# QCD with chemical potential in a small hyperspherical box

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# Outline

- Motivation of the formulation of QCD at non-zero quark chemical potential on $S^1 \times S^3$ using perturbation theory.
- Action of QCD on $S^1 \times S^3$ and results for the fermion number and Polyakov lines for $N = 3$ at low temperature and zero quark mass.
- $N = \infty$ theory and results for fermion number and Polyakov lines and resulting phase diagram.
- Preliminary $N = 2$ lattice results.

# Partition Function of QCD

The partition function of QCD at finite temperature $T = 1/\beta$, for $N_f$ quark flavors, each with a mass $m_f$ and coupled to a chemical potential $\mu_f$ is:

$$Z_{QCD} = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3\mathbf{x}\mathcal{L}_{QCD}}$$

where $\psi$ and $\bar{\psi}$ are the fundamental and anti-fundamental fermion fields, respectively, and $A$ is the $SU(N)$ gauge field, $A_\mu = A_\mu^a T^a$.

The Lagrangian is

$$\mathcal{L}_{QCD} = \frac{1}{4g^2}\mathrm{Tr}_F\left(F_{\mu\nu}F_{\mu\nu}\right) + \sum_{f=1}^{N_f} \bar{\psi}_f \left(\not{D}_F(A) - \gamma_0\mu_f + m_f\right)\psi_f,$$

with covariant derivative

$$D_\mu(A) \equiv \partial_\mu - A_\mu.$$

# What makes QCD at non-zero $\mu$ so difficult?

## The Sign Problem:

QCD at finite quark chemical potential $\mu$ has a complex action:

$$e^{S_f} = \exp \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3\mathbf{x}\bar{\psi}\left(\not{D}_F(A)-\gamma_0\mu+m\right)\psi}$$

$$= \log\det\left(\not{D}_F(A) - \gamma_0\mu + m\right) \sim \sum_{n=1}^{\infty}\left[e^{n\beta\mu}e^{i\theta_i n} + e^{-n\beta\mu}e^{-i\theta_i n}\right]$$

- The boltzmann weight $e^{-S}$ is complex so it is not possible to perform lattice simulations which use importance sampling.
- The sign problem also complicates large $N$ analysis: In the large $N$ limit the saddle point approximation becomes valid, but the stationary point of a complex action with respect to the angles of the Polyakov line $P = \mathscr{P}e^{\int_0^\beta \mathrm{d}t\, A_0(x)} = \mathrm{diag}\{e^{i\theta_1}, ..., e^{i\theta_N}\}$ lies in the space where the angles are complex. Therefore the eigenvalues of the Polyakov line lie off the unit circle on an arc in the complex plane.

$\Longrightarrow$ Need to generalize our techniques to handle a complex action.

# Region of validity of 1-loop calculations

Properties of $SU(N)$ gauge theories on $S^1 \times S^3$

- Valid for $\min[R_{S^1}, R_{S^3}] \ll \Lambda_{QCD}^{-1}$
  - $\mathbb{R}^3 \times S^1$, small $S^1$:
    - ⋆ Good: Allows study at any $N$ and in the limit of large 3-volume. YM/QCD: $m = 0, \mu = 0$: Gross, Pisarski, Yaffe (Rev.Mod.Phys.53:43,1981),
    - ⋆ Bad: Have to be in the limit of high temperatures (or small $S^1$)
  - $S^3 \times S^1$, small $S^3$:
    - ⋆ Good: Allows study at any temperature (or any $S^1$). YM/$\mathcal{N}=4$ SYM: Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk (hep-th/0310285 (JHEP)),
    - ⋆ Bad: Must be in small 3-volume. Well-defined transitions do not occur for finite $N$.

