A Ginsparg-Wilson approach to lattice $\mathcal{CP}$ symmetry

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- I start from the premise that it is possible to construct overlap lattice QCD from the RG (c.f. arXiv:0903.5521)
- In the continuum, introduce a new fermion field ψ_1
- Construct a new generating functional

$$Z_0(J_0,\overline{J}_0) = \int d\psi_0 d\overline{\psi}_0 dU \exp\left(-\frac{1}{2g_0^2}F_{\mu\nu}^2 - \overline{\psi}_0 D_0\psi_0 + \overline{\psi}_0 J_0 + \overline{J}_0\psi_0\right)$$

$$\times \int d\psi_1 d\overline{\psi_1} \exp\left((\overline{\psi_1} - \overline{\psi}_0\overline{B}^{-1})\alpha(\psi_1 - B^{-1}\psi_0)\right)$$

$$= Z_1(J_1,\overline{J}_1)$$

- For example set $\alpha = \Lambda\mathbb{1}$, $\Lambda \to \infty$, then

$$\psi_1(y) = \int d^4x B^{-1}(y,x)\psi_0(x) \quad \overline{\psi}_1(y) = \int d^4x\overline{\psi}_0(x)\overline{B}^{-1}(x,y)$$

$$J_1(x) = \int d^4y\overline{B}(x,y)J_0(y) \qquad \overline{J}_1(x) = \int d^4y\overline{J}_0(y)B(y,x)$$

$$D_1(x,y) = \int d^4x' d^4y' \overline{B}(x,x')D_0(x',y')B(y',y)$$

- Consider the blocking

$$B(x,y) = \overline{B}^{\dagger} = \sum_n \tilde{U}_{xn}\tilde{U}_{ny}$$

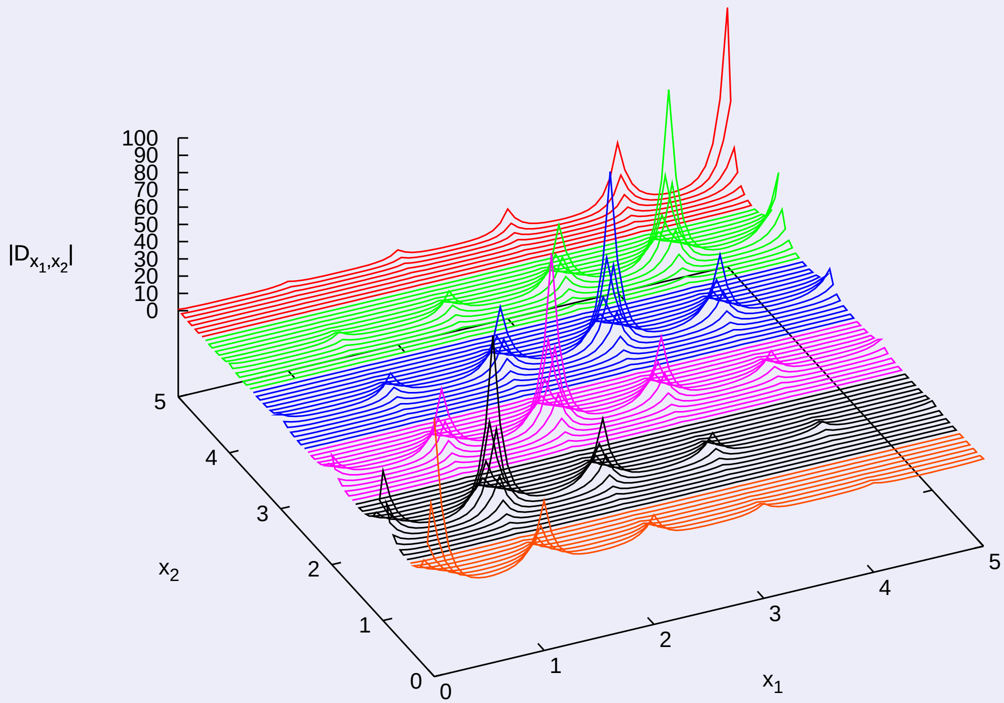
$$\left[\frac{1}{\zeta^4}e^{-\zeta\sum_\mu|x_\mu - n_\mu|}e^{-\zeta\sum_\mu|y_\mu - n_\mu|}\prod_\nu \theta(|x_\nu - n_\nu| - a/2)\theta(|y_\nu - n_\nu| - a/2)\right.$$

$$\left.+\frac{\Lambda}{a}\left(\delta^{(4)}(x-y) - \frac{1}{\zeta^4}e^{-\zeta\sum_\mu|y_\mu - n_\mu|}e^{-\zeta\sum_\mu|y_\mu - n_\mu|}\right)\right]$$

