Computing the long-distance contribution to second order weak amplitudes

Lattice 2010

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Outline

- Introduction
- Naïve lattice 2nd order self-energy
- Correct the short distance contribution
- Reduce finite volume errors
- Conclusion

Introduction

- Time evolution of $K^0 - \overline{K^0}$ system given by familiar Wigner-Weisskopf formula:

$$i\frac{d}{dt}\begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix} = \left\{ \begin{pmatrix} M_{K^0K^0} & M_{K^0\overline{K}^0} \\ M_{\overline{K}^0K^0} & M_{\overline{K}^0\overline{K}^0} \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \Gamma_{K^0K^0} & \Gamma_{K^0\overline{K}^0} \\ \Gamma_{\overline{K}^0K^0} & \Gamma_{\overline{K}^0\overline{K}^0} \end{pmatrix} \right\} \begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix}$$

where:

$$\Gamma_{ij} = 2\pi \sum_\alpha \int_{2m_\pi}^\infty dE \langle i|H_W|\alpha(E)\rangle\langle\alpha(E)|H_W|j\rangle \delta(E - m_K)$$

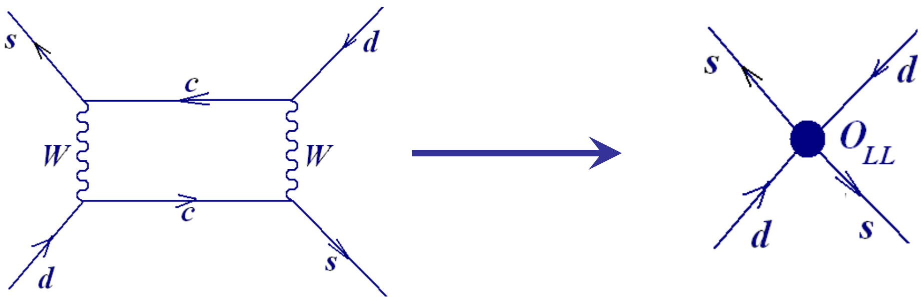
$$M_{ij} = \sum_\alpha \mathcal{P} \int_{2m_\pi}^\infty dE \frac{\langle i|H_W|\alpha(E)\rangle\langle\alpha(E)|H_W|j\rangle}{m_K - E}$$

- Neglecting CP violation:

$$m_{K_S} - m_{K_L} = 2M_{K^0\overline{K}^0}$$

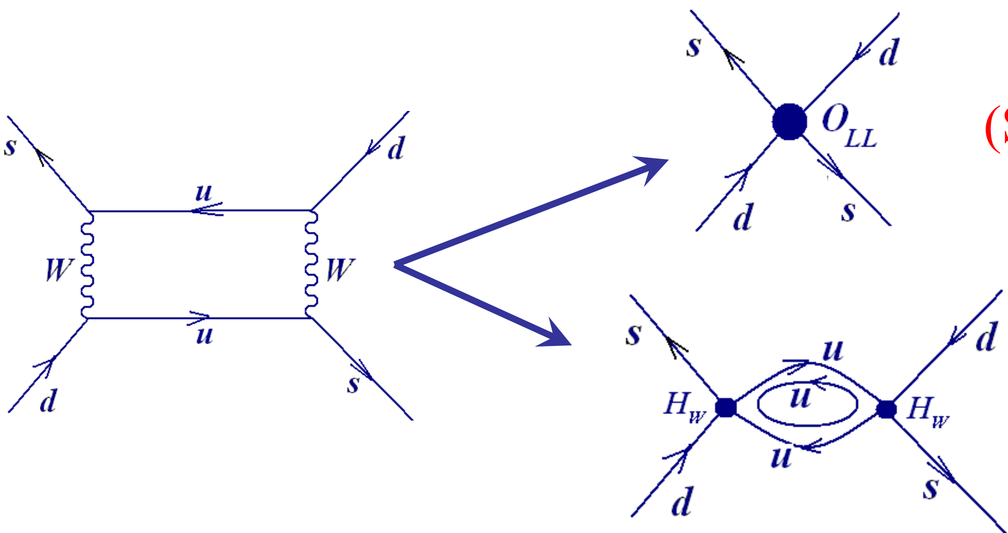
Contributions to m_{K_S} - m_{K_L}

- Charm part expected to be largest:



(Short distance)

- Possible $\Delta I = ½$ enhanced π - π contribution:

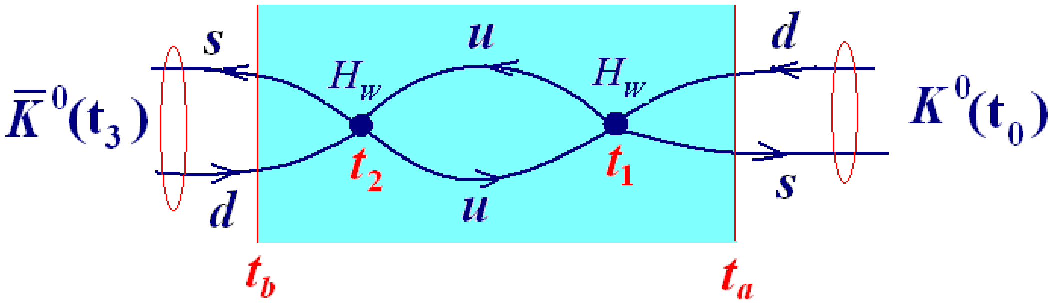


(Short distance)

(Long distance)

Naïve Lattice Perturbation Theory

- Begin with standard 2nd order perturbation theory:



$$\frac{1}{2}\Big\langle (K^0)^\dagger(t_3)\Big[\int_{t_a}^{t_b} dt_2 \int_{t_a}^{t_b} dt_1 H_W(t_2) H_W(t_1)\Big](K^0)^\dagger(t_0)\Big\rangle$$

$$= \Big[\sum_{\alpha E_n} \frac{\langle \overline{K}^0|H_W|\alpha(E_n)\rangle\langle\alpha(E_n)|H_W|K^0\rangle}{m_K - E_n}(t_b - t_a)\Big]\Big|\langle K^0|K^0(0)^\dagger|0\rangle\Big|^2 e^{-m_K(t_3-t_0)}$$

- If box size is tuned to make $E_0 = m_K$:

$$= \Big[\sum_{\alpha, E_n \neq E_0} \frac{\langle \overline{K}^0|H_W|\alpha(E_n)\rangle\langle\alpha(E_n)|H_W|K^0\rangle}{m_K - E_n}(t_b - t_a)$$
$$+\frac{1}{2}\langle \overline{K}^0|H_W|\alpha(E_0)\rangle\langle\alpha(E_0)|H_W|K^0\rangle(t_b - t_a)^2\Big]\Big|\langle K^0|K^0(0)^\dagger|0\rangle\Big|^2 e^{-m_K(t_3-t_0)}$$

Naïve Lattice Perturbation Theory

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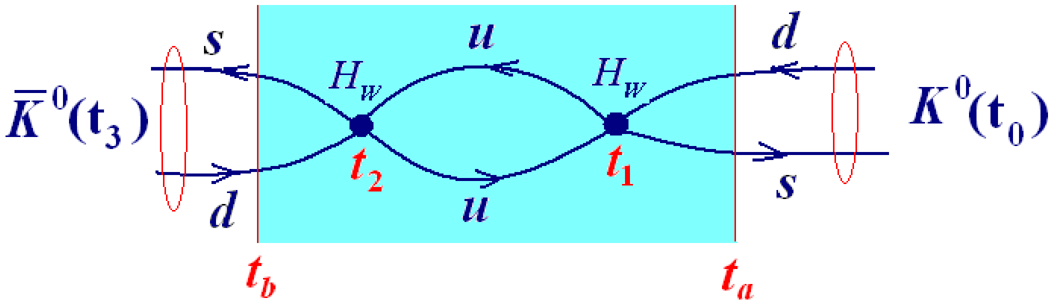

$$\frac{1}{2}\Big\langle (K^0)^\dagger(t_3)\Big[\int_{t_a}^{t_b} dt_2 \int_{t_a}^{t_b} dt_1 H_W(t_2) H_W(t_1)\Big](K^0)^\dagger(t_0)\Big\rangle$$

