end{pmatrix}$$

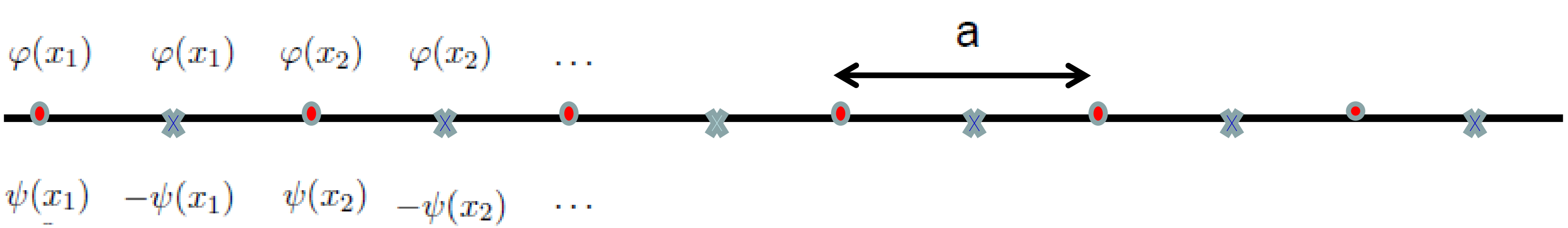
$$\alpha=\begin{pmatrix}\alpha&0\\0&-\alpha\end{pmatrix}$$

1-dim. Lattice N=1

$$\varphi = \begin{pmatrix} \varphi(x_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varphi(x_2) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varphi(x_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varphi(x_N) \end{pmatrix}$$

$$\varphi = \begin{pmatrix} \begin{pmatrix} \varphi(x_1) & 0 \\ 0 & \varphi(x_1) \end{pmatrix} & 0 & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} \varphi(x_2) & 0 \\ 0 & \varphi(x_2) \end{pmatrix} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \begin{pmatrix} \varphi(x_N) & 0 \\ 0 & \varphi(x_N) \end{pmatrix} \end{pmatrix}$$

$$\psi = \begin{pmatrix} \begin{pmatrix} \psi(x_1) & 0 \\ 0 & -\psi(x_1) \end{pmatrix} & 0 & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} \psi(x_2) & 0 \\ 0 & -\psi(x_2) \end{pmatrix} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \begin{pmatrix} \psi(x_N) & 0 \\ 0 & -\psi(x_N) \end{pmatrix} \end{pmatrix}$$



role of supercoordinate

Lattice superfield: $\Phi(x) = \varphi(x) + \frac{1}{2}(-1)^{\frac{2x}{a}}\psi(x)$ $\quad (x = \frac{na}{2})$

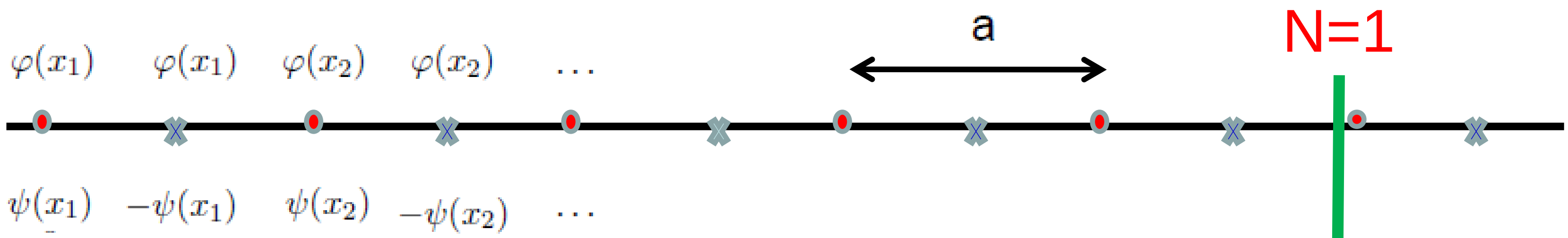
SUSY transformation: $\delta\Phi(x) = a^{-\frac{1}{2}}\alpha(-1)^{\frac{2x}{a}}\left(\Phi(x+\frac{a}{2}) - \Phi(x)\right)$

hermiticity

$$\delta\varphi(x) = \frac{i\alpha}{2}\left[\psi(x+\frac{a}{4}) + \psi(x-\frac{a}{4})\right] \to i\alpha\psi(x)$$

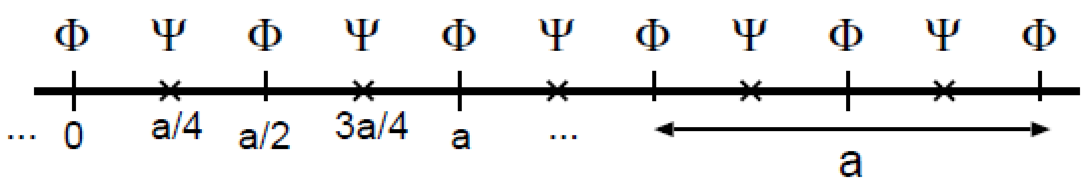
$$\delta\psi(x) = 2a^{-1}\alpha\left[\varphi(x+\frac{a}{4}) - \varphi(x-\frac{a}{4})\right] \to \alpha\frac{\partial\varphi(x)}{\partial x}$$

alternating sign $\longrightarrow$ key of the lattice SUSY



N=1

Momentum representation



N=2

$$\sum_x e^{ipx}(-1)^{\frac{2x}{a}}\Psi(x) = \sum_x e^{i(p-\frac{2\pi}{a})x}\Psi(x) = \Psi(p-\frac{2\pi}{a}) \longrightarrow \Psi(\frac{2\pi}{a}-p)$$

$$(-1)^{\frac{2x}{a}} = e^{\pm\frac{2\pi i}{a}}$$

species doubler

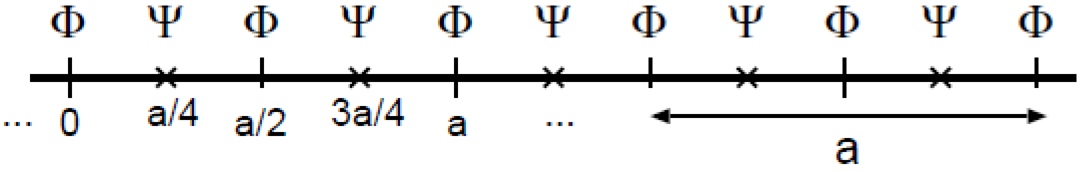
D=1 N=2 Lattice SUSY

Fourier transform

$$\Phi(p) = \frac{1}{2}\sum_{x=\frac{na}{2}} e^{ipx}\Phi(x), \qquad \Psi(p) = \frac{1}{2}\sum_{x=\frac{na}{2}+\frac{a}{4}} e^{ipx}\Psi(x)$$