- $\tilde{U}_{xn}$ is the link between the position x and the lattice site n,

$$\tilde{U}_{xn} = P[e^{ig\int_x^n A_\mu^a T^a ds_\mu}]$$

- This generates the naive lattice Dirac operator
- Additional degrees of freedom are projected to infinite mass



- To remove the fermion doublers, the blockings must contain Dirac structure.
- To remove the e^{ζ} terms from the Fourier representation, D_1 must be constructed from the matrix sign function.

- Valid blockings exist for the overlap operator.
- Use the Ginsparg-Wilson procedure to derive lattice chiral symmetry (arXiv:0903.5521,arXiv:1003.3991)

$$\psi_1 \to e^{i\epsilon\gamma_R}\psi_1 \qquad \gamma_L^{(\eta)} D + D\gamma_R^{(\eta)} = 0 \qquad \overline{\psi}_1 \to \overline{\psi}_1 e^{i\epsilon\gamma_L}$$

$$\gamma_R^{(\eta)} = \overline{B}^{(\eta)}\gamma_5 \left(\overline{B}^{(\eta)}\right)^{-1} = \text{sign}\Big[\gamma_5 \cos\left((1+\eta)\left(\frac{\pi}{2} - \theta\right)\right) + \text{sign}(\gamma_5(D_1^\dagger - D_1)) \sin\left((1+\eta)\left(\frac{\pi}{2} - \theta\right)\right)\Big]$$

$$\gamma_L^{(\eta)} = \left(B^{(\eta)}\right)^{-1}\gamma_5 B^{(\eta)} = \text{sign}\Big[\gamma_5 \cos\left((1-\eta)\left(\frac{\pi}{2} - \theta\right)\right) - \text{sign}(\gamma_5(D_1^\dagger - D_1)) \sin\left((1-\eta)\left(\frac{\pi}{2} - \theta\right)\right)\Big]$$

$$\tan\theta = 2\frac{\sqrt{1 - a^2 D_1^\dagger D_1/4}}{\sqrt{a^2 D_1^\dagger D_1}}$$

Properties of γ_L and γ_R

- γ_L and γ_R reduce to γ_5 in the continuum limit
- $\gamma_L D_1 + D_1 \gamma_R = 0$
- $(\gamma_R^{(\eta)})^2 = (\gamma_L^{(\eta)})^2 = 1$
- $\gamma_R^{(\eta)} = \gamma_5 \gamma_L^{(-\eta)} \gamma_5$
- $\gamma_R^{(-1)} = \gamma_L^{(1)} = \gamma_5,$
- $\gamma_R^{(1)} = \gamma_5 \gamma_L^{(-1)} \gamma_5 = \gamma_5(1 - D_1)$
- $\gamma_5 \gamma_R^{(-\eta-1)} \gamma_R^{(-\eta)} \gamma_R^{(-\eta-1)} \gamma_5 = \gamma_R^{(\eta)}$
- $\gamma_5 \gamma_R^{(-\eta-1)} D_1 = D_1 \gamma_5 \gamma_R^{(-\eta-1)}$
- For odd integer η, $\gamma_R^{(\eta)}$ and $\gamma_L^{(\eta)}$ are local

- $\mathcal{CP}$ symmetry in the continuum:

$$
\begin{aligned}
\mathcal{CP}: \psi(x) =& - W^{-1}\overline{\psi}^T(\overline{x})\\
\mathcal{CP}: \overline{\psi}(x) =& \psi(\overline{x})^T W\\
\mathcal{CP}: U_\mu(x) =& U_\mu(\overline{x})^{CP}\\
W\gamma_4 W^{-1} =& -\gamma_4^T\\
W\gamma_i W^{-1} =& \gamma_i^T\\
W\gamma_5 W^{-1} =& -\gamma_5^T\\
\mathcal{CP}: D_0[U](x,y) =& W^{-1} D_0[U^{CP}](\overline{x},\overline{y})^T W\\
\mathcal{CP}: \overline{\psi} D_0 \psi =& \overline{\psi} D_0 \psi\\
\mathcal{CP}: \overline{\psi} D_0 (1-\gamma_5)\psi =& \overline{\psi} D_0 (1-\gamma_5)\psi
\end{aligned}
$$

