# Hunting for the self-energy renormalon with NSPT

Clemens Bauer and Gunnar Bali, University of Regensburg
Antonio Pineda, UAB Barcelona

LATTICE 2010

## Stochastic Quantization

alternative way of calculating expectation values in Euclidian FT
Parisi, Wu (1981)

1. additional, fictitious time coordinate $\tau$:

$$\phi(x) \rightarrow \phi(x,\tau)$$

2. Langevin equation

$$\frac{\partial}{\partial\tau}\phi(x,\tau) = -\frac{\delta S}{\delta\phi(x,\tau)} + \eta(x,\tau)$$

3. expectation values via

$$\overline{\mathcal{O}[\phi]} = \lim_{T\rightarrow\infty}\frac{1}{T}\int_0^T d\tau \int \mathcal{D}[\eta]\, P[\eta]\, \mathcal{O}[\phi]$$

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## Numerical Stochastic Perturbation Theory (NSPT)

Stochastic Quantization implemented numerically for Lattice Gauge Theory

1. Langevin equation for Gauge fields:

$$\frac{\partial}{\partial\tau}U_\mu(x,\tau) = -i\left(\nabla_{x,\mu}S_{G,L}[U] + t^a\eta^a_\mu(x,\tau)\right)U_\mu(x,\tau)$$

2. Gauge fields as perturbative expansion

$$U = \mathbf{1}+\beta^{-\frac{1}{2}}U^{(1)}+\beta^{-1}U^{(2)}+\cdots+\beta^{-\frac{M}{2}}U^{(M)};\quad \beta^{-\frac{1}{2}} = \frac{g_0}{\sqrt{2N_c}}$$

$\rightarrow$ calculation cost $\propto M^2$
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3. Technicality 1: Stochastic Gauge Fixing
4. Technicality 2: zero mode treatment

perturbation theory: generic observable $R$ as series

$$R = \sum_n c_n \alpha^n$$

coefficients $c_n$ grow fast:

$$c_n \overset{n\to\infty}{\sim} K a^n n!\, n^b$$

$\longrightarrow$ perturbative expansion does not converge,
at best it is an asymptotic series

## Renormalons

- certain pattern of factorial growth of $c_n$
- e.g. arise when inserting "bubble" chains in Feynman diagrams
- small and large momentum behaviour origin:
  UV and IR renormalons

tool: summation via Borel transform and Borel integral

$$R \sim \sum_{n=0}^{\infty} r_n \alpha^{n+1} \implies B[R](t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}$$

$$\tilde{R} = \int_0^{\infty} dt\, e^{-t/\alpha}\, B[R](t)$$

→ behaviour of perturbative expansion R dictated by:
the closest singularity

$$u = -t\beta_0$$

to the origin of its Borel transform
→ leading renormalon

First observable: The plaquette

$$\langle P\rangle \equiv \langle 1-\frac{1}{3}\,\mathrm{tr}\,U_P\rangle = \sum_n c_n\alpha^n + \frac{\pi^2}{36}\,C_{GG}(\alpha)\,a^4\,\langle\frac{\alpha}{\pi}GG\rangle_{\mathrm{latt}} + O(a^6)$$

- contains gluon condensate $\langle G_{\mu\nu}G^{\mu\nu}\rangle$
- leading IR renormalon at $u = 2$

$\longrightarrow$ calculate plaquette to high orders with NSPT

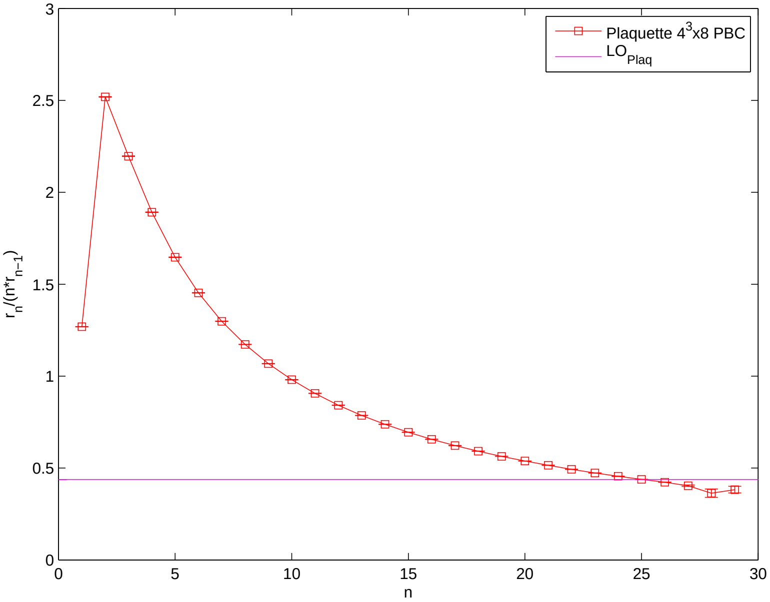
$\longrightarrow$ early works with NSPT, e.g.