$$\Phi(x) = a\int_0^{\frac{4\pi}{a}} \frac{dp}{2\pi}\Phi(p)e^{-ipx}, \qquad \Psi(x) = a\int_0^{\frac{4\pi}{a}} \frac{dp}{2\pi}\Psi(p)e^{-ipx}$$

$$\Phi(p+\frac{4\pi}{a}) = \Phi(p), \qquad \Psi(p+\frac{4\pi}{a}) = -\Psi(p)$$



$$\delta_1 \Phi(p) = i \cos \tfrac{ap}{4} \alpha \Psi(p)$$

$$\delta_1 \Psi(p) = -4i \sin \tfrac{ap}{4} \alpha \Phi(p)$$

$$\delta_1 \Phi(x) = \tfrac{i\alpha}{2} \left[\Psi(x + \tfrac{a}{4}) + \Psi(x - \tfrac{a}{4}) \right] \qquad x = \tfrac{na}{2}$$

$$\delta_1 \Psi(x) = 2\alpha \left[\Phi(x + \tfrac{a}{4}) - \Phi(x - \tfrac{a}{4}) \right] \qquad x = \tfrac{na}{2} + \tfrac{a}{4}$$

$$\delta_2 \Phi(p) = \cos \tfrac{ap}{4} \alpha \Psi(\tfrac{2\pi}{a} - p)$$

$$\delta_2 \Psi(\tfrac{2\pi}{a} - p) = 4 \sin \tfrac{ap}{4} \alpha \Phi(p)$$

$$\delta_2 \Phi(x) = \tfrac{i\alpha}{2} (-1)^n \left[\Psi(-x + \tfrac{a}{4}) - \Psi(-x - \tfrac{a}{4}) \right]$$

$$\delta_2 \Psi(x) = 2\alpha (-1)^n \left[\Phi(-x + \tfrac{a}{4}) - \Phi(-x - \tfrac{a}{4}) \right]$$

bi-local nature of SUSY transformation

$$\Phi(p) \sim \varphi(p) \qquad \Phi(\frac{2\pi}{a} - p) \sim D(p)$$

$$\Psi(p) \sim \psi_1(p) \qquad \Psi(\frac{2\pi}{a} - p) \sim \psi_2(p)$$

N=2

species doubler
as supermultiplets

Kinetic term

$$S_{kin} = 4a\int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp}{2\pi}\left[2\sin^2\frac{ap}{4}\Phi(-p)\Phi(p) - \frac{1}{4}\sin\frac{ap}{2}\Psi(-p)\Psi(p)\right]$$

Mass term

$$S_{mass} = 4am_0\int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp}{2\pi}\left[\Phi(p+\frac{2\pi}{a})\Phi(p) + \frac{1}{4}\Psi(p+\frac{2\pi}{a})\Psi(p)\right]$$

Interaction term

Lattice mom. conservation

$$S^{(n)} = g_0^{(n)} a^n \frac{4}{n!}\int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp_1}{2\pi}\cdots\frac{dp_n}{2\pi} 2\pi\delta\left(\sum_{i=1}^{n}\sin\frac{ap_i}{2}\right)$$

$$\times\left(\prod_{i=1}^{n}\cos\frac{ap_i}{2}\right)\left[2\sin^2\frac{ap_1}{4}\ \Phi(p_1)\Phi(p_2)\cdots\Phi(p_n) +\right.$$

$$\left. + \frac{n-1}{4}\ \sin\frac{a(p_1-p_2)}{4}\ \Psi(p_1)\Psi(p_2)\Phi(p_3)\cdots\Phi(p_n)\right]$$

exact lattice SUSY invariance under δ_1 and δ_2

Momentum representation

$$\delta_1 = \alpha\sqrt{a}Q_1, \quad \delta_2 = \alpha\sqrt{a}Q_2$$

$$Q_1^2 = Q_2^2 = \frac{2}{a}\sin\frac{ap}{2}, \qquad \{Q_1, Q_2\} = 0$$

$$\frac{2}{a}\sin\frac{ap}{2} \longleftrightarrow i\hat{\partial}$$

$$Q_1^2 = Q_2^2 = i\hat{\partial}, \qquad \{Q_1, Q_2\} = 0$$

Since difference operator does not satisfy Leibniz rule
" How can algebra be consistent ?"

Symmetric difference operator → No Leibniz rule

$$\hat{\partial}(F(x)G(x)) = \frac{1}{a}\left(F(x+\tfrac{a}{2})G(x+\tfrac{a}{2}) - F(x-\tfrac{a}{2})G(x-\tfrac{a}{2})\right)$$

$$= \hat{\partial}F(x)G(x+\tfrac{a}{2}) + F(x-\tfrac{a}{2})\hat{\partial}G(x)$$

$$= \hat{\partial}F(x)G(x-\tfrac{a}{2}) + F(x+\tfrac{a}{2})\hat{\partial}G(x)$$

Possible solutions:

(1) Link approach: Hopf algebraic symmetry

$$Q_1(F(x)G(x)) = Q_1F(x)G(x+\tfrac{a}{4}) + F(x-\tfrac{a}{4})Q_1G(x)$$

$$Q_2(F(x)G(x)) = Q_2F(x)G(x-\tfrac{a}{4}) + F(x+\tfrac{a}{4})Q_2G(x)$$

$$Q_1^2 = Q_2^2 = i\hat{\partial}, \qquad \{Q_1, Q_2\} = 0$$

(2) New star product: → Leibniz rule ?

$$\hat{\partial}(F(x) * G(x)) = (\hat{\partial}F(x)) * G(x) + F(x) * (\hat{\partial}G(x))$$

$$(F \cdot G)(p) = \int dp_1 dp_2 F(p_1) G(p_2) \delta(p - p_1 - p_2)$$

Lattice momentum: $$(F * G)(p) = \int d\hat{p}_1 d\hat{p}_2 F(p_1) G(p_2) \delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$ $\hat{p} = \frac{2}{a} \sin \frac{ap}{2}$

