A study of the complex action problem in a simple model for dynamical compactification in superstring theory using the factorization method

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Outline

Motivation to ... stay awake

- Study a model with strong complex action problem
- Apply a method of wide applicability with a promise to solve the overlap problem and improve chances to extrapolate results to larger systems
- Method used in SUSY matrix models [Nishimura-K.N.A ('01)], statistical models [Azcoiti-Di Carlo-Galante-Laliena ('02)], random matrix theory [Ambjorn-K.N.A-Nishimura-Verbaarschot ('02)], finite density QCD [Fodor-Katz-Schmidt ('07)]
- Attempt to explain when and how the method could be useful

Outline

Large–N Reduction

- Large N $U(N)$ gauge theory on D-dim torus **[Eguchi–Kawai ('82)]**

$$S = -N\beta \sum_n \mathrm{tr} \left(U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu} \right)$$

$$\rightarrow \quad S = N\beta\, \mathrm{tr} \left(U_\mu U_\nu U^\dagger_\mu U^\dagger_\nu \right)$$

- If $U(1)^D$ symmetry $U_\mu \rightarrow e^{i\alpha} U_\mu$ not spontaneously broken

Large–N Reduction

- A continuum version of the large–N reduced model

[Gross-Kitazawa ('82), Gonzalez-Aroyo & Korthals-Altes ('83)]

$$
\begin{aligned}
S &= \frac{1}{4g^2}\int d^D x\, \mathrm{tr}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\right)^2 \\
\to &\quad S = -\frac{1}{4g^2}\,\mathrm{tr}\left([A_\mu, A_\nu]\right)^2
\end{aligned}
$$

  - If $U(1)^D$ symmetry $A_\mu \to A_\mu + \alpha_\mu$ not spontaneously broken

- Revival of interest in this model+SUSY (and its 1-dim counterparts) in the context of
  - Non-perturbative string theory
  - Gauge/Gravity duality

IKKT or IIB Matrix Model

- $D = 10$, $\mathcal{N} = 1$, $SU(N)$ SYM at zero volume limit as a non–perturbative formulation of type IIB superstring theory (conjecture) **[Ishibashi-Kawai-Kitazawa-Tsuchiya ('96)]**

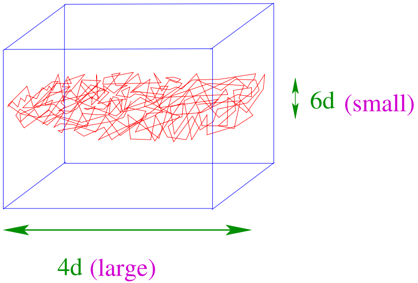
$$S = -\frac{1}{g^2}\text{tr}\left\{\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\psi_\alpha\left(\Gamma_\mu\right)_{\alpha\beta}[A_\mu, \psi_\beta]\right\}$$

$A_\mu(\mu = 1, \ldots, 10)$, $\psi_\alpha(\alpha = 1, \ldots, 16)$ $\quad N \times N \quad$ Hermitian

  - ψ_α are Majorana–Weyl in the *adjoint*

Quantum Space–time

- Eigenvalues of A_μs interpreted as space–time coordinates $\Rightarrow$space–time dynamically generated
- Quantum (fuzzy) space–time: A_μ not simultaneously diagonalizable on generic configurations.
- Posibility of dynamical compactification of extra dimensions



Dynamical Compactification

- SSB of $SO(10)$: Order parameter the 10×10 real symmetric "moment of inertia"

$$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$$

with eigenvalues

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{10}$$

- e.g $SO(10) \to SO(4)$ given by

$$\langle\lambda_1\rangle = \langle\lambda_2\rangle = \langle\lambda_3\rangle = \langle\lambda_4\rangle \gg \langle\lambda_5\rangle$$

in the large N limit

- Evidence that this is possible to realize using gaussian expansion methods

[Nishimura-Sugino ('01), Kawai-Kawamoto-Kuroki-Matsuo-Shinohara ('02)]

