Wilson Fermions, Random Matrix Theory and the Aoki Phase

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Acknowledgments

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P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Microscopic Spectrum of the Wilson Dirac Operator, arXiv:1001.2937 [hep-th].

G. Akemann, P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Spectrum of the Wilson Dirac Operator at Finite Lattice Spacing, to be published.

Contents

I. Motivation

Wilson Dirac Operator

Lattice Results

Wilson Dirac operator

Wilson introduced the Wilson term to eliminate doublers.

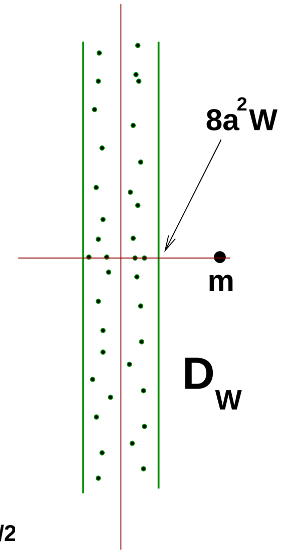
$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{1}{2}a\nabla^*_\mu\nabla^*_\mu.$$

$$\{D_W, \gamma_5\} \neq 0.$$

Block structure

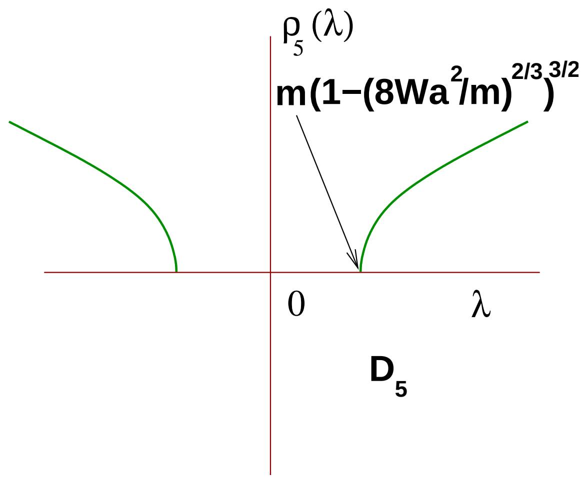
$$D_W = \begin{pmatrix} aA & id \\ id^\dagger & aB \end{pmatrix}$$

with $A^\dagger = A,\ B^\dagger = B.$

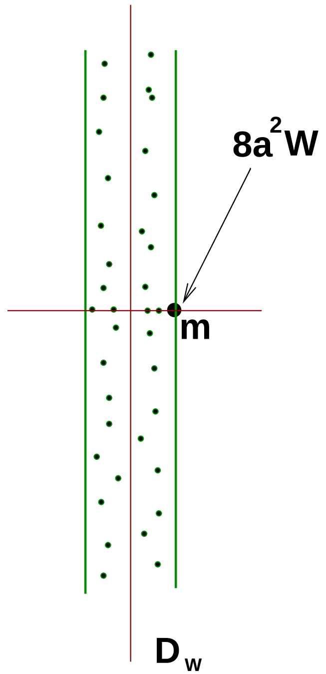


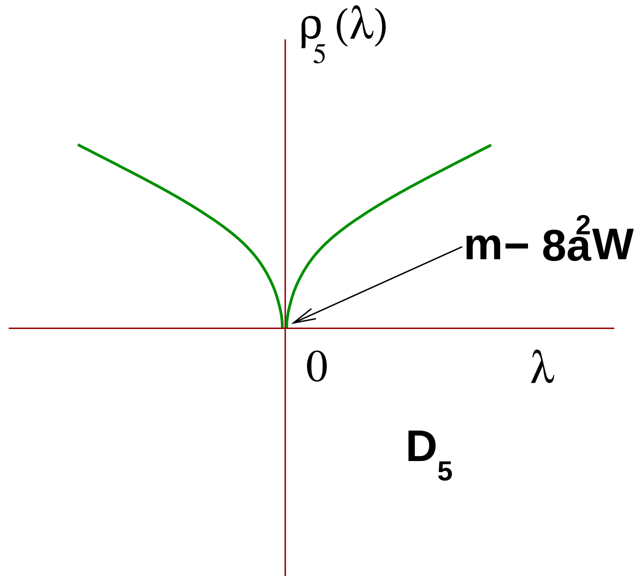
$$D_W = \gamma_5 D_W^\dagger \gamma_5.$$