Leibniz rule in mom.

$$\hat{p}\ (F * G)(p) = \int\ d\hat{p}_1\ d\hat{p}_2\, [\hat{p}_1\ F(p_1)\ G(p_2) + F(p_1)\ \hat{p}_2\ G(p_2)]\ \delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$ $d\hat{p} = d\tilde{p} \cos \tilde{p}$

New star product in coordinate space:

$$\begin{aligned}(F * G)(x) = F(x) * G(x) &= a \int \frac{d\hat{p}}{2\pi} e^{-ipx}\ (F * G)(p) \\ &= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} d\tilde{p}\ \cos \tilde{p}\ e^{-ipx} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\tilde{p}_1}{2\pi} \frac{d\tilde{p}_2}{2\pi} \cos \tilde{p}_1\ \cos \tilde{p}_2 \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{i\tau(\sin \tilde{p} - \sin \tilde{p}_1 - \sin \tilde{p}_2)} \\ &\quad \times \sum_{y,z} e^{i(m\tilde{p}_1 + l\tilde{p}_2)} F(y) G(z) \\ &= \int_{-\infty}^{\infty} d\tau J_{n\pm1}(\tau) \sum_{m,l} J_{m\pm1}(\tau) J_{l\pm1}(\tau) F(y) G(z)\end{aligned}$$

$x = \frac{na}{2}, y = \frac{ma}{2}, z = \frac{la}{2}$

$\tilde{p} = \frac{ap}{2}$

$$\delta\left(\frac{2}{a} \sin \tilde{p}_i\right) = \frac{a}{4\pi} \int_{-\infty}^{\infty} d\tau e^{i\tau \sin \tilde{p}_i}$$

$$J_n(\tau) = \frac{1}{2\pi} \int_{\alpha}^{2\pi+\alpha} e^{i(n\theta - \tau \sin \theta)} d\theta$$

$$J_{n\pm1}(\tau) = \frac{1}{2}(J_{n+1}(\tau) + J_{n-1}(\tau))$$

commutative $$F(x) * G(x) = G(x) * F(x)$$

$$
\begin{aligned}
i\hat{\partial}(F(x) * G(x)) &= a \int \frac{d\hat{p}}{2\pi}\, i\hat{\partial}_x\, e^{-ipx}\, (F * G)(p) \\
&= \frac{a^2}{4} \int d\hat{p}\, e^{-ipx} \sum_{y,z} \int \frac{d\hat{p}_1}{2\pi} \frac{d\hat{p}_2}{2\pi}\, e^{ip_1 y + ip_2 z} \\
&\quad \times \Big((i\hat{\partial}_y\, F(y))G(z) \; + \; F(y)\, (i\hat{\partial}_z\, G(z)) \Big)\; \delta(\hat{p} - \hat{p}_1 - \hat{p}_2) \\
&= (i\hat{\partial}F(x)) * G(x) + F(x) * (i\hat{\partial}G(x)).
\end{aligned}
$$

star product ⟷ Leibniz rule

$$
\hat{\partial}(F(x) * G(x)) = (\hat{\partial}F(x)) * G(x) + F(x) * (\hat{\partial}G(x))
$$

Exact SUSY invariance in coordinate space on star product actions

$$S^{(2)} = \sum_x \left[\Phi(x) * \left(2\Phi(x) - \Phi(x+\frac{a}{2}) - \Phi(x-\frac{a}{2})\right) + \frac{i}{2}\Psi(x+\frac{3a}{4}) * \Psi(x+\frac{a}{4})\right]$$

$$S^{(n)} = \frac{4}{n!}g_0^{(n)} \sum_x \Big[\left(2\Phi(x) - \Phi(x+\frac{a}{2}) - \Phi(x-\frac{a}{2})\right) * \Phi(x)^{n-1}(x) + \frac{(n-1)i}{2}\Psi(x+\frac{3a}{4}) * \Psi(x+\frac{a}{4}) * \Phi^{n-2}(x)\Big]$$

$$Q_j(F(x) * G(x)) = Q_j F(x) * G(x) + (-1)^{|F|} F(x) * Q_j G(x)$$

Conclusions

★ Exactly SUSY invariant formulation in mom. and coordinate space is found.

★ Lattice SUSY algebra is exactly fulfilled on the star product in coordinate space.

★ Ward-Takahashi id. is fulfilled.

An interesting possibility:
Link approach and star product formulation is equivalent !

Another solution to chiral fermion problem

Species doublers are physical

$$\Phi_1(x) = c_1 \quad , \quad \Phi_2(x) = (-1)^{\frac{2x}{a}} c_2$$

$$\delta_1 = \alpha\sqrt{a}Q_1, \quad \delta_2 = \alpha\sqrt{a}Q_2$$

$$Q_1^2 = Q_2^2 = \tfrac{2}{a}\sin\tfrac{ap}{2}, \qquad \{Q_1, Q_2\} = 0$$

$$Q_1^2 = Q_2^2 = i\hat{\partial}, \qquad \{Q_1, Q_2\} = 0$$

$$\tfrac{2}{a}\sin\tfrac{ap}{2} \longleftrightarrow i\hat{\partial}$$

$$\begin{aligned}
\hat{\partial}(F(x)G(x)) &= \tfrac{1}{a}\left(F(x+\tfrac{a}{2})G(x+\tfrac{a}{2}) - F(x-\tfrac{a}{2})G(x-\tfrac{a}{2})\right) \\
&= \hat{\partial}F(x)G(x+\tfrac{a}{2}) + F(x-\tfrac{a}{2})\hat{\partial}G(x) \\
&= \hat{\partial}F(x)G(x-\tfrac{a}{2}) + F(x+\tfrac{a}{2})\hat{\partial}G(x)
\end{aligned}$$

"How can the lattice supersymmetry algebra be consistent since the difference operator does not satisfy Leibniz rule while the super charges satisfy Leibniz rule ?"