- Di Renzo et al. hep-th/9502095, 8 loops,
- Di Renzo et al. hep-lat/0011067, 10 loops,
- Rakow hep-lat/0510046, 16 loops

at leading order:

$$\text{notation}: \qquad \sum_{n=0}^{\infty} r_n \alpha^{n+1},$$

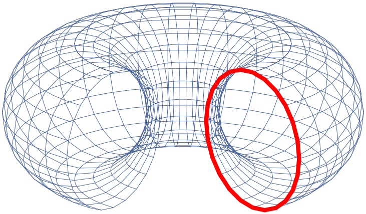
$$\mathrm{LO}_{\mathrm{Plaq}} = \lim_{n\to\infty} \frac{r_n}{r_{n-1}} = \frac{11}{8\pi} n$$


Plaquette 4^3x8 PBC
LO_Plaq
r_n/(n*r_n−1)
n
0
0.5
1
1.5
2
2.5
3
0
5
10
15
20
25
30

second observable:

Polyakov Loop

$$\mathcal{P}(\vec{x}) = \frac{1}{3} \mathrm{Tr} \prod_{\tau=0}^{L_4-1} U_4(\vec{x}, \tau)$$



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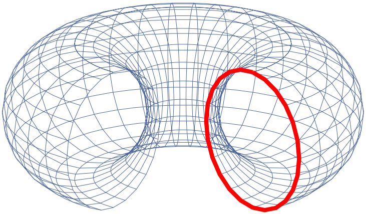


gives access to static quark self energy $V_{\mathrm{self}}$:

$$V_{\mathrm{self}} = \lim_{L_4\to\infty}\left(-\frac{1}{L_4}\ln\langle\mathcal{P}\rangle\right),$$

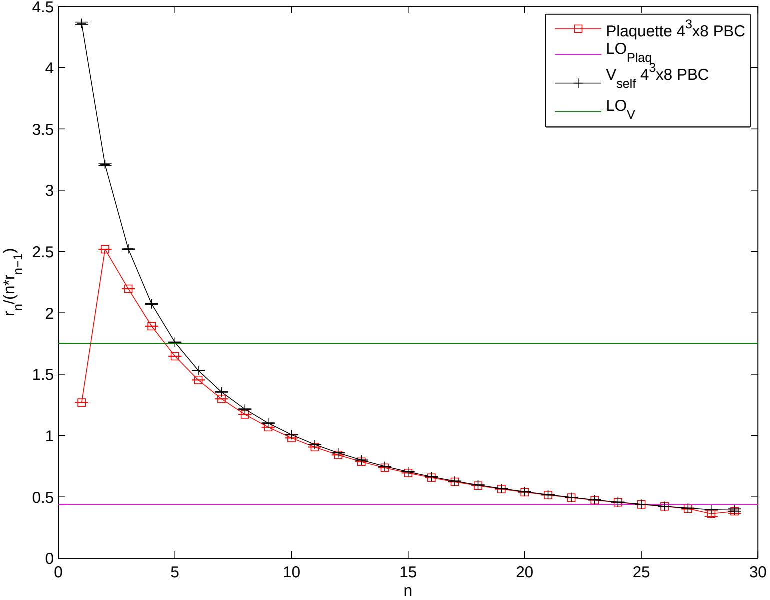
$$V_{\mathrm{self}} = \sum V_{\mathrm{self}}^{(n)}\alpha^n$$

- $V_{\rm self}$ of an infinitely heavy quark is linearly UV divergent
- leading UV Renormalon appears already at $u = 1/2$

→ renormalon should emerge four times faster as for plaquette
→ at leading order

$$\text{notation}: \qquad \sum_{n=0}^{\infty} r_n \alpha^{n+1},$$

$$\mathrm{LO_V} = \lim_{n\to\infty} \frac{r_n}{r_{n-1}} = 4\ \mathrm{LO_{Plaq}}$$