Related Models Expected to Realize $SO(D)$ SSB

- 6d IKKT model [Nishimura-Vernizzi ('00) Nishimura-K.N.A ('01)]
  $A_\mu(\mu = 1, \dots, 16)$, $\psi_\alpha(\alpha = 1, \dots, 4)$ (6d Weyl, adjoint)
  $SO(6) \to SO(3)$ [Nishimura-Okubo-Sugino in prep]
- 4d toy model (non-SUSY)[Nishimura ('01)]
  $A_\mu(\mu = 1, \dots, 4)$,
  $\psi^f_\alpha(\alpha = 1, \dots, 4; f = 1, \dots, N_f)$ (4d Weyl, *fundamental*)

$$Z = \int dA d\psi d\bar{\psi}\, \mathrm{e}^{-S_B - S_F}$$

$$S_B = \frac{1}{2} N \,\mathrm{tr}(A_\mu)^2 \qquad S_F = -\bar{\psi}^f_\alpha (\Gamma_\mu)_{\alpha\beta} A_\mu \psi^f_\beta$$

$N \to \infty$, $r = N_f/N$ fixed $\Rightarrow SO(4) \to SO(2)$ [Nishimura-Okubo-Sugino ('04)]

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Outline

Monte Carlo Simulations

- Complex fermionic partition function:

$$S_F = \frac{1}{2g^2}(\Gamma_\mu)_{\alpha\beta}\mathrm{tr}\,\{\psi_\alpha[A_\mu,\psi_\beta]\} = \Psi_i \mathcal{M}_{ij}\Psi_j \text{ IKKT}$$

$\mathcal{M}_{ij}$ is $16(N^2-1)\times 16(N^2-1)$ ($\mathcal{O}(N^6)$ comp effort)

$$S_F = -\bar{\psi}^f_\alpha(\Gamma_\mu)_{\alpha\beta}A_\mu\psi^f_\beta = \Psi_i\mathcal{D}_{ij}\Psi_j \text{ 4d toy model}$$

$\mathcal{D}_{ij}$ is $4N\times 4N$ ($\mathcal{O}(N^3)$ comp effort) which gives
$Z_F[A] = \mathrm{Pf}\mathcal{M}[A]$ (IKKT), $Z_F[A] = (\det\mathcal{D}[A])^{N_f}$ (toy model)
and

$$Z = \int dA\, \mathrm{e}^{-S_B[A]}\, Z_F[A]$$

where $Z_F[A]$ is in general complex
$\Rightarrow$(strong) complex action problem

Dynamical Compactification

d–dimensional configurations $\{A_\mu\}$

$$\exists O \in SO(10) \qquad \text{s.t.} \qquad A'_\mu = O_{\mu\nu} A_\nu$$
$$A'_{d+1} = A'_{d+2} = \ldots = A'_{10} = 0$$

$\{A_\mu\}$ is 9–dimensional	$\Gamma = 0 \mod \pi$	Pf$\mathcal{M}(A) \in \mathbb{R}$
$\{A_\mu\}$ is 8–dimensional	$\frac{\partial \Gamma}{\partial A_{\mu_1}} = 0$	
$\{A_\mu\}$ is 7–dimensional	$\frac{\partial^2 \Gamma}{\partial A_{\mu_1} \partial A_{\mu_2}} = 0$	
$\{A_\mu\}$ is 6–dimensional	$\frac{\partial^3 \Gamma}{\partial A_{\mu_1} \partial A_{\mu_2} \partial A_{\mu_3}} = 0$	Pf$\mathcal{M}(A) \geq 0$
$\ldots$	$\ldots$	$\ldots$

[Nishimura-Vernizzi ('00)]

- Stationarity of phase increases for lower d, compensates entropy loss
- Low dimensional configurations ($d \leq 6$ for IKKT $d \leq 3$ for 4d toy model) have $Z_F[A] \geq 0$!

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Reweighting...

- Fluctuations of complex phase crucial in the realization of the SSB scenario [Nishimura–K.N.A ('01)]
  ... but difficult to simulate
- Simulate phase quenched model

$$Z_0 = \int dA\, e^{-S_0}$$

$$Z_F[A] = |Z_F[A]| e^{\imath\Gamma} \qquad S_0 = S_B - \ln |Z_F[A]|$$

$$\langle \lambda_n \rangle = \frac{\langle \lambda_n e^{\imath\Gamma} \rangle_0}{\langle e^{\imath\Gamma} \rangle_0}$$

Density of States

- Principal moments of inertia

$$\tilde{\lambda}_n \equiv \frac{\lambda_n}{\langle \lambda_n \rangle_0}$$

(deviation from 1: effect of phase)