$$Q_1(F(x)G(x)) = Q_1F(x)G(x+\tfrac{a}{4}) + F(x-\tfrac{a}{4})Q_1G(x)$$

$$Q_2(F(x)G(x)) = Q_2F(x)G(x-\tfrac{a}{4}) + F(x+\tfrac{a}{4})Q_2G(x)$$

$$Q_1(F(x)G(x)) = Q_1F(x)G(x+\tfrac{a}{4}) + F(x-\tfrac{a}{4})Q_1G(x)$$

$$= Q_1(G(x)F(x)) = Q_1G(x)F(x+\tfrac{a}{4}) + G(x-\tfrac{a}{4})Q_1F(x)$$

$$Q_1G(x)F(x+\tfrac{a}{4}) = F(x-\tfrac{a}{4})Q_1G(x)$$

$$Q_1F(x)G(x+\tfrac{a}{4}) = G(x-\tfrac{a}{4})Q_1F(x)$$

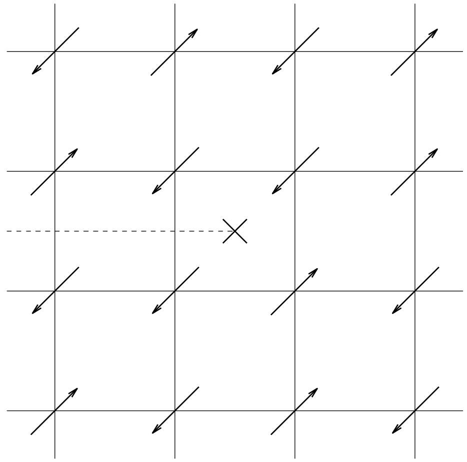
夏の学校2009

超対称性の格子上での定式化とその背景

河本昇

北海道大学

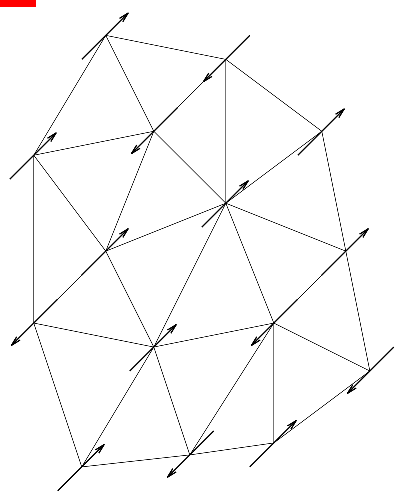
Motivations



$$\int \overline{\psi}\ \partial\!\!\!/\psi$$

Majorana fermion

$\sigma_i = \pm 1$



$$\int e\overline{\psi}\gamma^a e_a^\mu D_\mu \psi$$

fermion + gravity

Boulatov &Kazakov

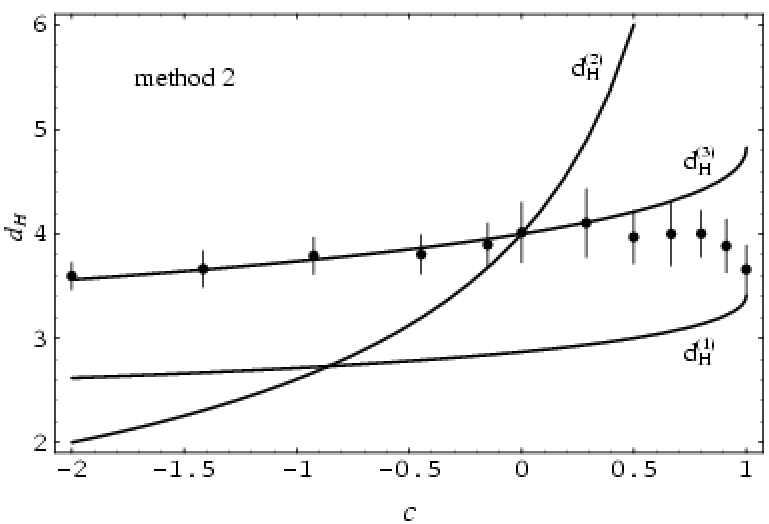
Fractal Structure of 2D Quantum Gravity

$$(\#\text{ of triangles in radius } r) \sim r^{d_H}$$

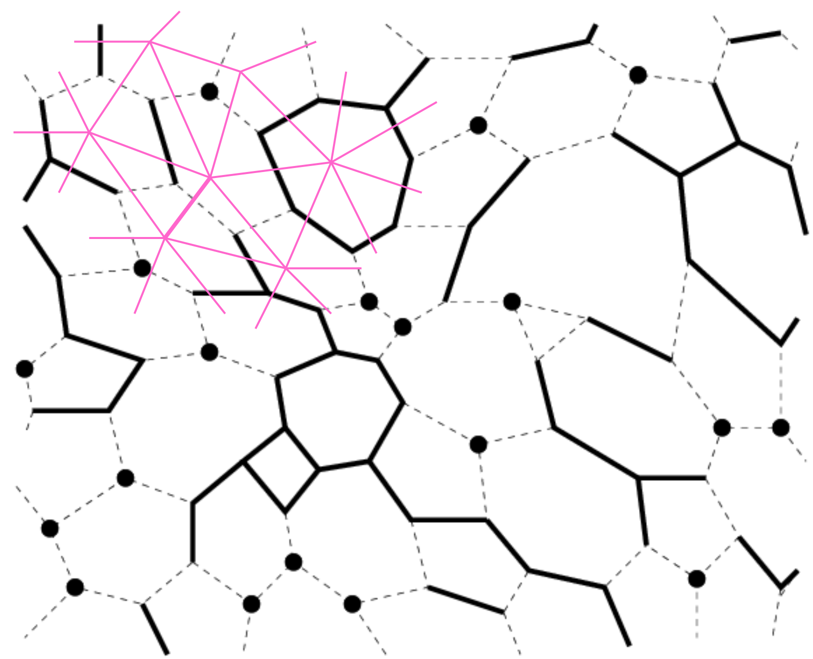
$$d_H^{(3)}(c) = 2 \times \frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$$

(c: central charge $\sim$ matter)

N.K. & Watabiki



N.K. & Yotsuji



Q state Potts model on random surface

Quantization and Twisted SUSY

Tsukioka, N.K.,
Kato, Miyake, Uchida

Continuum

$$S = \int d^2x \phi \epsilon^{\mu\nu} \partial_\mu \omega_\nu$$

$$\delta\phi = 0, \quad \delta\omega_\mu = \partial_\mu v$$

(Two dimensional Abelian BF)

$$S = \int d^2x [\epsilon^{\mu\nu} \phi \partial_\mu \omega_\nu + \underline{b \partial^\mu \omega_\mu - i\bar{c}\partial^\mu \partial_\mu c} - i\underline{\lambda}\rho]$$

$$= \int d^2x s\tilde{s}\frac{1}{2}\epsilon^{\mu\nu} s_\mu s_\nu(-i\bar{c}c)$$

Auxiliary field
Off-shell invariance

$$\underline{s^2} = \{s, \tilde{s}\} = \tilde{s}^2 = \{s_\mu, s_\nu\} \underline{= 0},$$
$$\{s, s_\mu\} = -i\partial_\mu, \{\tilde{s}, s_\mu\} = i\epsilon_{\mu\nu}\partial^\nu$$