$$D_5 \equiv \gamma_5(D_W + m) = D_5^\dagger$$

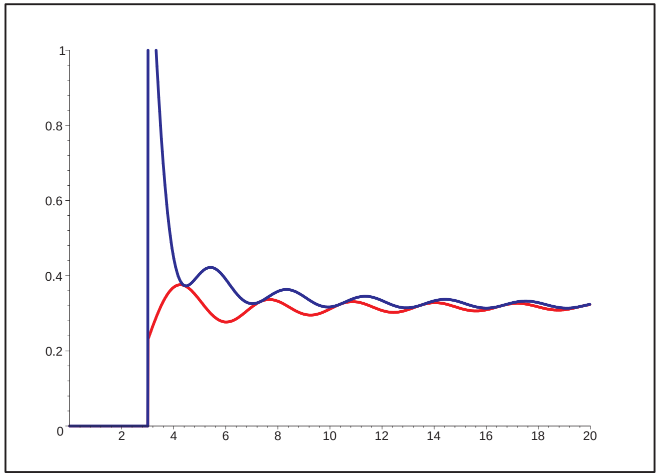


Onset of the Aoki Phase





Spectrum of D_5 for $a = 0$



The microscopic spectral density of $\gamma_5(D_W + m)$ for $m = 3$, $\nu = 0$ and $a = 0$ for different number of flavors. Results are shown for $N_f = 0$ (blue) and $N_f = 2$ (red).

See talk of Poul Damgaard for effect of dynamical quarks at $a \neq 0$.

Motivation

Lattice studies of the distribution of the smallest eigenvalue of the Wilson Dirac operator. Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005

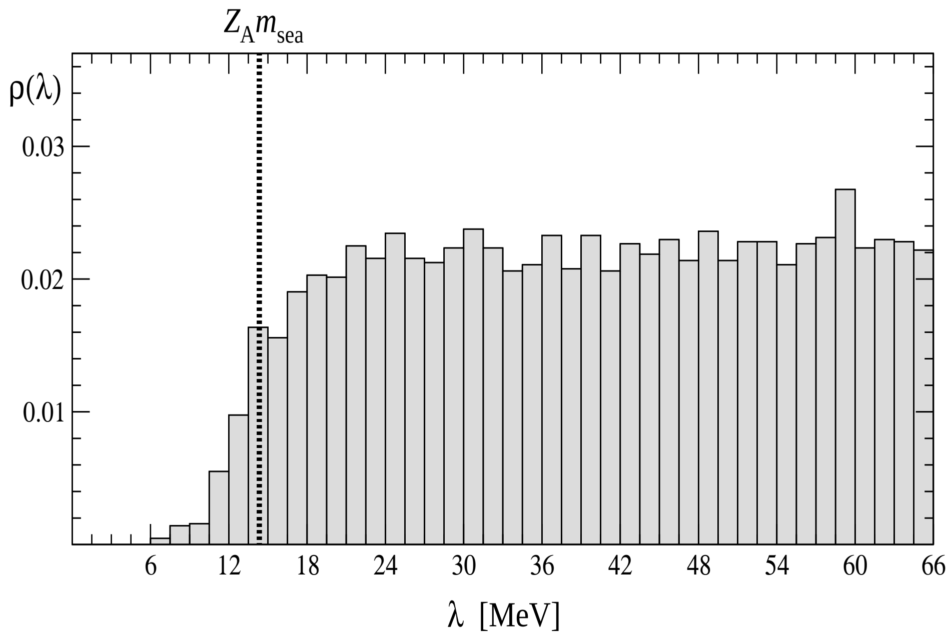
Mean fields studies of the spectrum of the Wilson Dirac operator based on chiral perturbation theory. Sharpe-2006

Lifshitz tail states in superconducting quantum dots with magnetic impurities. Lamacraft-Simons-1996

Discussions in the literature on the existence of the Aoki Phase.
Sharpe-Singleton-1998, Shindler-2009, Azcoiti-Di Carlo-Follana-Vaquero-2009

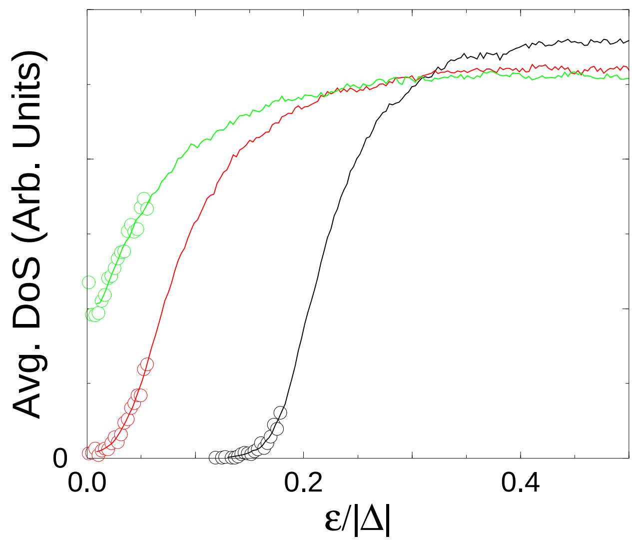
The existence of a gap in the spectrum of D_5 is important to evaluate its inverse efficiently.