Plaquette $4^3$x8 PBC
$LO_{Plaq}$
$V_{self}$ $4^3$x8 PBC
$LO_V$
$r_n/(n*r_{n-1})$
n
0
0.5
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Periodic boundary conditions (PBC): zero momentum modes appear
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## Twisted Boundary Conditions

$$U_\mu(x + L\hat{\nu}) = \Omega_\nu U_\mu(x) \Omega_\nu^\dagger$$

- at least two directions must be twisted
- constant twist matrices $\Omega_\nu$ yield

$$\Omega_\mu \Omega_\nu = \eta \Omega_\nu \Omega_\mu, \quad \eta \in Z(N)$$

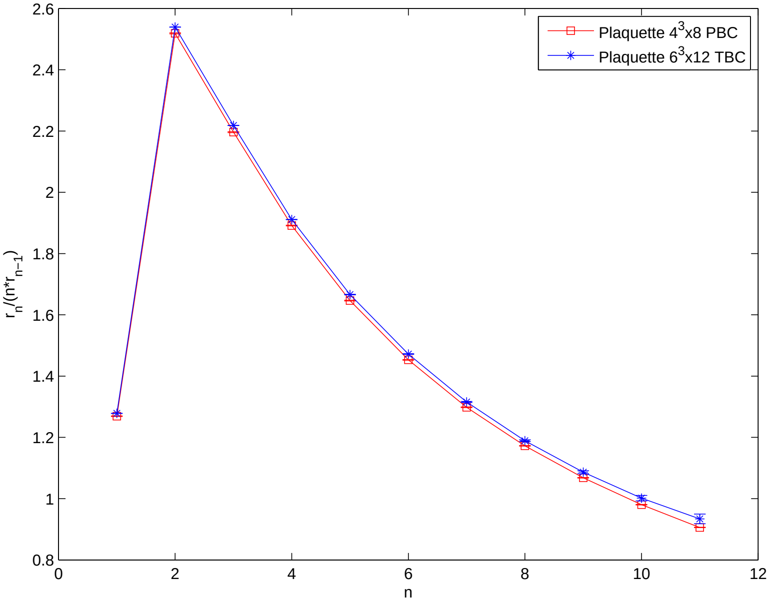
  for instance $\eta = e^{2\pi i/3}$ for $SU(3)$
- 2 options for implementation: either explicit choice of $\Omega_\nu$, or phase factors for certain plaquettes at corners of twisted planes

Effects of Twisted Boundary Conditions:

1. elimination of zero modes → no subtraction needed
2. momenta $k_\nu$ in twisted directions quantized as if the $SU(N)$ gauge fields lived on a lattice of size $L \times N$ instead of $L$:

$$k_\nu = \begin{cases} \dfrac{2\pi}{LN} n_\nu, & \nu = \text{twisted direction,} \\[2ex] \dfrac{2\pi}{L} n_\nu, & \nu = \text{periodic direction.} \end{cases}$$

→ reduction of finite size effects


Plaquette 4^3x8 PBC
Plaquette 6^3x12 TBC
r_n/(n*r_{n−1})
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2.6
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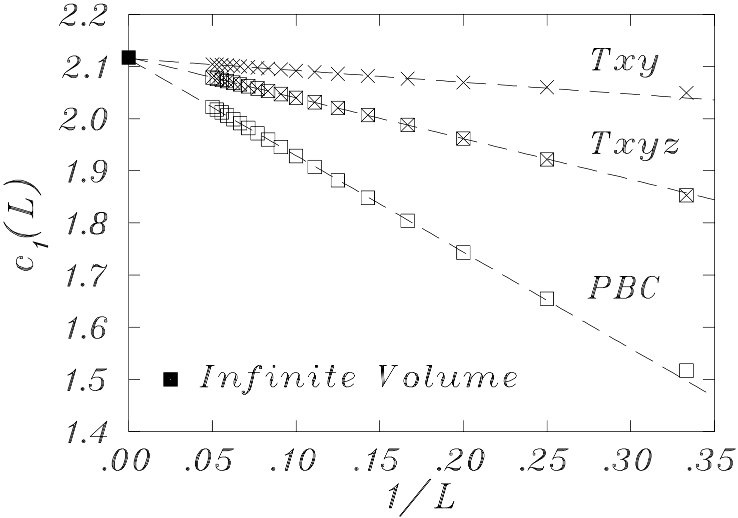
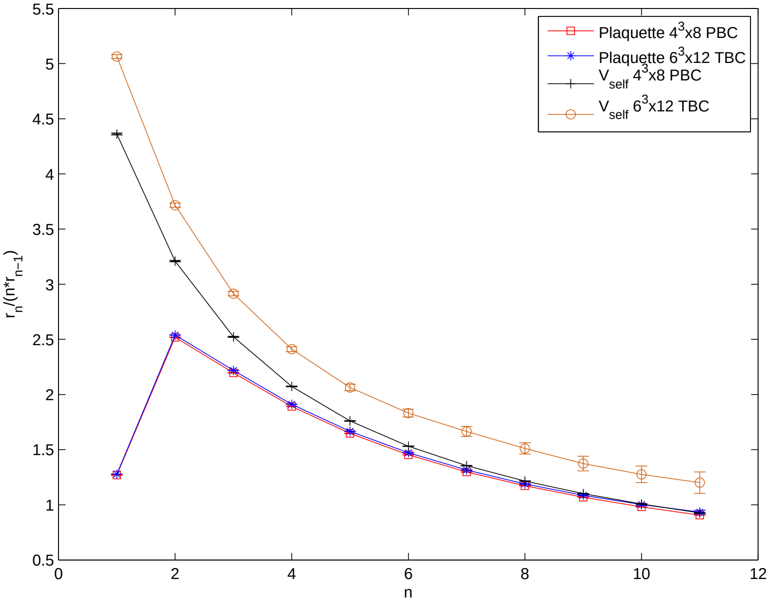
from Trottier et al. hep-lat/0110051:

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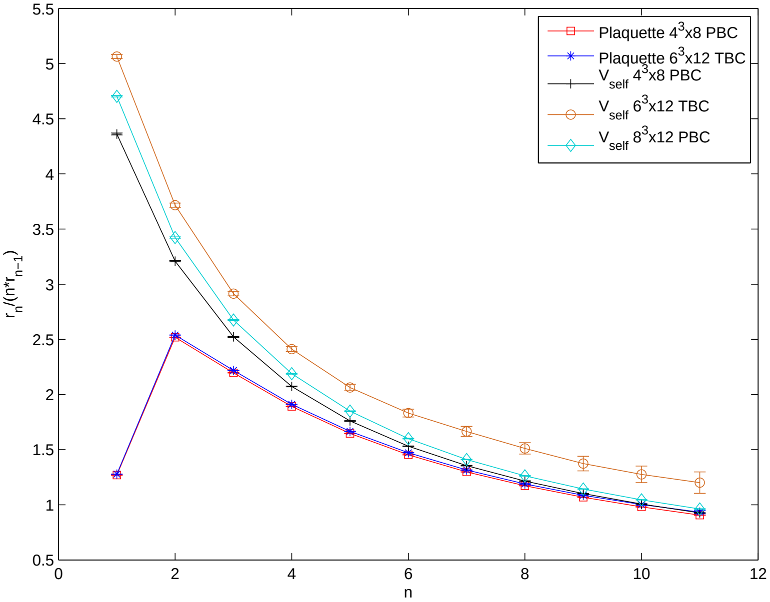
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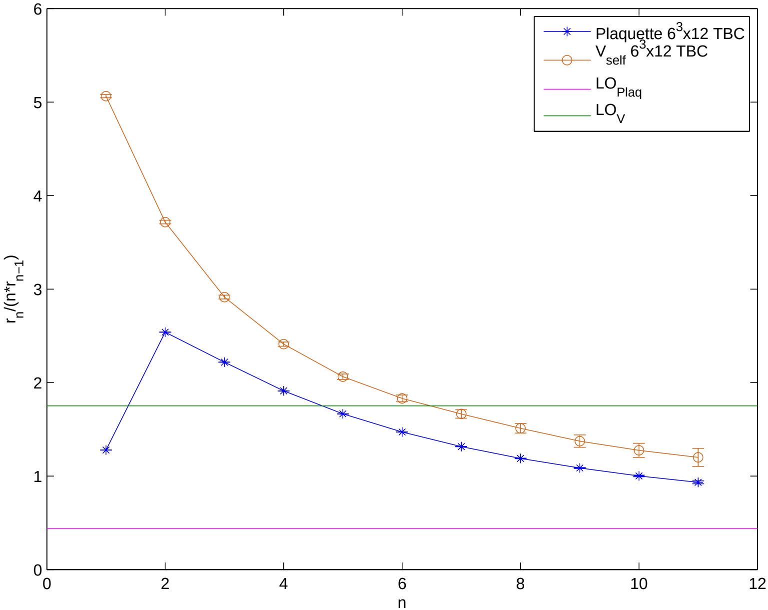
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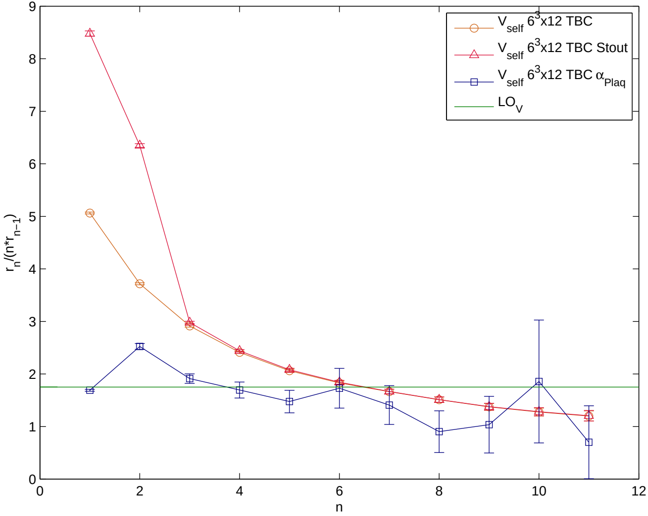