Nilpotency of BRS charge s

ϕ^A	$s\phi^A$	$s_\mu\phi^A$	$\tilde{s}\phi^A$
ϕ	$i\rho$	$-\epsilon_{\mu\nu}\partial^\nu\bar{c}$	0
ω_ν	$\partial_\nu c$	$-i\epsilon_{\mu\nu}\lambda$	$-\epsilon_{\nu\rho}\partial^\rho c$
c	0	$-i\omega_\mu$	0
$\bar{c}$	$-ib$	0	$-i\phi$
b	0	$\partial_\mu\bar{c}$	$-i\rho$
λ	$\epsilon^{\mu\nu}\partial_\mu\omega_\nu$	0	$-\partial_\mu\omega^\mu$
ρ	0	$-\partial_\mu\phi - \epsilon_{\mu\nu}\partial^\nu b$	0

N=D=2 Twisted SUSY

Kato,N.K.&Uchida

Fermionic Link Fields

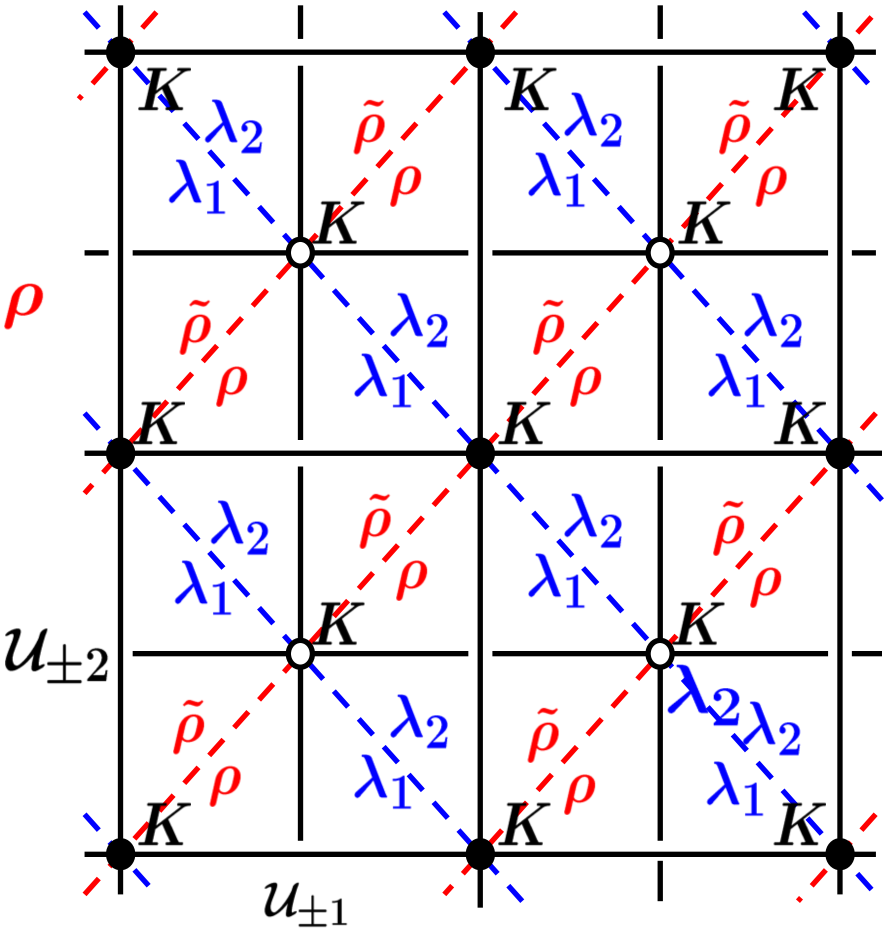
$$[\nabla_\mu, \mathcal{U}_{+\nu}] \equiv -\epsilon_{\mu\nu}\tilde{\rho}$$

$$[\nabla_\mu, \mathcal{U}_{-\nu}] \equiv -\delta_{\mu\nu}\rho$$

$$\epsilon_{\mu\nu}[\nabla, \mathcal{U}_{-\nu}] \equiv -\epsilon_{\mu\nu}\lambda_\nu$$

Auxiliary Field

$$K = \frac{1}{2}\{\nabla_\mu, \lambda_\mu\}$$



Twisted N=2 Lattice SUSY Transformation

$$s_A(\varphi) \equiv [\nabla_A, \varphi\}_{x+a_A+a_\varphi,x}$$

Shifts of Fields

∇	$\tilde{\nabla}$	∇_μ	$\mathcal{U}_{\pm\mu}$	ρ	$\tilde{\rho}$	λ_μ	K
a	$\tilde{a}$	a_μ	$\pm n_\mu$	$-a$	$-\tilde{a}$	$-a_\mu$	0

	s	$\tilde{s}$	s_μ
$\mathcal{U}_{+\nu}$	0	$+\epsilon_{\nu\rho}\lambda_\rho$	$-\epsilon_{\mu\nu}\tilde{\rho}$
$\mathcal{U}_{-\nu}$	$-\lambda_\nu$	0	$-\delta_{\mu\nu}\rho$
λ_ν	0	0	$-i[\mathcal{U}_{+\mu},\mathcal{U}_{-\nu}] + \delta_{\mu\nu}(K + \frac{i}{2}[\mathcal{U}_{+\rho},\mathcal{U}_{-\rho}])$
ρ	$-\frac{i}{2}[\mathcal{U}_{+\rho},\mathcal{U}_{-\rho}] - K$	$+\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]$	0
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho},\mathcal{U}_{+\sigma}]$	$+\frac{i}{2}[\mathcal{U}_{+\rho},\mathcal{U}_{-\rho}] - K$	0
K	$+\frac{i}{2}[\mathcal{U}_{+\rho},\lambda_\rho]$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho},\lambda_\sigma]$	$-\frac{i}{2}[\mathcal{U}_{+\mu},\rho] - \frac{i}{2}\epsilon_{\mu\nu}[\mathcal{U}_{-\nu},\tilde{\rho}]$