Lattice Results for the Wilson Dirac Spectrum



Spectral density of $\gamma_5(D_W + m)$ on a 48×24^3 lattice.

Lüscher-2007

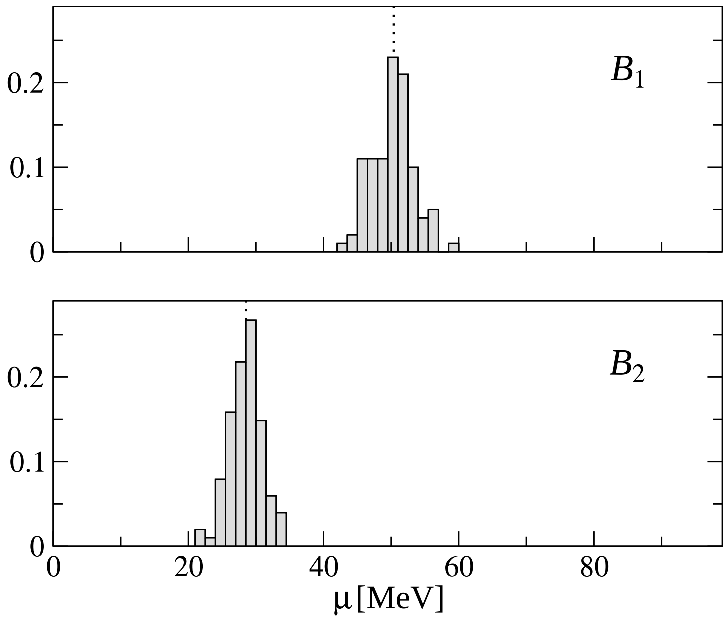


Spectral density of a Bogulubov Hamiltonian in the presence of magnetic impurities.

Lamacraft-Simons-2001

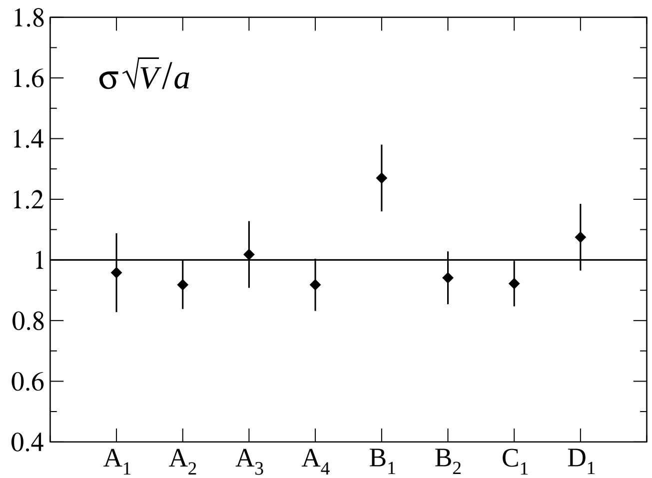
- ✓ Dirac spectrum has a gap.
- ✓ A Gaussian tail intrudes inside the gap.

Distribution of the Smallest Eigenvalue



Distribution of the smallest eigenvalue of the Hermitian Wilson Dirac operator on a 64×32^3 lattice for two different values of the quark mass.

Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005



Scaling of the width of the distribution.

Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005

Questions

- √ Can we obtain analytical results for the Dirac spectrum in the microscopic domain?
  - ⋆ Effect of topology?
  - ⋆ Effect of the fermion determinant?
  - ⋆ Distribution of the smallest eigenvalue?
- √ How does the probability to find more than ν real eigenvalues scale with the volume?
- √ Is there a Random Matrix Theory that describes the discretization errors of the Wilson Dirac operator?

III. Chiral Lagrangian for the Dirac Spectrum

Chiral Lagrangian.

γ_5 -Hermiticity and the Sign of W_8 .

Chiral Lagrangian for the Generating Function of the Wilson Dirac Spectrum.

Microscopic Spectral Density

Remarks

In order to access the spectrum of $\gamma_5(D_W + m)$ we introduce a twisted mass in the fermion determinant

$$\det[D_W + m + \gamma_5 z] = \det[\gamma_5(D_W + m) + z].$$

The low energy limit of the corresponding partition function is given by a chiral Lagrangian that up to low energy constants is uniquely determined by symmetries.