- Physical observables are also invariant under $\mathcal{CP}$.
- Consider the generating functional in the Glashow-Weinberg-Salam model for electro-weak interactions

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\overline{\psi}D_0(1+\gamma_5)\psi + \frac{1}{2}\overline{\psi}D_0(1-\gamma_5)\psi + \\ & \frac{1}{4}\overline{\psi}(1+\gamma_5)\phi(1+\gamma_5)\psi + \frac{1}{4}\overline{\psi}(1-\gamma_5)\phi^\dagger(1-\gamma_5)\psi + \\ & \phi^\dagger(D_0^2+\mu^2)\phi - \lambda(\phi^\dagger\phi)^2 + \\ & \overline{\chi}\psi + \overline{\psi}\chi\end{aligned}$$

- Propagators: $(\upsilon = \langle\phi\rangle)$, $\upsilon_{LR} = (1-\gamma_5)\upsilon(1-\gamma_5)$

$$G_{LL} = \frac{1}{4}(1-\gamma_5)\frac{1}{D_{LL} - \upsilon_{LR}D_{RR}^{-1}\upsilon_{RL}^{\dagger}}(1+\gamma_5)$$

$$G_{RR} = \frac{1}{4}(1+\gamma_5)\frac{1}{D_{RR} - \upsilon_{RL}^{\dagger}D_{LL}^{-1}\upsilon_{LR}}(1-\gamma_5)$$

$$G_{LR} = \frac{1}{4}(1-\gamma_5)\frac{1}{\upsilon_{RL}^{\dagger} - D_{RR}{\upsilon_{LR}}^{-1}D_{LL}}(1-\gamma_5)$$

$$G_{RL} = \frac{1}{4}(1+\gamma_5)\frac{1}{\upsilon_{LR} - D_{LL}({\upsilon_{RL}}^{\dagger})^{-1}D_{RR}}(1+\gamma_5)$$

- These are invariant under $\mathcal{CP}$.

- Previously, the continuum $\mathcal{C}$ and $\mathcal{P}$ transformations are applied directly to the lattice fields
- The first to notice that there was a problem of $\mathcal{CP}$ on the lattice were Peter Hasenfratz/Martin Lüscher (Hasenfratz, Berlin 2001)
- The chiral gauge Lagrangian is $\overline{\psi}D(1-\gamma_R^{(\eta)})\psi$
- Under $\mathcal{CP}$ this transforms to

$$-\psi^T D^T(1+(\gamma_L^{(-\eta)})^T)\overline{\psi}^T = \overline{\psi}(1+\gamma_L^{(-\eta)})D\psi = \overline{\psi}D(1-\gamma_R^{(-\eta)})\psi$$

- Two no-go theorems say that it is impossible to construct a $\mathcal{CP}$-invariant lattice chiral gauge theory, Fujikawa et al arXiv:hep-lat/0209007; Jahn, Pawlowski hep-lat/0205005.

The propagators on the lattice ($\upsilon = \langle\phi\rangle$), $\upsilon_{LR} = (1-\gamma_5)\upsilon(1-\hat{\gamma}_5)$, $\hat{\gamma}_5 = \gamma_5(1-D)$

$$G_{LL} = \frac{1}{4}(1-\hat{\gamma}_5)\frac{1}{D_{LL} - \upsilon_{LR} D^{-1}_{RR} \upsilon^{\dagger}_{RL}}(1+\gamma_5)$$

$$G_{RR} = \frac{1}{4}(1+\hat{\gamma}_5)\frac{1}{D_{RR} - \upsilon^{\dagger}_{RL} D^{-1}_{LL} \upsilon^{\dagger}_{LR}}(1-\gamma_5)$$

$$G_{LR} = \frac{1}{4}(1-\hat{\gamma}_5)\frac{1}{\upsilon^{\dagger}_{RL} - D_{RR} \upsilon_{LR}{}^{-1} D_{LL}}(1-\gamma_5)$$

$$G_{RL} = \frac{1}{4}(1+\hat{\gamma}_5)\frac{1}{\upsilon_{LR} - D_{LL} (\upsilon_{RL}{}^{\dagger})^{-1} D_{RR}}(1+\gamma_5)$$

The propagators are not invariant under $\mathcal{CP}$.