# 1-loop effective action

Following [Aharony et al (hep-th/0310285)] the effective action of the Polyakov line order parameter $P = \mathscr{P} e^{\int_0^\beta \mathrm{d}t\, A_0(x)} = \mathrm{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ is

$$
\begin{aligned}
S(P) &= -\log Z(P) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - z_b(n\beta/R)\right) \mathrm{Tr}_A P^n \\
&\quad + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} N_f z_f(n\beta/R, mR) \left[ e^{n\beta\mu} \mathrm{Tr}_F P^n + e^{-n\beta\mu} \mathrm{Tr}_F P^{\dagger n} \right],
\end{aligned}
$$

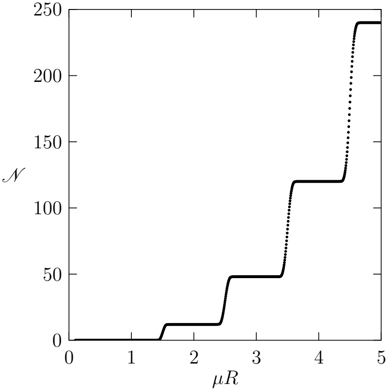
where $\beta = 1/T$, $R = R_{S^3}$, $m =$ quark mass,

$$z_b(\beta/R) = \sum_{\ell=1}^{\infty} d_\ell^{(v,T)} e^{-\beta \varepsilon_\ell^{(v,T)}} = 2 \sum_{\ell=1}^{\infty} \ell(\ell+2) e^{-n\beta(\ell+1)/R}$$

$$z_f(\beta/R, mR) = \sum_{\ell=1}^{\infty} d_\ell^{(f)} e^{-\beta \varepsilon_\ell^{(f,m)}} = 2 \sum_{\ell=1}^{\infty} \ell(\ell+1) e^{-\beta \sqrt{(\ell+\frac{1}{2})^2 + m^2 R^2}/R}$$

For the pure Yang-Mills theory the weak-coupling analogue of the deconfinement transition temperature was calculated in the large $N$ limit: $T_d R \simeq 0.759$ or $\beta_d/R \simeq 1.317$ [Aharony et al (hep-th/0310285)].

# Fermion number at low $T$ ($N = 3$, $N_f = 1$, $mR = 0$)



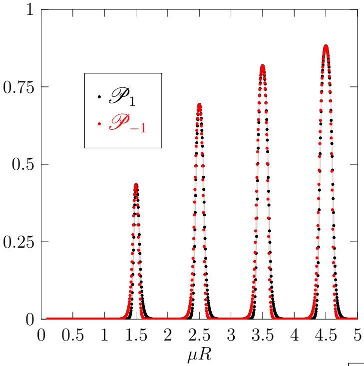

The fermion number expectation value for QCD on $S^1 \times S^3$ (low T) is

$$\mathcal{N} = \frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial \mu}\right) = \frac{-1}{\beta Z}\int [\mathrm{d}\theta]\, e^{-S}\left(\frac{\partial S}{\partial \mu}\right)$$

$$\mathcal{N} \xrightarrow[\beta\to\infty]{} \frac{2N_f}{Z}\int [\mathrm{d}\theta]\, e^{-S}\sum_{l=1}^{\infty}\sum_{i=1}^{N} l(l+1)\left[\frac{e^{\beta\mu}}{e^{\beta\mu}+e^{-i\theta_i+\beta(l+1/2)/R}}\right]$$

The level structure results from the Fermi-Dirac distribution function obtained in the $\mu$-derivative.

# Polyakov lines at low $T$ ($N = 3$, $N_f = 1$, $mR = 0$)



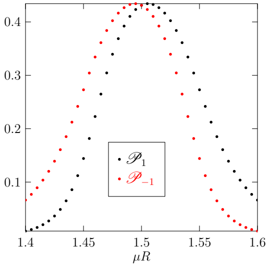

Each level transition in the fermion number corresponds to a spike in $\mathscr{P}_1$ and $\mathscr{P}_{-1}$:

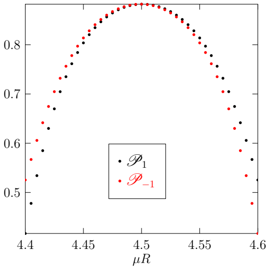
$$\mathscr{P}_1 = \langle \mathrm{Tr} P \rangle \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \sum_{i=1}^{N} e^{i\theta_i}}{Z},$$

$$\mathscr{P}_{-1} = \langle \mathrm{Tr} P^\dagger \rangle \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \sum_{i=1}^{N} e^{-i\theta_i}}{Z}.$$

$\mathscr{P}_1 \neq \mathscr{P}^*_{-1}$ for non-zero $\mu$.