In the microscopic domain, where the combinations

$$mV, \qquad zV, \qquad a^2 V$$

are kept fixed in the thermodynamic limit, the m , z and a dependence of the chiral Lagrangian resides in the zero momentum part of the partition function that factorizes form the nonzero momentum part.

Chiral Lagrangian

In the microscopic domain the QCD partition function in the sector of topological charge ν is given by

$$Z_\nu^\chi(m,z,a) = \int_{U\in U(N_f)} dU \det{}^\nu U e^{\frac{1}{2}mV\Sigma \mathrm{Tr}(U+U^\dagger)+\frac{1}{2}zV\Sigma\mathrm{Tr}(U-U^\dagger)-a^2VW_8\mathrm{Tr}(U^2+U^{-2})}.$$

Sharpe-Singleton-1998, Rupak-Shoresh-2002, Bär-Rupak-Shoresh-2004, Damgaard-Splittorff-JV-2010

- $W_8 > 0$

  The mean field result of the spectrum of D_5 has a gap $[-z_c, z_c]$ given by $z_c = m[1-(8a^2W_8/m\Sigma)^{2/3}]^{3/2}$.

- $W_8 < 0$

  Changing the sign of W_8 corresponds to $a \to ia$. Then the continuum Wilson Dirac operator is anti-Hermitian with eigenvalues on the imaginary axis, but the spectrum of $\gamma_5(D_W+m)$ becomes complex.

  Then. this chiral Lagrangian cannot be used to calculate the spectrum of D_5 .

γ_5-Hermiticity and Sign of W_8

Because of $D_5^\dagger = D_5$ we have the QCD inequality

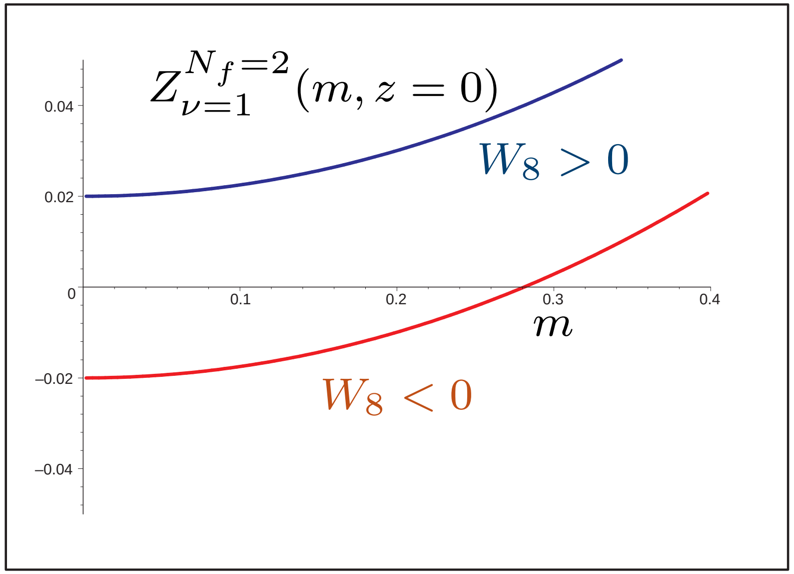
$$Z_\nu^{\mathrm{QCD},N_f=2}(m,z) = \langle \det{}^2(\gamma_5(D_W+m)+z)\rangle > 0 \qquad \text{for} \quad m,z \quad \text{real.}$$

By changing variables $U \to iU$ it follows that

$$Z_\nu^{\chi\, N_f}(0,0,W_8) = (i)^{N_f \nu} Z_\nu^{\chi\, N_f}(0,0,-W_8).$$

Therefore, the partition function for $W_8 < 0$ changes sign as a function of m .

Akemann-Damgaard-Splittorff-JV-2010



Generating Function for Wilson Dirac spectrum

The generating function for the Wilson Dirac spectrum is given by

$$Z(m, z, z', a) = \left\langle \det(D_W + m) \frac{\det(D_W + m + \gamma_5 z)}{\det(D_W + m + \gamma_5 z')} \right\rangle .$$

Damgaard-Splittorff-JV-2010

The resolvent is equal to

$$G(z, m, a) = \lim_{z' \to z} \frac{d}{dz} Z(m, z, z', a) \Big|_{z'=z} ,$$

and the spectral density is given by

$$\rho(z) = \frac{1}{\pi} \mathrm{Im}\, G(z), \qquad \text{for} \quad z \quad \text{real}.$$

For $z \ll \Lambda_{\rm QCD}$ the z -dependence of the generating function is given by a chiral Lagrangian that is uniquely determined by symmetries that should be compatible with the convergence of the bosonic integrals. In the microscopic domain the partition function reduces to a super-unitary matrix integral.