Fujikawa, Ishibashi and Suzuki, hep-lat/0203016

- Five recent papers discuss the lattice $\mathcal{CP}$ problem:
  - Gattringer, Pak arXiv:0802.2496 (A. Hasenfratz, von Allmen) – double the number of degrees of freedom – Can we easily relate this to the usual theory?
  - Igarashi, Pawlowski arXiv:0902.4783 – modify $\mathcal{CP}$ symmetries on the lattice – but why should we do this?
  - NC, arXiv:0903.5521 – use a different formulation of lattice chiral symmetry – but is this formulation local?
  - NC, arXiv:1003.3991 – the subject of this talk
  - Poppitz, Shang, arXiv:1003.5896 – using mirror fermions.

Igarashi, Pawlowski arXiv:0902.4783

- Re-write charge conjugation for Weyl fermions as

$$\mathcal{C} : \overline{\psi}(1 \pm \gamma_5) \rightarrow \psi^T C(1 \pm \hat{\gamma}_5^T)$$

$$\mathcal{C} : (1 \pm \hat{\gamma}_5)\psi \rightarrow -(1 \pm \gamma_5^T)C^{-1}\overline{\psi}^T$$

$$\hat{\gamma}_5 = \gamma_5(1 - D)$$

- This solves the problems of Weyl and Majoranna actions
- How does charge conjugation act on the fields ψ and $\overline{\psi}$?

Ginsparg Wilson approach to $\mathcal{CP}$ on the lattice

Under $\mathcal{CP}$,

$$\gamma_R^{(\eta)} \to -W\gamma_L^{(-\eta)}W^{-1} \qquad \gamma_L^{(\eta)} \to -W\gamma_R^{(-\eta)}W^{-1}$$

$$B^{(\eta)} \to W(\overline{B}^{(-\eta)}[U^{CP}])^T W^{-1} \qquad \overline{B}^{(\eta)} \to W B^{(-\eta)}[U^{CP}])^T W^{-1}$$

We know that

$$\psi_1 = (B^{(\eta)})^{-1}\psi_0; \qquad \overline{\psi}_1 = \overline{\psi}_0(\overline{B}^{(\eta)})^{-1}$$

The lattice fermion fields transform as

$$\mathcal{CP} : \overline{\psi}_1 = ((B^{(-\eta)})^{-1}B^{(\eta)}\psi_1)^T W = (\gamma_R^{(-\eta-1)}\gamma_5\psi_1)^T W$$

$$\mathcal{CP} : \psi_1 = -W^{-1}(\overline{\psi}_1\overline{B}^{(\eta)}(\overline{B}^{(-\eta)})^{-1})^T$$