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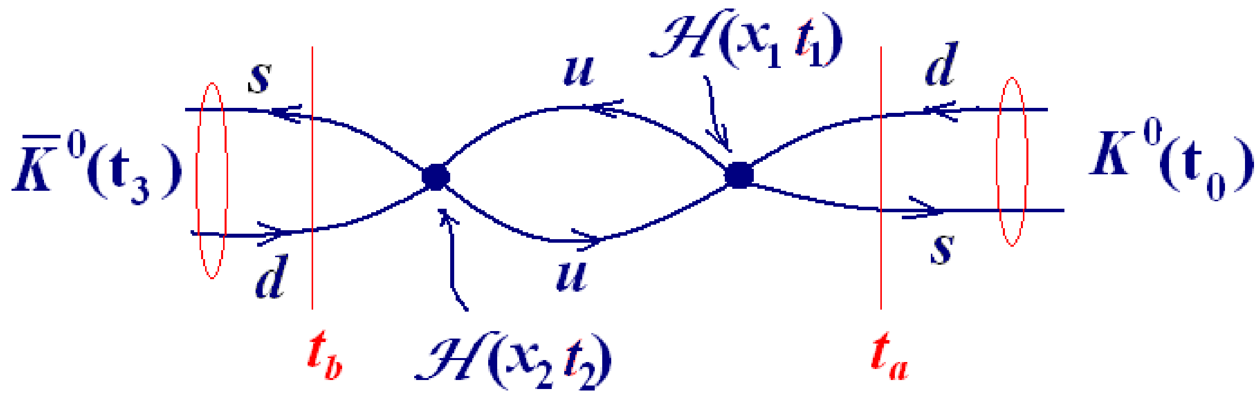
$$+\frac{1}{2}\langle \overline{K}^0 | H_W | \alpha(E_0)\rangle\langle\alpha(E_0)|H_W|K^0\rangle(t_b - t_a)^2\Big]\Big|\langle K^0|K^0(0)^\dagger|0\rangle\Big|^2 e^{-m_K(t_3-t_0)}$$

Correct short distance component

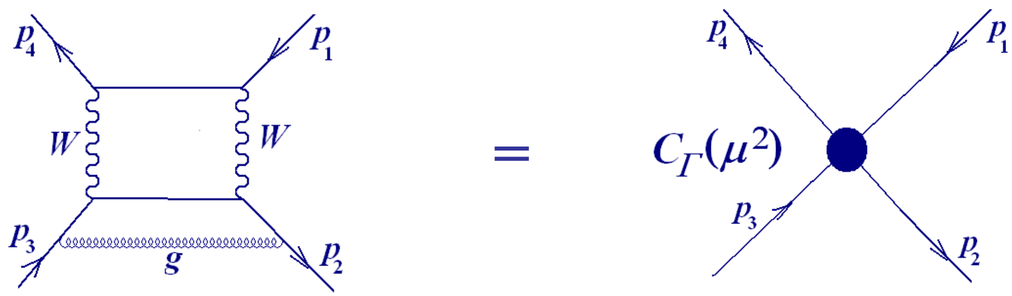
- Naïve 2nd order calculation fails when $(x_1\, t_1) \rightarrow (x_2\, t_2)$



- Use RI/MOM normalized subtraction to replace unphysical with physical short distance part.