- density of states

$$\langle \tilde{\lambda}_n \rangle = \int_0^\infty dx\, x\, \rho_n(x) \qquad \rho_n(x) = \langle \delta(x - \tilde{\lambda}_n) \rangle$$

- Factorization:

$$\rho_n(x) = \frac{1}{\langle \mathrm{e}^{\imath\Gamma} \rangle_0} \rho_n^{(0)}(x) w_n(x)$$

$$\rho_n^{(0)}(x) = \langle \delta(x - \tilde{\lambda}_n) \rangle_0 \quad w_n(x) \equiv \langle \mathrm{e}^{\imath\Gamma} \rangle_{n,x}$$

$$\langle \cdot \rangle_{n,x} \to Z_{n,x} = \int dA\, \mathrm{e}^{-S_0} \delta(x - \tilde{\lambda}_n)$$

Solution

- Minimize the "free energy"

$$\mathcal{F}_n(x) = -\ln \rho_n(x)$$

by solving the saddle point equation

$$\frac{1}{N^2} f_n^{(0)}(x) \equiv \frac{d}{dx} \ln \rho_n^{(0)}(x) = -\frac{1}{N^2} \frac{d}{dx} \ln w_n(x) \equiv -\frac{d}{dx} \Phi_n(x)$$

- $\frac{1}{N^2} f_n^{(0)}(x)$: No complex action problem, calculate at large N.
- $\Phi_n(x)$: Complex action problem, obtain by FSS
- Error in determining $\langle \lambda_n \rangle$ *does not propagate exponentially in* N.

Implementation

- Simulate the system

$$Z_{n,V} = \int dA\, \mathrm{e}^{-S_0 - V(\lambda_n)} \qquad \text{e.g.} \quad V(z) = \frac{1}{2}\gamma(z-\xi)^2$$

$$\gamma \sim 10^3 - 10^7, \quad \xi \sim [-10\langle\lambda_n\rangle_0, +100\langle\lambda_n\rangle_0]$$

- Consider the distribution function:

$$\rho_{n,V}(x) = \langle \delta(x - \tilde{\lambda}_n)\rangle_{n,V} \propto \rho_n^{(0)}(x)\mathrm{e}^{-V(x\langle\lambda_n\rangle_0)}$$

- Solve

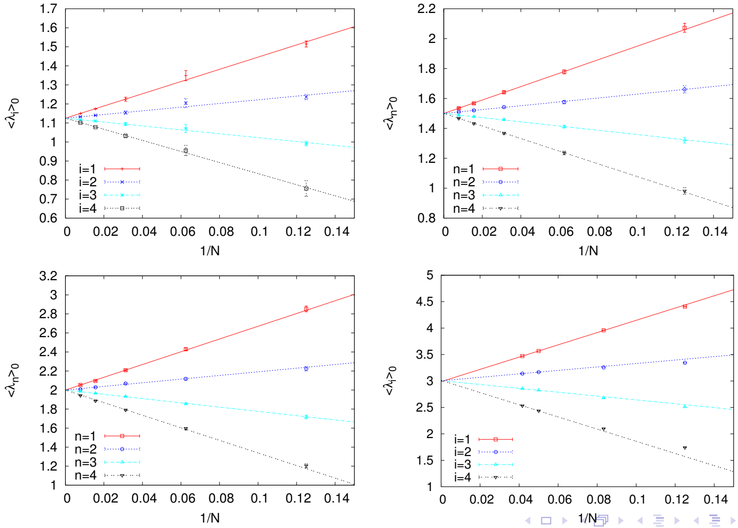
$$0 = \frac{d}{dx} \ln \rho_{n,V}(x)|_{x=x_p} = f_n^{(0)}(x_p) - \langle\lambda_n\rangle_0 V'\left(x_p\langle\lambda_n\rangle_0\right)$$

- Use estimator $x_p = \langle\tilde{\lambda}_n\rangle_{n,V}$ and

$$w_n(x_p) = \langle\cos\Gamma\rangle_{n,V} \qquad f_n^{(0)}(x_p) = \gamma\langle\lambda_n\rangle_0\left(\langle\lambda_n\rangle_{n,V} - \xi\right)$$

Outline

No SSB: $\langle \lambda_n \rangle_0 = 1 + \frac{r}{2}$, $r = 1/4, 1, 2, 4$ [Nishimura ('01)]