As $\mu$ increases the peaks of $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ get wider indicating that the regions of deconfinement become larger with increasing $\mu$.



# $N = \infty$ theory at low $T$

In the large $N$ limit the saddle point method is valid and it is possible to solve for several observables analytically. Considering a single level transition in the low $T$ limit and performing the sum over $n$ the action reduces to

$$S(\theta_i) = -\frac{1}{2}\sum_{i,j=1}^{N} \log\sin^2\left(\frac{\theta_i - \theta_j}{2}\right) + N\sum_{i=1}^{N} V(\theta_i)$$

$$V(\theta) = i\mathcal{N}\theta - \sigma \log\left(1 + \xi e^{i\theta}\right)$$

- $\mathcal{N}$ is a Lagrange multiplier necessary to satisfy the $\det P = 1$ constraint: $\sum_{i=1}^{N} \theta_i = 0$.
- $\sigma \equiv \sigma_l \equiv 2l(l+1)\frac{N}{N_f}$
- $\xi \equiv \exp\left(\beta(\mu - \varepsilon)\right)$
- $\varepsilon \equiv \varepsilon_l \equiv \sqrt{m^2 + (l+1/2)^2 R^{-2}}$

# Large $N$ formalism

As $\mu$ increases from 0 the Polyakov line eigenvalues are continuously distributed along a closed contour $\mathcal{C}$ in the $z$-plane up to some critical value when a gap opens up. It is useful to consider a map into the complex $z$-plane of the Polyakov line eigenvalues:

$$\frac{1}{N}\sum_{i=1}^{N} \longrightarrow \int_{-\pi}^{\pi} \frac{ds}{2\pi} = \int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) \ .$$

The contour $\mathcal{C}$ on which the Polyakov line eigenvalues lie is given by the inverse map $z(s)$, which can be obtained by solving

$$i\frac{ds}{dz} = \varrho(z).$$

The distribution must satisfy the normalization condition

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) = 1$$

and the det $P = 1$ constraint

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) \log z = 0.$$

# Equation of Motion for Polyakov line eigenvalues

The EOM found from taking $\partial S/\partial\theta_i = 0$ has the integral form

$$zV'(z) = \mathfrak{P}\int_{\mathcal{C}} \frac{dz'}{2\pi i}\,\varrho(z')\frac{z+z'}{z-z'}\ , \qquad zV'(z) = \mathcal{N} - \frac{\sigma\xi z}{1+\xi z}\ .$$

where $\mathfrak{P}$ indicates principal value and closing the contour allows for evaluation of the right-hand side using Cauchy's theorem. In the confined phase the most general form of the density allowed by the EOM is then:

$$\varrho(z) = \frac{c_1}{z} + \frac{c_2\sigma\xi}{1+\xi z} \propto V'(z).$$

For the deconfined (gapped) phase the resolvent is defined by

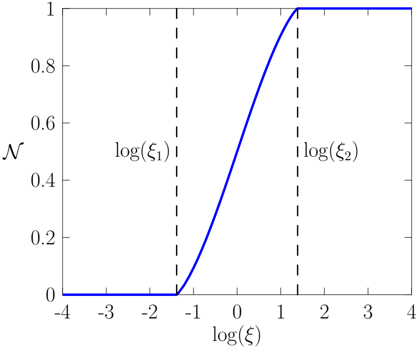
$$\omega(z) = -\frac{1}{N}\sum_j \frac{z+z_j}{z-z_j} = -\int_{\mathcal{C}} \frac{dz'}{2\pi i}\varrho(z')\frac{z+z'}{z-z'}$$