$$= -W^{-1}(\overline{\psi}_1\gamma_5\gamma_R^{(-\eta-1)})^T$$

- $\mathcal{CP} : \overline{\psi}_1 = (\gamma_R^{(-\eta-1)}\gamma_5\psi_1)^T W$
- $\mathcal{CP} : \psi_1 = -W^{-1}(\overline{\psi}_1\gamma_5\gamma_R^{(-\eta-1)})^T$
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- $\mathcal{CP} : \gamma_R^{(\eta)} = -W(\gamma_L^{(-\eta)})^T W^{-1}$
- $\mathcal{CP} : \gamma_L^{(\eta)} = -W(\gamma_R^{(-\eta)})^T W^{-1}$
- $\mathcal{CP} : D = W D^T W^{-1}$
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- $\mathcal{CP} : \overline{\psi}_1 = (\gamma_R^{(-\eta-1)}\gamma_5\psi_1)^T W$
- $\mathcal{CP} : \psi_1 = -W^{-1}(\overline{\psi}_1\gamma_5\gamma_R^{(-\eta-1)})^T$
- $\mathcal{CP} : \gamma_R^{(\eta)} = -W(\gamma_L^{(-\eta)})^T W^{-1}$
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- $\mathcal{CP} : D = W D^T W^{-1}$
- $\gamma_5\gamma_R^{(-\eta-1)} D \gamma_R^{(-\eta-1)}\gamma_5 = D$
- $\gamma_5\gamma_R^{(-\eta-1)}\gamma_R^{(-\eta)}\gamma_R^{(-\eta-1)}\gamma_5 = \gamma_R^{(\eta)}$
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- $\mathcal{CP} : \overline{\psi}_1 = (\gamma_R^{(-\eta-1)}\gamma_5\psi_1)^T W$
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- $\mathcal{CP}: \overline{\psi}_1 = (\gamma_R^{(-\eta-1)}\gamma_5\psi_1)^T W$
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- $\mathcal{CP}: \gamma_R^{(\eta)} = -W(\gamma_L^{(-\eta)})^T W^{-1}$
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- $\mathcal{CP}: D = WD^T W^{-1}$
- $\gamma_5\gamma_R^{(-\eta-1)} D \gamma_R^{(-\eta-1)}\gamma_5 = D$
- $\gamma_5\gamma_R^{(-\eta-1)}\gamma_R^{(-\eta)}\gamma_R^{(-\eta-1)}\gamma_5 = \gamma_R^{(\eta)}$
- $\mathcal{CP}: \overline{\psi} D\psi = \overline{\psi} D\psi$
- $\mathcal{CP}: \overline{\psi} D(1-\gamma_R^{(\eta)})\psi = \overline{\psi}\gamma_R^{(-\eta-1)}\gamma_5(1+\gamma_L^{(-\eta)})D\gamma_5\gamma_R^{(-\eta-1)}\psi$

$$= \overline{\psi} D(1-\gamma_R^{(-\eta-1)}\gamma_5\gamma_R^{(-\eta)}\gamma_5\gamma_R^{(-\eta-1)})\psi$$

$$= \overline{\psi} D(1-\gamma_R^{(\eta)})\psi$$

- $\mathcal{CP}: \overline{\psi}_1 = (\gamma_R^{(-\eta-1)}\gamma_5\psi_1)^T W$
- $\mathcal{CP}: \psi_1 = -W^{-1}(\overline{\psi}_1\gamma_5\gamma_R^{(-\eta-1)})^T$
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- $\gamma_5\gamma_R^{(-\eta-1)} D \gamma_R^{(-\eta-1)}\gamma_5 = D$
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- $\mathcal{CP}: \overline{\psi}D\psi = \overline{\psi}D\psi$
- $$\begin{aligned}\mathcal{CP}: \overline{\psi}D(1-\gamma_R^{(\eta)})\psi &= \overline{\psi}\gamma_R^{(-\eta-1)}\gamma_5(1+\gamma_L^{(-\eta)})D\gamma_5\gamma_R^{(-\eta-1)}\psi \\ &= \overline{\psi}D(1-\gamma_R^{(-\eta-1)}\gamma_5\gamma_R^{(-\eta)}\gamma_5\gamma_R^{(-\eta-1)})\psi \\ &= \overline{\psi}D(1-\gamma_R^{(\eta)})\psi\end{aligned}$$

For $\eta = \pm 1$, this gives the lattice $\mathcal{CP}$ transformations suggested by Igarashi, Pawlowski.

- The standard and chiral gauge actions are invariant under lattice $\mathcal{CP}$
- Lattice $\mathcal{CP}$ invariant Weyl and Majoranna actions can be defined with a lattice $\mathcal{CP}$ and gauge invariant measure
- The propagators in the chiral gauge theory are invariant under lattice $\mathcal{CP}$
- The Higgs field can be treated in the same way
- Instead of naively applying the continuum $\mathcal{CP}$ symmetry on the lattice, it is more natural to use a modified lattice $\mathcal{CP}$ symmetry derived from the Ginsparg-Wilson equation.
- *Just as the overlap action, $\overline{\psi}D\psi$, obeys multiple chiral symmetries, it also obeys multiple $\mathcal{CP}$ symmetries, including the continuum symmetry. Does this ambiguity create a problem?*