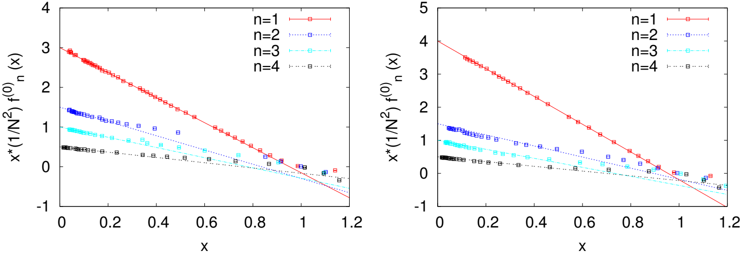


Scaling of $f_n^{(0)}(x)$ for small x: $d \approx (n-1)$

- Expect from measure dA ($A_\mu \sim \sqrt{x}$): $\rho_n^{(0)}(x) \sim (\sqrt{x})^{(5-n)N^2}$

$$\frac{1}{N^2} f_n^{(0)}(x) \approx \left\{ \frac{1}{2}(5-n) + r\delta_{n,1} \right\} \frac{1}{x} + a_n$$

(for $n = 1$ also contribution from fermion determinant)

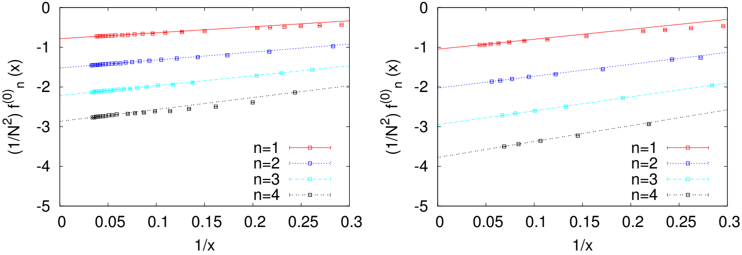


$x \frac{1}{N^2} f_n^{(0)}(x)$ for $r = 1$ (left) $r = 2$ (right), $N = 64$

Scaling of $f_n^{(0)}(x)$ for large x: $d \approx n$

- Leading term from $\rho_N^{(0)}(x) \sim \exp\left(-\langle \frac{1}{2} N \mathrm{tr}(A_\mu)^2 \rangle_{n,V}\right) = \exp\left(-\frac{1}{2} N^2 \sum_{k=1}^{n} \langle \lambda_k \rangle_{n,V}\right) \sim \exp\left(-\frac{1}{2} N^2 n x \langle \lambda_n \rangle_0\right)$ plus subleading term from measure and fermionic determinant:

$$\frac{1}{N^2} f_n^{(0)}(x) \approx -\frac{1}{2} n \langle \lambda_n \rangle_0 + \left(\frac{n}{2} + r\right) \frac{1}{x}$$



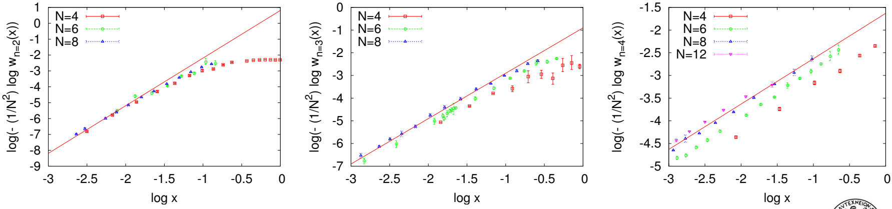
$\frac{1}{N^2} f_n^{(0)}(x)$ for $r = 1$ (left) $r = 2$ (right), $N = 64$

Outline

Scaling of $\Phi_n(x)$ for small x: $d \approx (n-1)$

- Since $A_\mu \sim \sqrt{x}$, fluctuations of phase has width $\delta\Gamma \sim (\sqrt{x})^{5-n}$ and assuming the distribution is gaussian $(-\ln w_n(x) = \delta\Gamma^2/2)$

$$\Phi_n(x) = \frac{1}{N^2} \ln w_n(x) \simeq -c_n x^{5-n}, \qquad x \ll 1,\ n = 2, 3, 4$$



$\ln(-\Phi_n(x))$ for $r = 1$ $n = 2, 3, 4$, $N = 4, 6, 8$ Fit to $N = 8$ w/r to c_n