Twisted SUSY Algebra closes off-shell

$$\begin{aligned}
\{s, s_\mu\}(\varphi)_{x+a_\varphi,x} &= +i[\mathcal{U}_{+\mu},\varphi]_{x+a_\varphi+n_\mu,x} \\
\{\tilde{s}, s_\mu\}(\varphi)_{x+a_\varphi,x} &= +i\epsilon_{\mu\nu}[\mathcal{U}_{-\nu},\varphi]_{x+a_\varphi-n_\nu,x} \\
s^2(\varphi)_{x+a_\varphi,x} &= \tilde{s}^2(\varphi)_{x+a_\varphi,x} = 0 \\
\{s,\tilde{s}\}(\varphi)_{x+a_\varphi,x} &= \{s_\mu, s_\nu\}(\varphi)_{x+a_\varphi,x} = 0
\end{aligned}$$

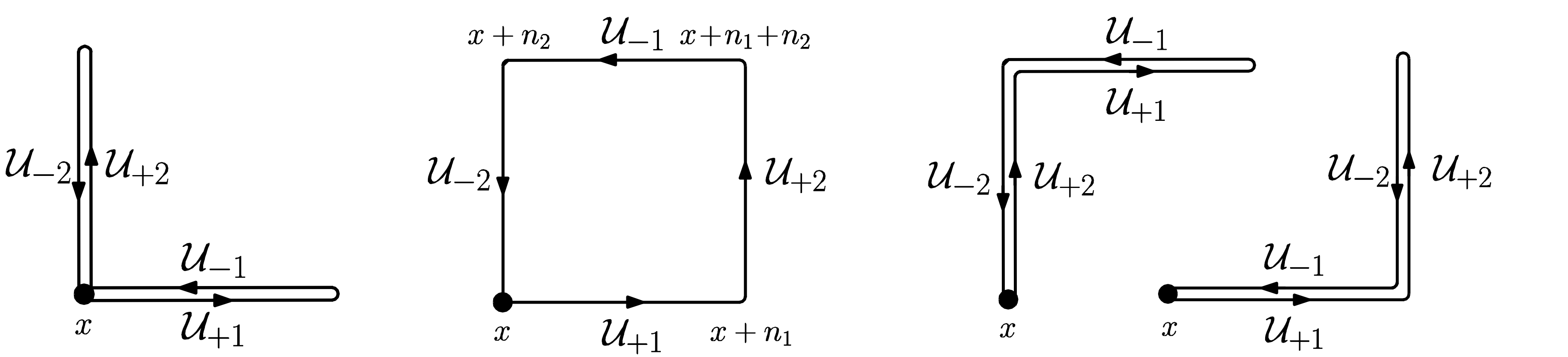
Twisted N=2 Super Yang-Mills Action

Action has twisted SUSY exact form. $\longrightarrow$ Off-shell SUSY invariance for all twisted super charges.

$$
\begin{aligned}
S &\equiv \frac{1}{2}\sum_x \mathrm{Tr}\ s\tilde{s}s_1s_2\ \mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \\
&= S_B + S_F \\
S_B &= \sum_x \mathrm{Tr}\Big[\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K^2_{x,x} \\
&\quad -\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x}\Big] \\
S_F &= \sum_x \mathrm{Tr}\Big[-i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x-a}(\rho)_{x-a,x} \\
&\quad - i(\tilde{\rho})_{x,x+\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu},\lambda_\nu]_{x+\tilde{a},x}\Big]
\end{aligned}
$$

Bosonic part of the Action

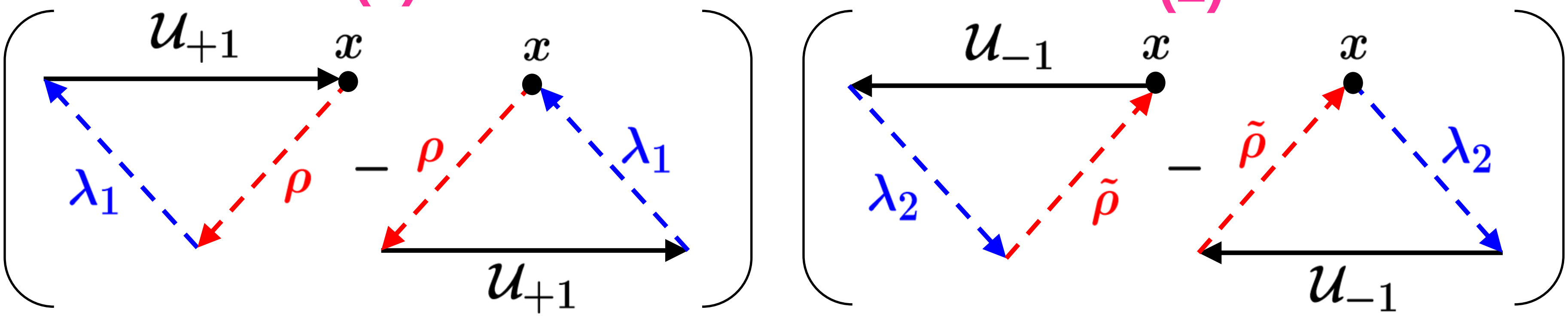
$$S_B = \sum_x \mathrm{Tr}\Big[\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K_{x,x}^2 - \frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x}\Big]$$