$$\alpha_{\rm Plaq} \equiv -\frac{3}{4\pi}\ln\Box = \alpha_{\rm L}+3.3726\,\alpha_{\rm L}^2+17.694\,\alpha_{\rm L}^3+\cdots+4.571\cdot10^9\alpha_{\rm L}^{12}$$

Lastly: with little additional expense, calculate static self energy in octet representation:

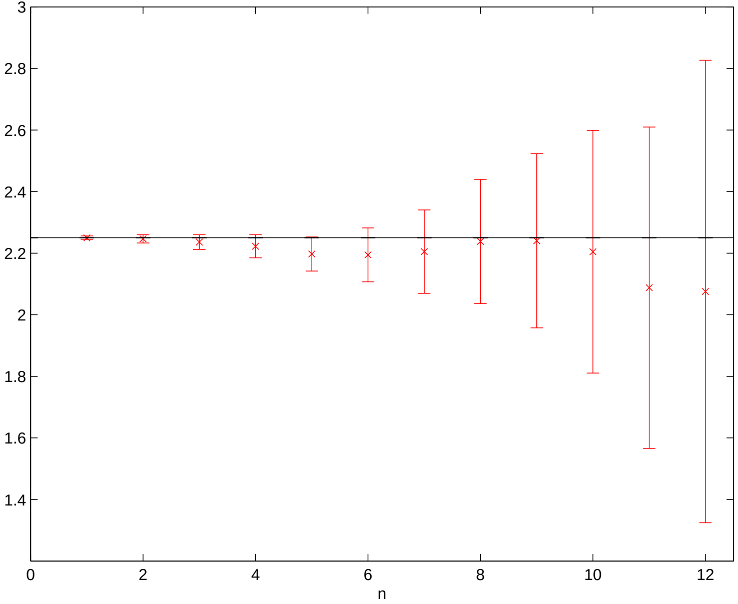
$$V_{\mathrm{O,self}} = \sum V^{(n)}_{\mathrm{O,self}} \alpha^n$$

Known so far (e.g. Bali, Pineda (hep-ph/0310130)):

- renormalon structure is equal to the singlet case
- Casimir Scaling
  for $n = 1, 2$ exact and approximately for $n = 3$:

$$\frac{V^{(n)}_{\mathrm{O,self}}}{V^{(n)}_{\mathrm{self}}} = \frac{C_A}{C_f} = 2.25$$

recently: Anzai et al. arXiv:1004.1562v1 [hep-ph]:
also $n = 3$ exact, at $n = 4$ first violation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# Summary

- NSPT is a powerful tool for high-order calculations
  $\longrightarrow$ renormalon physics
- $V_{\rm self}$ renormalon should emerge a lot earlier than the usual candidate from the gluon condensate
- $V_{\rm self}$ severely affected by finite size effects when using periodic boundary conditions
- twisted boundary conditions clearly reduce finite size effects: distinct ratio curves at moderate lattices
- further simulations on larger lattice volumes needed to decide on renormalon existence
- preliminary data suggest that the relation
  $V^{(n)}_{\rm O,self}/V^{(n)}_{\rm self} = C_A/C_F$
  is approximately valid well beyond $n = 3$