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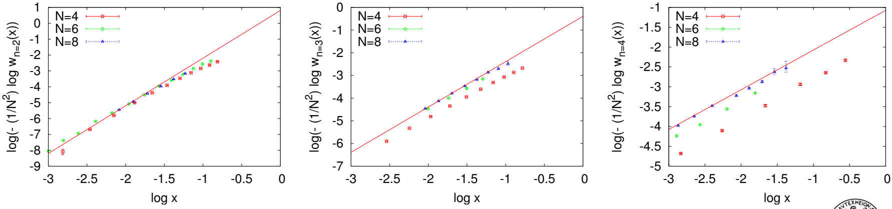
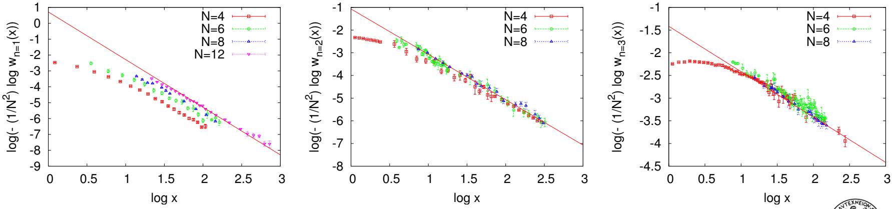


$\ln(-\Phi_n(x))$ for $r = 2$ $n = 2,3,4$, $N = 4,6,8$ Fit to $N = 8$ w/r to c_n

Scaling of $\Phi_n(x)$ for large x: $d \approx n$

- Since $A_\mu \sim \sqrt{x}$, fluctuations of phase has width $\delta\Gamma \sim (1/\sqrt{x})^{4-n}$ and assuming the distribution is gaussian $(-\ln w_n(x) = \delta\Gamma^2/2)$

$$\Phi_n(x) = \frac{1}{N^2} \ln w_n(x) \simeq -d_n x^{-(4-n)}, \qquad x \gg 1,\ n = 1,2,3$$



$\ln(-\Phi_n(x))$ for $r = 1$ $n = 1, 2, 3$, $N = 4, 6, 8, 12$ Fit to largest N w/r to d_n

Scaling of $\Phi_n(x)$ for large x: $d \approx n$

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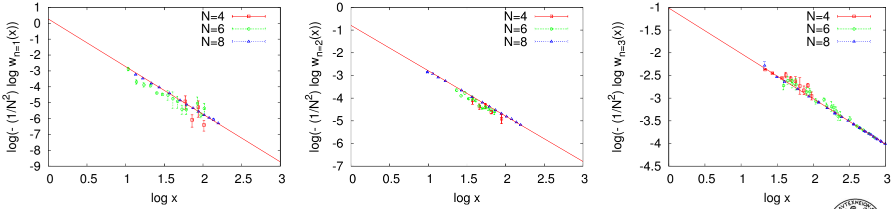
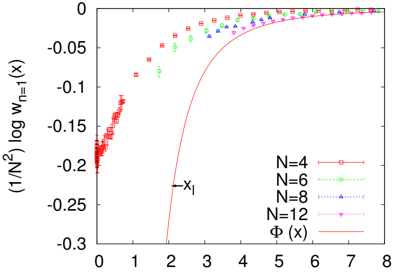
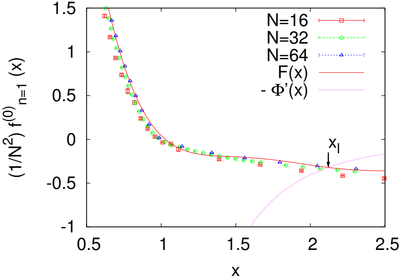


$\ln(-\Phi_n(x))$ for $r = 2$ $n = 2, 3, 4$, $N = 4, 6, 8$ Fit to $N = 8$ w/r to d_n

Solution: Double peak structure for $n = 2, 3$

- Using c_n and d_n we extrapolate...





$r = 1$ $n = 1$: Large dimension dominates

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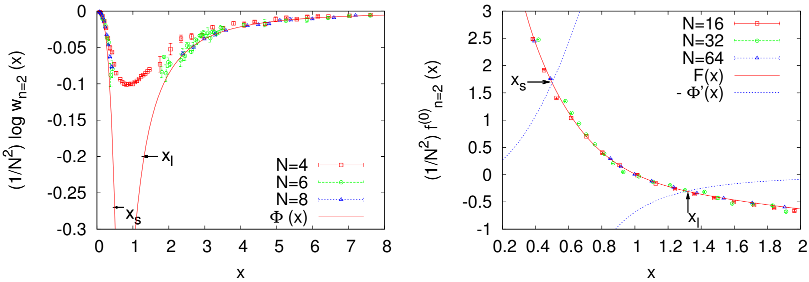