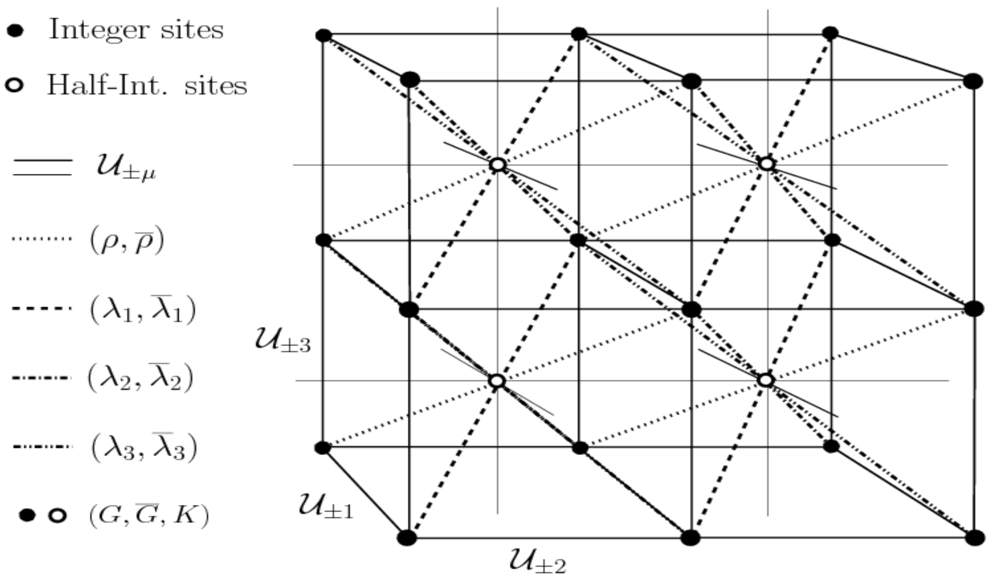
Fermionic part of the Action

$$
\begin{aligned}
S_F \;=\; \sum_x \mathrm{Tr}\Big[& -i[\mathcal{U}_{+\mu}, \lambda_\mu]_{x,x-a}(\rho)_{x-a,x} && \cdots \text{(1)} \\
& - i(\tilde{\rho})_{x,x+\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu}, \lambda_\nu]_{x+\tilde{a},x}\Big] && \cdots \text{(2)}
\end{aligned}
$$

(1) (2)



Higer dimensional extension is possible:



3-dim. N=4 super Yang-Mills

Two Problems

$$Q_A(\phi_1(x)\phi_2(x)) = (Q_A\phi_1(x))\phi_2(x) + \phi_1(x+a_A)Q_A\phi_2(x)$$

$$Q_A(\phi_2(x)\phi_1(x)) = (Q_A\phi_2(x))\phi_1(x) + \phi_2(x+a_A)Q_A\phi_1(x)$$

Bruckmann
Kok

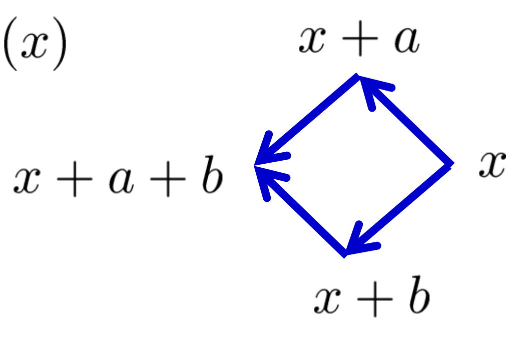
When $\phi_1(x)\phi_2(x) = \phi_2(x)\phi_1(x)$ "inconsistency"

but if we introduce the following "mild non-commutativity":

$$(Q_A\phi_i(x))\phi_j(x) = \phi_j(x+a_A)(Q_A\phi_i(x)) \quad i,j = 1,2$$

then $$Q_A(\phi_1(x)\phi_2(x)) = Q_A(\phi_2(x)\phi_1(x))$$

In general $$\phi_a(x+b)\phi_b(x) = \phi_b(x+a)\phi_a(x)$$



Modified Leibniz rule
+
Mild non-commutativity

→ Hopf algebraic Field Theory

Concrete representation of this non-commutativity

$$\{X_1, X_2\} = \frac{i}{2\pi N b}, \qquad p_\mu = \frac{2\pi}{N} k_\mu, \quad k_\mu \in \mathbf{Z}$$

$$\hat{\Delta}(\xi) = \frac{1}{N} \sum_p \exp i p_\mu \left(X_\mu - \xi_\mu \right)$$

$$\varphi_a \diamond \varphi_b(\xi) = N^{-1} Tr \varphi_a \varphi_b \hat{\Delta}(\xi) = N \varphi_a(\xi_1 - \tfrac{b}{2}, \xi_2) \varphi_b(\xi_1 + \tfrac{a}{2}, \xi_2)$$

Orbifold condition $\varphi_a \Omega = \omega^a \Omega \varphi_a$ $\qquad \left(\Omega = \exp -i \frac{2\pi}{N} X_1 = \omega^{X_1} \right)$

Lattice version of Moyal product

∇_A operation makes link holes and thus loses gauge invariance.

A possible solution

We claim: if there is covariantly constant super parameter η_A which has opposite shift of ∇_A and commutes with all the super covariant derivatives:

$$\{\eta_A, \nabla_B\} = 0$$

$$\{\eta_A, \varphi\} = 0$$

$\longrightarrow$

$$\{\eta_A \nabla_A, S\} = 0$$

lattice SUSY and gauge invariant !

η_A compensates the link holes.

$\longrightarrow$ η_A gets coordinate dependence

super gravity

Gauge Theory on the Random Lattice

Form

0 ϕ

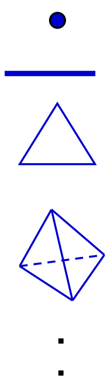
1 $\omega = \omega_\mu dx^\mu$

2 $B = B_{\mu\nu} dx^\mu dx\nu$

3 $\Omega = \Omega_{\mu\nu\rho} dx^\mu dx^\nu dx\rho$

$\vdots \qquad \vdots$

Simplex



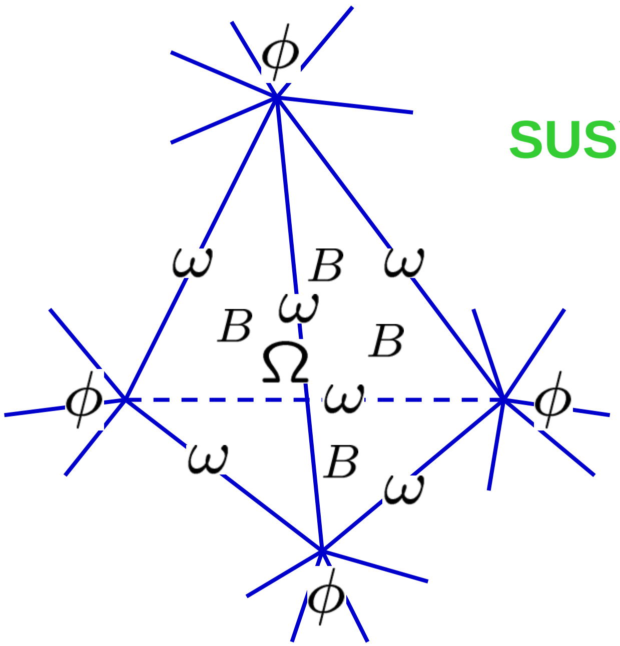
$$\mathcal{A} = \mathbf{j}(\overset{1}{\omega} + \overset{3}{\Omega} + \cdots) + \mathbf{k}(\overset{0}{\phi} + \overset{2}{B} + \cdots)$$
$$+\mathbf{1}(\chi^{(1)} + \chi^{(3)} + \cdots) + \mathbf{i}(\chi^{(0)} + \chi^{(2)} + \cdots)$$

$$S = \int Tr(\tfrac{1}{2}\mathcal{A}Q\mathcal{A} \ + \ \tfrac{1}{3}\mathcal{A}^3)$$

Gauge Theory + Gravity ?