and the contour must be peeled off the distribution to enclose the surrounding poles. The spectral density $\varrho(z)$ can then be obtained from

$$z\varrho(z) = \frac{1}{2}\big[\omega(z+\epsilon) - \omega(z-\epsilon)\big]\ , \qquad z \in \mathcal{C}\ .$$

by collecting residues and applying the normalization condition.

# Effective fermion number $\mathcal{N}$ at a level transition




Adding the $\partial S/\partial\theta_i$ for all the $\theta_i$s we find that the Lagrange multiplier is

$$\mathcal{N} \xrightarrow[N\to\infty]{} \frac{1}{N}\sum_i \frac{\sigma\xi z_i}{1+\xi z_i} = \frac{T}{N^2}\frac{\partial \log Z}{\partial\mu}$$

which is the effective fermion number, $\mathcal{N} = \mathscr{N}/N^2$, valid in the large $N$ limit.

Notice : As $\xi \to 0$ $\quad \mathcal{N} \to 0,$

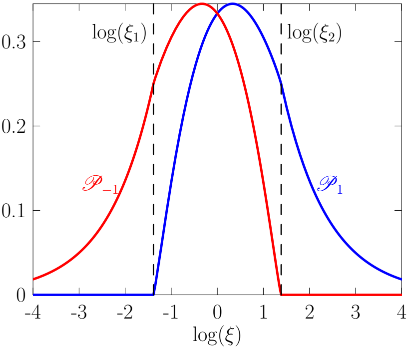
As $\xi \to \infty$ $\quad \mathcal{N} \to \sigma.$

The confining regions with levels at $\mathcal{N}=0$ and $\mathcal{N}=\sigma$ are separated by 3rd order phase transitions from the deconfined region where $\mathcal{N}$ is found by applying the $\det P = 1$ constraint to obtain

$$\xi = \frac{(\sigma-\mathcal{N})^{\sigma-\mathcal{N}}(1+\mathcal{N})^{1+\mathcal{N}}}{\mathcal{N}^{\mathcal{N}}(1+\sigma-\mathcal{N})^{1+\sigma-\mathcal{N}}}\ .$$

# Polyakov lines $\mathscr{P}_1$, $\mathscr{P}_{-1}$ at a level transition

The density of Polyakov line eigenvalues $\varrho(z)$ can be obtained from the EOM and allows for calculation of the Polyakov lines $\mathcal{P}_1$, $\mathcal{P}_{-1}$ as a function of $\mu$.




The Polyakov lines in the confined and deconfined regions are obtained from

$$\mathscr{P}_1 = \int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) z \ ,$$

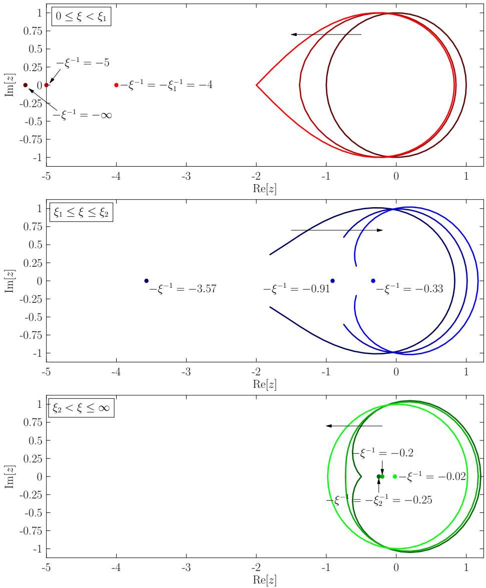
$$\mathscr{P}_{-1} = \int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) \frac{1}{z} \ .$$