$r = 1$ $n = 2$: Two peak structure

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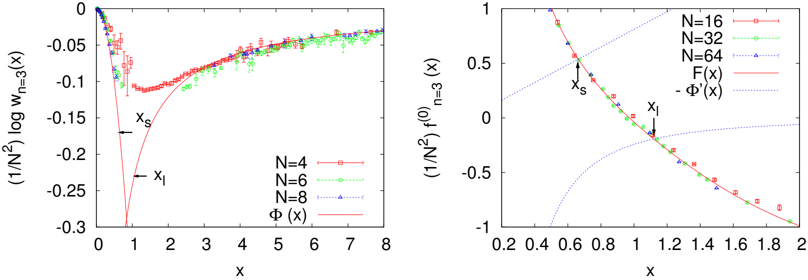

$r = 1$ $n = 3$: Two peak structure

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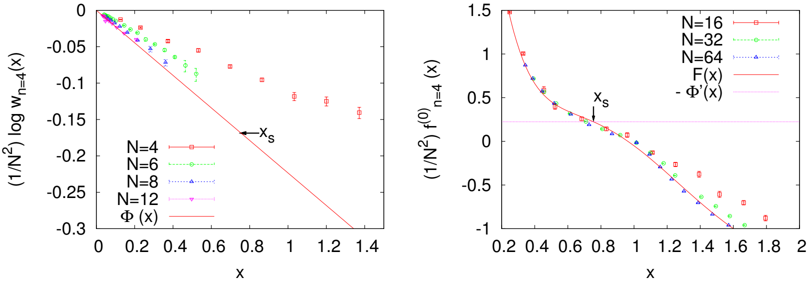
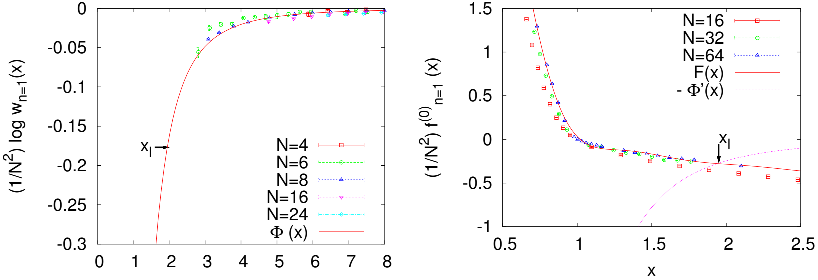


$r = 1$ $n = 4$: Small dimension dominates

Solution: Double peak structure for $n = 2, 3$

- Using c_n and d_n we extrapolate...



$r = 2$ $n = 1$: Large dimension dominates

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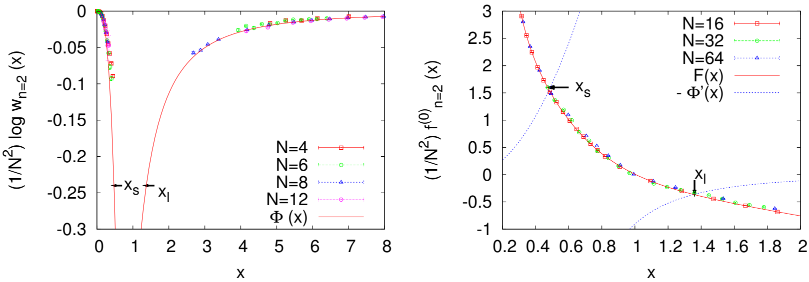
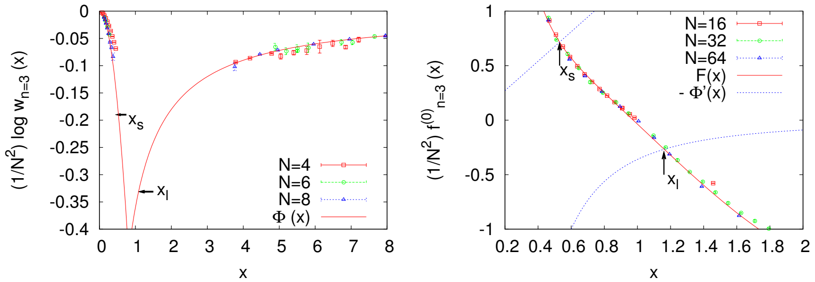
- Using c_n and d_n we extrapolate...