Generating Function for the Wilson Dirac Spectrum

The generating function for the Wilson Dirac spectrum is given by

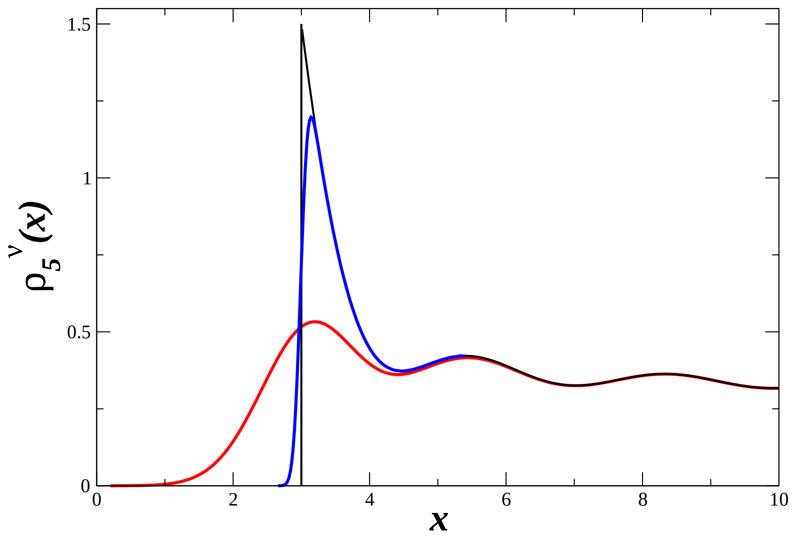
$$Z(m,z,a) = \int_{U\in Gl(N_f+1|1)} dU e^{i\frac{1}{2}mV\Sigma \mathrm{Tr}(U-U^\dagger)+i\frac{1}{2}V\Sigma\mathrm{Tr}(\zeta U+U^\dagger\zeta)-i^2a^2VW_8\mathrm{Tr}(U^2+U^{-2})}.$$

Here, $\zeta_3 = \mathrm{diag}(0,0,z,z')$.

Damgaard-Splittorff-JV-2010,Damgaard-Osborn-Toublan-JV-1998

- ✓ The transformation $U \to iU$ is required to get convergent integrals in the noncompact sector for $W_8 > 0$.
- ✓ For $N_f = 0$ the integral reduces to one dimensional integrals. See talk of Poul Damgaard for $N_f = 1$.

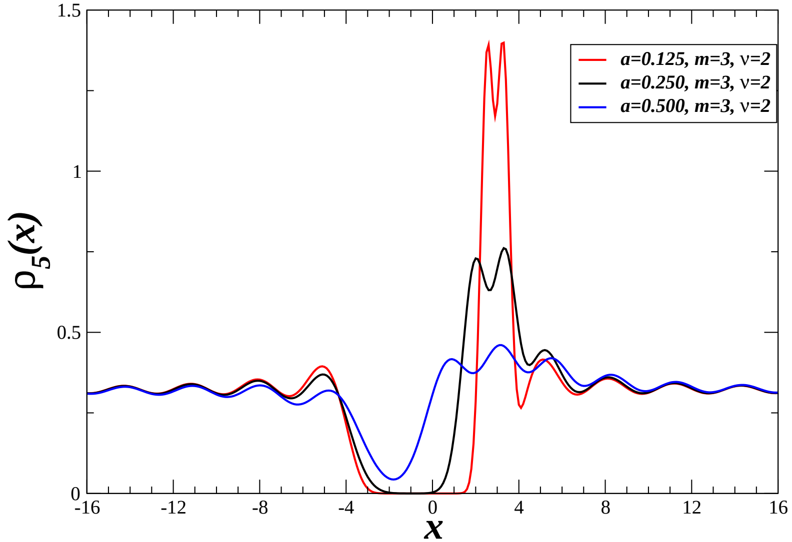
Plots of the Spectral Density



The microscopic spectrum of $\gamma_5(D_W + m)$ for $mV\Sigma = 3$, $\nu = 0$ and $a\sqrt{W_8 V} = 0$, 0.03 , and 0.250 . The $\nu = 0$ spectrum is reflection symmetric about $x = 0$.

Damgaard-Splittorff-JV-2010

The peak at $x = m$ in the left figure is due to the measure. The peak height for $a = 0$ is equal to $mV\Sigma$.



The microscopic spectrum of $\gamma_5(D_W + m)$ for $mV\Sigma = 3$, $\nu = 2$ and $a\sqrt{W_8 V} = 0.125,\ 0.250$ and 0.500 , respectively.