Boson ⟷ Fermion ?

SUSY ?



Generalized Gauge Theories in arbitrary dimensions

N.K.&Watabiki ‘91

gauge field	$A = T^a A^a_\mu dx_\mu$	$\mathcal{A} = \mathbf{1}\psi + \mathbf{i}\hat{\psi} + \mathbf{j}A + \mathbf{k}\hat{A}$
gauge parameter	$v = T^a v^a$	$\mathcal{V} = \mathbf{1}\hat{a} + \mathbf{i}a + \mathbf{j}\hat{\alpha} + \mathbf{k}\alpha$
derivative	$d = dx^\mu \partial_\mu$	$\mathcal{Q} = \mathbf{j}d$
curvature	$F = dA + A^2$	$\mathcal{F} = \mathcal{Q}\mathcal{A} + \mathcal{A}^2$
gauge trans.	$\delta A = dv + [A, v]$	$\delta \mathcal{A} = \mathcal{Q}\mathcal{V} + [\mathcal{A}, \mathcal{V}]$
Chern-Simons	$\int Tr(\frac{1}{2}AdA \; + \; \frac{1}{3}A^3)$	$\int Tr_{\mathbf{k}}(\frac{1}{2}\mathcal{A}\mathcal{Q}\mathcal{A} \; + \; \frac{1}{3}\mathcal{A}^3)$
Topological Yang-Mills	$\int Tr(FF)$	$\int Str_{\mathbf{1}}(\mathcal{F}\mathcal{F})$
Yang-Mills	$\int Tr(F \star F)$	$\int Tr_{\mathbf{1}}(\mathcal{F} \mathrm{v} \mathcal{F}) \star 1$

Puzzle 2

What is the role of "quaternion"
in generalized gauge theory ?

$$\mathcal{A} = \mathbf{1}\psi + \mathbf{i}\hat{\psi} + \mathbf{j}A + \mathbf{k}\hat{A}$$

$$\mathcal{V} = \mathbf{1}\hat{a} + \mathbf{i}a + \mathbf{j}\hat{\alpha} + \mathbf{k}\alpha$$

Single lattice translation as SUSY transformation

$$\Phi(x) = \varphi(x) + \frac{(-1)^{\hat{x}}}{\sqrt{N}}\psi(x) \quad (\eta(x) = \frac{(-1)^{\hat{x}}}{\sqrt{N}},\ \eta^2(x) = \frac{1}{N} \equiv a,\ \hat{x} = \frac{x}{a})$$

$$\delta\Phi(x) = \Phi(x+a) - \Phi(x) = \varphi(x+a) + \frac{(-1)^{\hat{x}}}{\sqrt{N}}\psi(x+a) - (\varphi(x)) + \frac{(-1)^{\hat{x}}}{\sqrt{N}}\psi(x))$$

$$= \eta(x)[-\psi(x+a) - \psi(x) + \eta(x)\frac{\varphi(x+a)-\phi(x)}{1/N}]$$

$$= \delta\varphi(x) + \eta(x)\delta\psi(x)$$

$$\begin{cases} \delta\varphi(x) = -\eta(x)(\psi(x+a) + \psi(x)) \equiv \eta Q\varphi(x) \\ \delta\psi(x) = \eta(x)\partial_+\varphi(x) \equiv \eta Q\psi(x) \end{cases}$$

$$\delta^2\varphi(x) = -\eta\delta(\psi(x+a) + \psi(x)) = -\eta^2(\partial\varphi(x+a) + \partial\varphi(x))$$

$$= -\eta^2\frac{\varphi(x+2a)-\varphi(x)}{1/N} = -\frac{2}{N}\partial_+\varphi(x)$$

Super parameter $\eta(x) = \frac{(-1)^{\hat{x}}}{\sqrt{N}}$

$$\delta^2 = (\eta Q)^2 = -\eta^2 Q^2 = -\frac{2}{N}\partial_+$$

SUSY algebra $\{Q, Q\} = \partial_+$

Matrix Representation

$$\Phi(x) = \varphi(x) + \frac{(-1)^{\hat{x}}}{\sqrt{N}}\psi(x)$$

$$\varphi = \begin{bmatrix} \varphi(x_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varphi(x_2) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varphi(x_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varphi(x_N) \end{bmatrix}$$

$$(-1)^{\hat{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \mathbf{1} \equiv \mathbf{k} \otimes \mathbf{1}$$

ϕ and ψ are diagonal.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ \cdots \end{bmatrix} \otimes \mathbf{1} = (\sigma_1 + i\sigma_2) \otimes \mathbf{1} \equiv (\mathbf{i} + \mathbf{j}) \otimes \mathbf{1}$$

TexPoint Display

$$\Delta_- \equiv Q^{-1} = \begin{bmatrix} 0 & & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \mathbf{1} = (\sigma_1 - i\sigma_2) \otimes \mathbf{1} \equiv (\mathbf{i} - \mathbf{j}) \otimes \mathbf{1}$$

Two step translation as SUSY transformation

$$\delta\Phi = Q^{-1}\Phi Q - \Phi = -Q^{-1}[Q, \Phi] \qquad Q^2 = \Delta_+^2$$

$$\Phi = \mathbf{1}\varphi + \frac{\mathbf{k}}{\sqrt{N}}\psi \qquad \mathcal{V} = \mathbf{1}\hat{a} + ia + j\hat{\alpha} + k\alpha$$

Partial answer to Puzzle 2

Quaternion may be fundamentally related to the lattice SUSY transformation. Chirality may play an important role in the transformation.

Differential form structure for Dirac-Kaeher mechanism should be essentially introduced to accommodate super gravity nature.