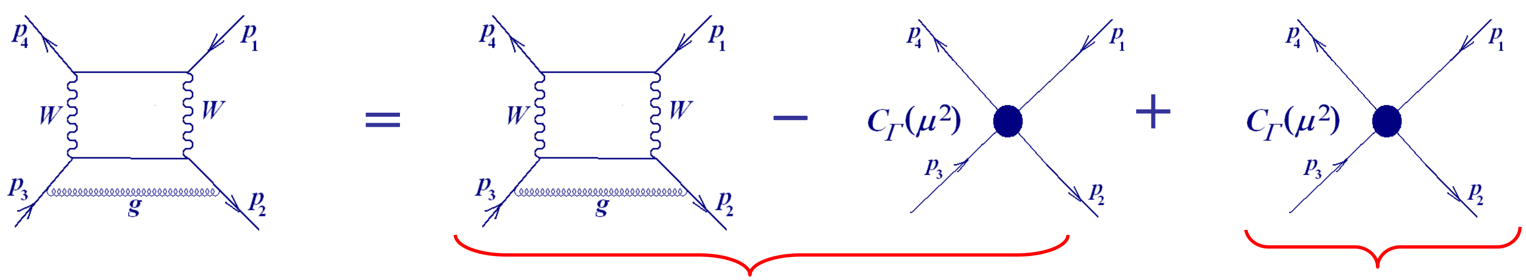
Recall extraction of short distance part

- Determine Wilson coefficient:



when evaluated at $p_a \cdot p_b = \mu^2\,(1 - 4\delta_{ab})$: $\quad \Lambda_{\mathrm{QCD}} < \mu < m_W$

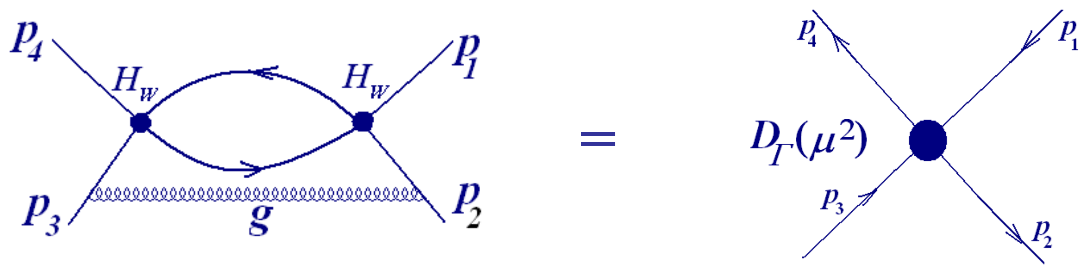
- Separate into short and long distance parts:



For internal momentum ***k*****:** $\qquad \boldsymbol{k \le \mu} \qquad\qquad \boldsymbol{\mu \le k}$

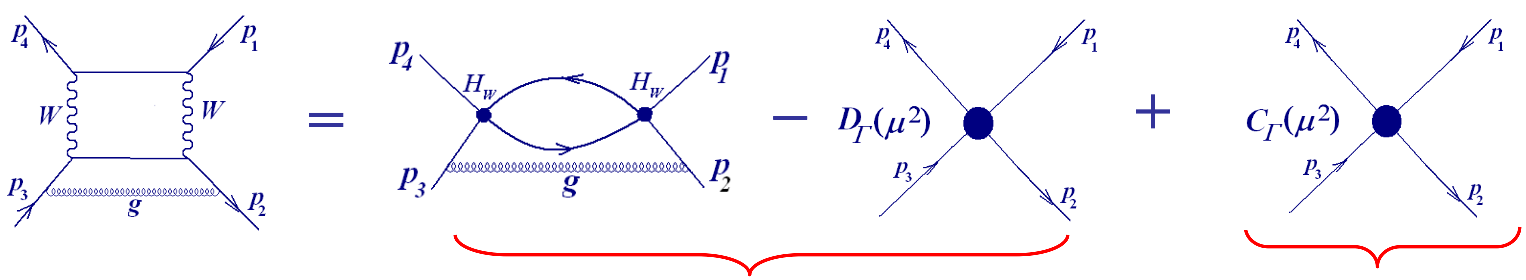
Replace long distance part with 2nd order PT