$r = 2$ $n = 2$: Two peak structure

Solution: Double peak structure for $n = 2, 3$

- Using c_n and d_n we extrapolate...



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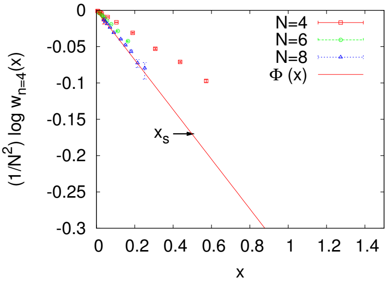

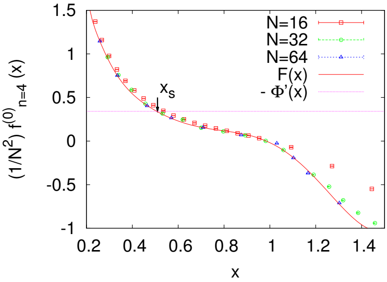

$r = 2$ $n = 4$: Small dimension dominates

Solution: $\langle \tilde{\lambda}_n \rangle$

	$r = 1$			$r = 2$		
n	x_s	x_l	$\langle \tilde{\lambda}_n \rangle_{\text{Gauss}}$	x_s	x_l	$\langle \tilde{\lambda}_n \rangle_{\text{Gauss}}$
1		2.12	1.4		1.94	1.7
2	0.49	1.29	1.4	0.48	1.36	1.7
3	*0.67*[1]	1.13	0.7	*0.53*[1]	1.16	0.5
4	0.75		0.5	0.51		0.1

$\langle \tilde{\lambda}_n \rangle_{\text{Gauss}}$ obtained by GEM **[Nishimura-Okubo-Sugino ('04)]**

[1] *assuming* $SO(4) \to SO(2)$

Solution:

- $\langle\tilde{\lambda}_1\rangle > 1 > \langle\tilde{\lambda}_4\rangle$ shows SSB
- Determine dominant peak for $n = 2, 3$ **[Nishimura-K.N.A ('01)]**

$$\Delta_n \stackrel{\text{def}}{=} \frac{1}{N^2}\left\{\log \rho_n(x_l) - \log \rho_n(x_s)\right\}$$
$$= \Phi_n(x_l) - \Phi_n(x_s) + \Xi_n$$
$$\Xi_n \stackrel{\text{def}}{=} \int_{x_s}^{x_l} dx \left\{\frac{1}{N^2} f_n^{(0)}(x)\right\}$$

Δ_n depends on $\Phi_n(x)$ only at x_l, x_s
Ξ_n computed by fitting $\frac{1}{N^2} f_n^{(0)}(x)$ to a continuous function in $[x_s, x_l]$

- $\Delta_n > 0 \Rightarrow x_l$ dominates; $\Delta_n < 0 \Rightarrow x_s$ dominates

Solution:

	$r = 1$				$r = 2$			
n	$\Phi_n(x_s)$	$\Phi_n(x_l)$	Ξ_n	Δ_n	$\Phi_n(x_s)$	$\Phi_n(x_l)$	Ξ_n	Δ_n
1		-0.23				-0.18		
2	-0.27	-0.20	0.27	0.34	-0.25	-0.25	0.25	0.25
3	-0.18	-0.22	0.07	0.03	-0.19	-0.31	0.12	0.00
4	-0.17				-0.17			

- $n = 2$ large x peak dominates
- $n = 3$ uncertain

Summary

4d toy matrix model:

- Dynamical compactification in matrix models: first time from direct, non perturbative calculations
- Importance of phase in SSB: No SSB if absent
- Double peak structure: Two dynamical length scales
- Consistent with gaussian expansion method

Factorization Method:

- Compute density of states (DOS) at regions where the competing phase fluctuations+entropy favour configurations using simple scaling properties of the phase and DOS of phase quenched model
- Expectation values computed by minimizing free energy, drastically reducing propagation of errors
- Applicable to a wide range of models

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Soon to come...

- Study dynamical compactification in 6D IKKT model using Monte Carlo and gaussian expansion method
- Improve results in 4d model:

$$\rho(x_1, x_2, x_3, x_4) = \langle \prod_k \delta(x_k - \tilde{\lambda}_k) \rangle$$