Damgaard-Splittorff-JV-2010

Distribution of "Topological" Eigenvalues

For $a = 0$ the eigenvalue density of D_5 can be decomposed as

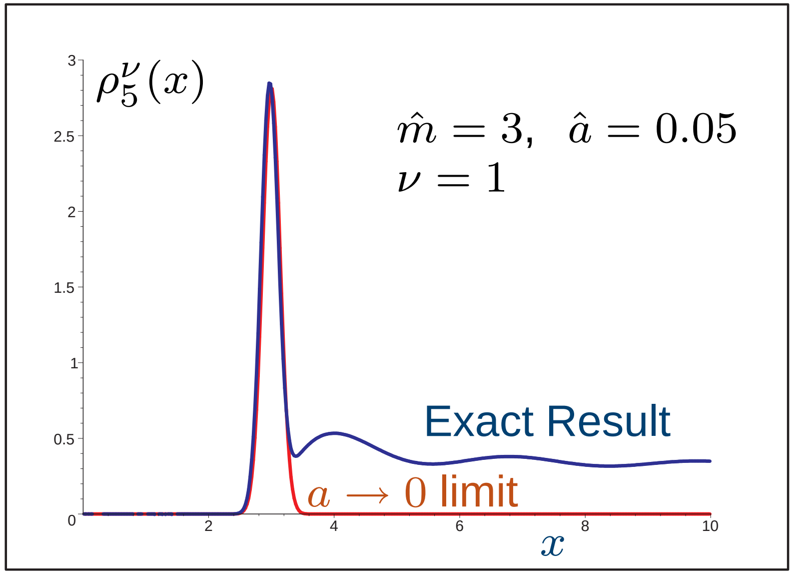
$$\rho_5^\nu(\lambda) = \nu\delta(\lambda - m) + \rho_{\lambda>m}(\lambda).$$

For $a \neq 0$ the width of the peak at $\lambda = m$ becomes finite.

- For small a, this distribution is exactly the spectral density of the real eigenvalues of D_W.
- For $\nu = 1$ the result is given by (see red curve in figure)

$$\rho_{5,\mathrm{topo}}^{\nu=1}(x) = \frac{1}{4a\sqrt{\pi V W_8}} e^{-\frac{V\Sigma^2(x-m)^2}{16a^2W_8}}.$$

- The low-energy constant W_8 can be obtained from the width of the distribution of the smallest eigenvalue.



Tail States

For $|z-m|/a$ fixed for $a \to 0$ the spectral density inside the gap can also be obtained from a saddle point analysis:

$$\rho(z) \sim e^{-\Sigma^2 V(z-m)^2/16a^2 W_8} \qquad \text{for} \quad 0 < z \ll m.$$

The width parameter is given by Damgaard-Splittorff-JV-2010

$$\sigma^2 = \frac{8a^2 W_8}{V\Sigma^2}, \qquad \frac{\sigma}{\Delta\lambda} = \frac{\sqrt{8}}{\pi} a\sqrt{W_8 V}.$$

This is exactly the scaling behavior found by Del Debbio-et al-2006.

- √ Typical lattice parameters are $mV\Sigma = 6$ and $a\sqrt{W_8 V} = 0.2 - 0.5$.
- √ For $mV\Sigma \gg 1$ and $a^2 W_8 V \gg 1$ the distribution of the smallest eigenvalue is given by the Tracy-Widom distribution.

V. Random Matrix Theory

Random Matrix Theory for the Wilson Dirac Operator

Random Matrix Theory for the Wilson Dirac Operator

Since the chiral Lagrangian is determined uniquely by symmetries, it also can be obtained from a random matrix theory with the same symmetries. In the sector of topological charge ν the random matrix partition function is given by

$$Z^{\nu}_{N_f} = \int dAdBdW \, \det{}^{N_f}(D_W + m + z\gamma_5) \, P(D_W),$$

with

$$D_W = \begin{pmatrix} aA & C \\ -C^\dagger & aB \end{pmatrix}. \qquad \text{and} \quad A^\dagger = A, \qquad B^\dagger = B.$$

A is a square matrix of size $n \times n$, and B is a square matrix of size $(n+\nu)\times(n+\nu)$. The matrix C is a complex $n \times (n+\nu)$ matrix. Damgaard-Splittorff-JV-2010

In the microscopic domain, the Random Matrix Theory partition function reduces to the chiral Lagrangian introduced before with $W_8 > 0$.