- Determine a new Wilson coefficient:



when evaluated at $p_a \cdot p_b = \mu^2\,(1 - 4\delta_{ab})$: $\Lambda_{\mathrm{QCD}} < \mu < m_W$

- Replace the long distance part:



For internal momentum $\boldsymbol{k}$: $\boldsymbol{k \le \mu}$ $\boldsymbol{\mu \le k}$

Short distance contribution to long distance part

- Physical amplitude:
  - Finite before subtraction $\sim G_F^2\, m_c^2$
  - Short distance contribution to long distance part $\sim G_F^2\, m_c^2\, \Lambda_{\mathrm{QCD}}^2/\mu^2$
- H_W^2 amplitude:
  - $\mu < m_c$
    - Divergent before subtraction $\sim \boldsymbol{G_F^2\, a^{-2}}$
    - Requires dimension 6 and 8 subtractions
  - $m_c < \mu$
    - Finite up to logs before subtraction (GIM)
    - Short distance contribution $\sim \boldsymbol{G_F^2\, m_c^2\, \Lambda_{\mathrm{QCD}}^2/\mu^2}$

Finite volume errors

- Singular energy denominator $1/(m_K - E_n)$ will introduce uncontrolled $1/L^3$ errors.
- Generalize the Lellouch-Luscher method to relate the finite and infinite volume mass shift:
  - Finite volume π-π energy gives $k = \sqrt{E_{\pi\pi}^2/4 - m_\pi^2}$
  - Infinite volume width and mass shift determine weak π-π resonant phase shift $\delta_W(k)$
  - Relate them by imposing Luscher condition

$$\phi\left(\frac{kL}{2\pi}\right)+\delta_0(k)+\delta_W(k) = n\pi$$

$$\tan(\phi(q)) = -\frac{\pi^{3/2}q}{Z_{00}(1,q^2)} \qquad Z_{00} = \frac{1}{\sqrt{4\pi}}\sum_{n\in Z^3}\frac{1}{(n^2-q^2)^s}$$

Finite volume energy

- Adjust volume so $E_{\pi\pi} = m_K$ and use degenerate perturbation theory.
- Energies given by eigenvalues of

$$\begin{pmatrix} m_K + \sum_{\alpha, E_n \neq m_K 0} \frac{|\langle \alpha(E_n)|H_W|K_S\rangle|^2}{m_K - E_n} & \langle K_S|H_W|\pi\pi\rangle \\ \langle \pi\pi|H_W|K_S\rangle & m_K + \sum_{\alpha, E_n \neq m_K} \frac{|\langle \alpha(E_n)|H_W|\pi\pi\rangle|^2}{m_K - E_n} \end{pmatrix}$$

$$\begin{aligned} E_{\pm} &= m_K \pm \langle K_S|H_W|\pi\pi\rangle \\ &+ \frac{1}{2}\Big[\sum_{\alpha, E_n \neq m_K} \frac{|\langle \alpha(E_n)|H_W|K_S\rangle|^2}{m_K - E_n} + \sum_{\alpha, E_n \neq m_K} \frac{|\langle \alpha(E_n)|H_W|\pi\pi\rangle|^2}{m_K - E_n} \Big] \end{aligned}$$