This was not the case for earlier attempts to formulate random matrix theories for the Wilson Dirac operator. Jurkiewicz-Nowak-Zahed-1996,Hehl-Schäfer-1999

VIII. Conclusions

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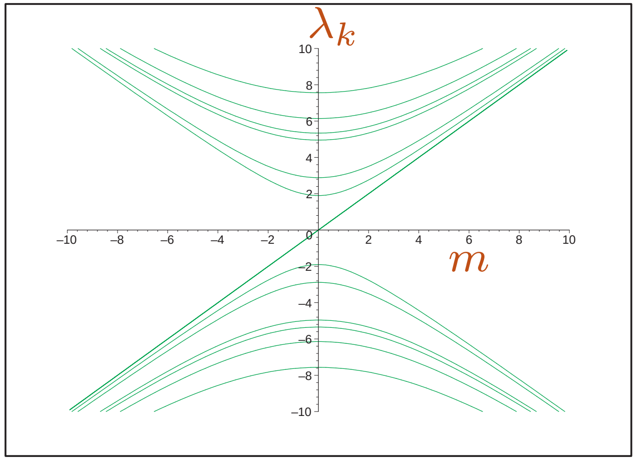
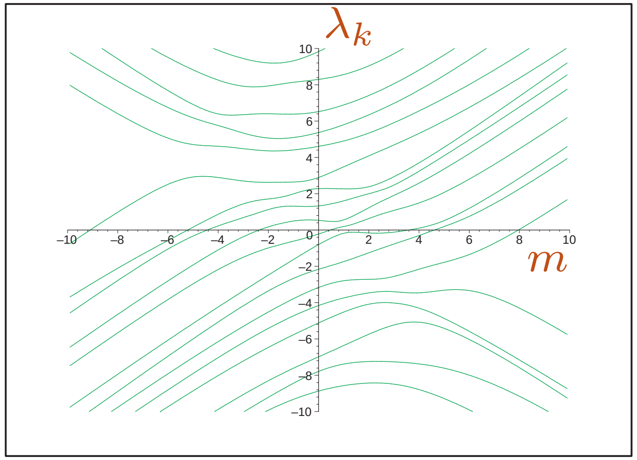
- The width of the distribution of the smallest eigenvalue scales as $a/\sqrt{V}$ suggested by lattice simulations.
- For small a the density of states inside the gap has a Gaussian tail.
- Random Matrix Theory has been so successful because it is based based on symmetries, universality and the separation of scales. Ideas of Ken Wilson have had a strong impact on its development. In fact, Wilson's first paper was in fact on Random Matrix Theory (J. Math. Phys. 1962).

Spectral Flow at Nonzero Topology





The topological charge is given by the difference of the number of positive eigenvalues for large positive charge and the number of positive eigenvalues for large negative mass.

If ϕ is an eigenfunction of a topological zero mode,then

$$a = 0: \qquad D_W\phi = 0, \qquad \gamma_5\phi = \phi, \Longrightarrow \gamma_5(D_W + m)\phi = m\phi.$$

$$a \neq 0: \qquad \gamma_5(D_W + m_k)\phi_k = 0 \Longrightarrow D_W\phi_k = -m_k\phi_k.$$

For $a \neq 0$ the flow-line may cross the x -axis more than once so that the number is real modes is larger than ν . This generically does not happen in the ϵ domain.

Dirac Spectrum for $a = 0$

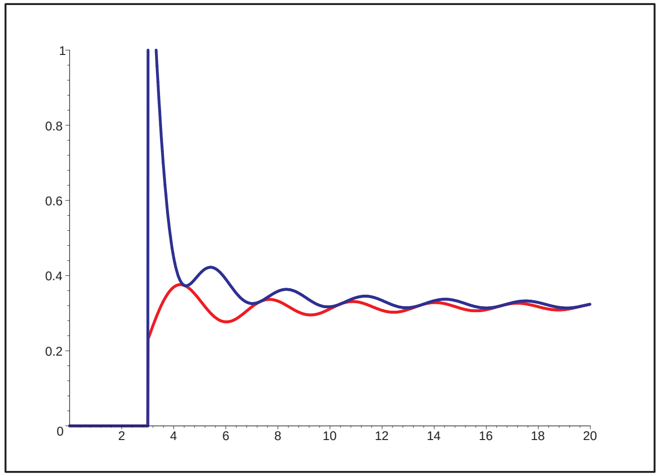
For $a = 0$ we have that

$$\gamma_5(D_W + m) = \begin{pmatrix} m & C \\ C^\dagger & -m \end{pmatrix}.$$

C can be brought to a diagonal form by a unitary transformation. This results in the spectrum of D_5

$$(\lambda_k - m)^2 - |c_k|^2 = 0 \quad \Longrightarrow \quad \lambda_k = \pm\sqrt{m^2 + |c_k|^2}$$

That is why $\gamma_5(D_W + m)$ has a gap $[-m, m]$ for $a = 0$. For $a \neq 0$ states intrude inside the gap.



The microscopic spectral density of $\gamma_5(D_W + m)$ for $m = 3$, $\nu = 0$ and $a = 0$ for different number of flavors. Results are shown for $N_f = 0$ (blue) and $N_f = 2$ (red).

Mean Field Result for the Gap

$W_8 > 0$

The spectral density of $\gamma_5(D_W + m)$ vanishes if the correspond resolvent is real. For large $mV\Sigma$ and a^2VW_8 the resolvent can be calculated by a saddle point approximation. The result for the spectral gap $[-z_c, z_c]$ is given by (in units where $\Sigma = 1$ and $W_8 = 1$)

$$z_c = m[1 - (8a^2/m)^{2/3}]^{3/2}.$$

The gap closes when $8a^2 = m$. This is the onset of the Aoki phase.

$W_8 < 0$

The spectrum of $D_5 = \gamma_5(D_W + m)$ is complex and this chiral Lagrangian cannot be used to calculate the spectrum of D_5 . To calculate the spectrum one has to introduce a partition function with quarks and conjugate anti-quarks as is the case for QCD at nonzero chemical potential (see Stephanov-1996).

Microscopic Spectral Spectral Density

The resolvent is given by

$$G^\nu(z,m;a) = \int_{-\infty}^{\infty} ds \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \frac{i}{2}\cos(\theta) e^{S_f+S_b} e^{(i\theta - s)\nu}$$
$$\times\, (-m\sin(\theta) + im\sinh(s) + iz\cos(\theta) + iz\cosh(s)$$
$$+4a^2[\cos(2\theta) + \cosh(2s) + (e^{i\theta+s} + e^{-i\theta-s})] + 1\Big)\,.$$

Here,

$$S_f = -m\sin(\theta) + iz\cos(\theta) + 2a^2\cos(2\theta),$$
$$S_b = -im\sinh(s) - iz\cosh(s) - 2a^2\cosh(2s).$$

The microscopic quenched spectral density is equal to

$$\rho_5^\nu(x,m;a) = \frac{1}{\pi}\mathrm{Im}[G^\nu(x,m;a))].$$

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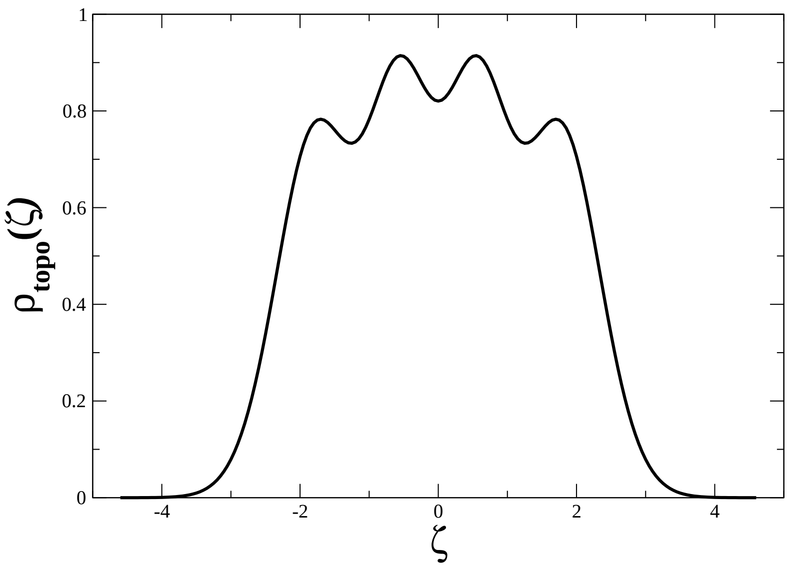
Density of Real Eigenvalues



The quenched spectral density of the topological real eigenvalues of D_W for $\nu = 4$ and $a = 0.25$.

Damgaard-Splittorff-JV-2010

The real eigenvalues of D_W give rise to a cut on top of the cloud of complex eigenvalues. This implies that the spectral density of the real modes is given by the discontinuity of the chiral condensate. Because $\Sigma(m)$ is real for real m the discontinuity is given by the imaginary part.

$$\rho^{\text{topological}}(m) = \frac{1}{\pi}\text{Im}\,\Sigma(m,a).$$

For large ν the distribution of real eigenvalues approaches a semicircle. This may be the first time that a semi-circular distribution of eigenvalues has been in a physical system. Note that since this result is derived from a chiral Lagrangian, it is universal.