Finite volume energy

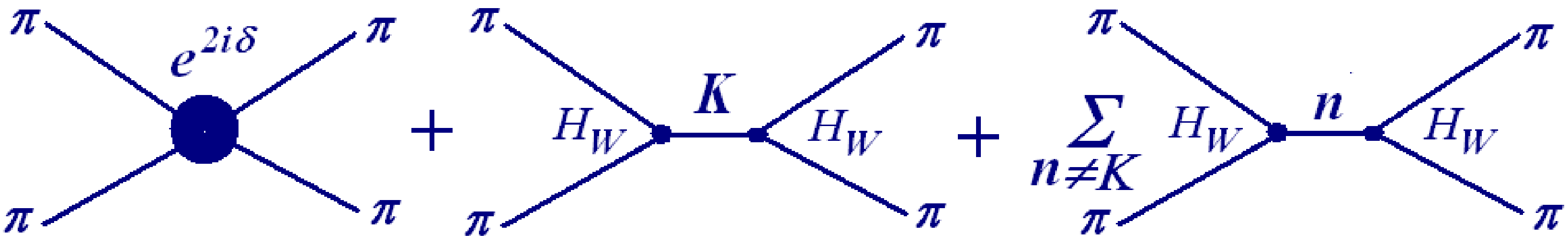
- Adjust volume so $E_{\pi\pi} = m_K$ and use degenerate perturbation theory.
- Energies given by eigenvalues of

$$\begin{pmatrix} m_K + \sum_{\alpha, E_n \neq m_K 0} \frac{|\langle \alpha(E_n)|H_W|K_S\rangle|^2}{m_K - E_n} & \langle K_S|H_W|\pi\pi\rangle \\ \langle \pi\pi|H_W|K_S\rangle & m_K + \sum_{\alpha, E_n \neq m_K} \frac{|\langle \alpha(E_n)|H_W|\pi\pi\rangle|^2}{m_K - E_n} \end{pmatrix}$$

$$\begin{aligned} E_\pm &= m_K \pm \langle K_S|H_W|\pi\pi\rangle \\ &\quad + \frac{1}{2}\Big[\sum_{\alpha, E_n \neq m_K} \frac{|\langle \alpha(E_n)|H_W|K_S\rangle|^2}{m_K - E_n} + \sum_{\alpha, E_n \neq m_K} \frac{|\langle \alpha(E_n)|H_W|\pi\pi\rangle|^2}{m_K - E_n}\Big] \end{aligned}$$

$$\hat{M}^{(2)}_{K_S}$$

Infinite volume scattering



- Total phase shift $\delta_{\rm tot}(k)$:

$$\delta_{\rm tot}(k) = \delta_0(k) + \arctan\left(\frac{\Gamma(k)/2}{m_K + M^{(2)}_{K_S} - E_k}\right) - \frac{E_k k}{2\pi}\sum_\alpha \int dE' \frac{\langle\pi\pi(k)|H_W|\alpha(E')\rangle\langle\alpha(E')|H_W|\pi\pi(k)\rangle}{E_k - E'}$$

- Require that:

$$\delta_{\rm tot}(k_\pm) + \phi\left(\frac{k_\pm L}{2\pi}\right) = n\pi$$

Infinite-finite volume relations

- Expand to 1st order in H_W

$$\Gamma = \frac{2E_k}{k}\frac{\partial}{\partial k}\left(\phi\left(\frac{kL}{2\pi}\right) + \delta_0(k)\right)|\langle\pi\pi(k)|H_W|K_S\rangle|^2$$

- Expand to 2nd order in H_W and subtract $M_{K_L}^{(2)} = \hat{M}_{K_L}^{(2)}$

$$M_{\overline{K}^0 K^0} = \hat{M}_{\overline{K}^0 K^0} + \frac{m_\pi^2}{2m_K k^2}|\langle\pi\pi(E)|H_W|K_S\rangle|^2 - \frac{\partial}{\partial E}|\langle\pi\pi(E)|H_W|K_S\rangle|^2$$

Conclusion

- With sufficient computing power a lattice calculation of m_{K_S} - m_{K_L} appears possible.
- Include valence charm quarks.
- Apply NPR methods to second order amplitudes.
- Use on-shell $K \rightarrow \pi\,\pi$ kinematics and remove $(t_b\text{-}t_a)^2$ π-π contribution from 2nd order self energy amplitude.
- Add known $1/L^3$ correction.