As $\xi$ increases :

$$\mathscr{P}_1 = 0 \ , \qquad \frac{\mathcal{N}}{\sigma + 1 - \mathcal{N}} \frac{1}{\xi} \ , \qquad \frac{\sigma}{\xi} \ .$$

$$\mathscr{P}_{-1} = \sigma\xi \ , \qquad \frac{\sigma - \mathcal{N}}{1 + \mathcal{N}} \xi \ , \qquad 0 \ .$$

# Distribution of Polyakov line eigenvalues at level transitions




The distribution $z(s)$ of the eigenvalues of the Polyakov line is found by inversion of $i\frac{ds}{dz} = \varrho(z)$. In the confined phase (red, green)

$$e^{is} = z(1+\xi z)^{\sigma} \ , \quad e^{is} = \frac{z^{1+\sigma}\xi^{\sigma}}{(1+\xi z)^{\sigma}},$$

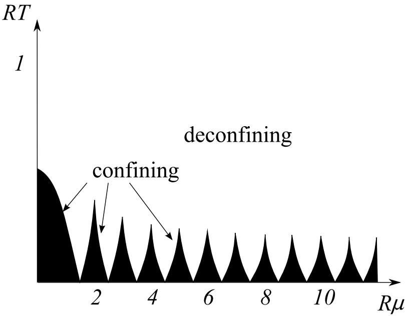
respectively. In the deconfined phase (blue) the distribution has a gap corresponding to zeros in

$$\varrho(z) = \frac{f(z)}{z}\sqrt{(z-\tilde{z})(z-\tilde{z}^*)} \ ,$$

where $\tilde{z}$, $\tilde{z}^*$ are the endpoints of the distribution and

$$f(z) = \frac{\sigma}{(1+\xi z)\left|\frac{1}{\xi}+\tilde{z}\right|} \ .$$

# Phase diagram in the $(\mu R, TR)$-plane in the large $N$ limit for $mR = 0$




Considering multiple levels allows for calculation of the boundaries of the confining regions as a function of temperature and chemical potential. The potential is:

$$V(\theta) = i\mathcal{N}\theta - \sum_{\ell} \sigma_\ell \log\left(1 + \xi_\ell e^{i\theta}\right)$$

So the density is generalized to

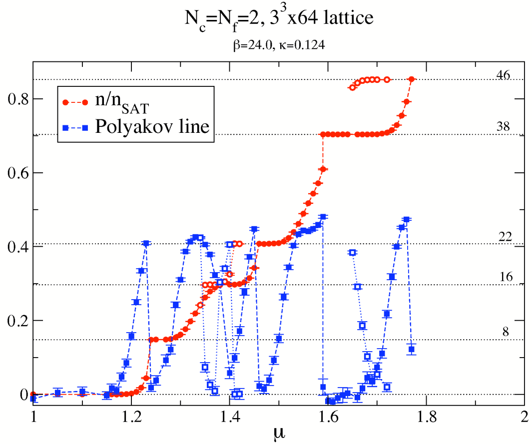
$$\varrho(z) = \frac{\mathcal{N}+1}{z} - \sum_{\kappa \le \ell} \frac{\xi_\kappa \sigma_\kappa}{1 + \xi_\kappa z} + \sum_{\kappa > \ell} \frac{\xi_\kappa \sigma_\kappa}{1 + \xi_\kappa z} \, .$$

The zeros of $\varrho(z)$ determine the $z$ values at which a gap forms. These are plugged into $i\frac{ds}{dz} = \varrho(z)$, in the form

$$e^{is} = z \frac{\prod_{\kappa > \ell} (1 + \xi_\kappa z)^{\sigma_\kappa}}{\prod_{\kappa \le \ell} ((\xi_\kappa z)^{-1} + 1)^{\sigma_\kappa}} \, ,$$

which will then give the critical lines in the $(\mu R, TR)$-plane.

# Preliminary lattice results from 2-color QCD




Simulation results for $N = 2$ QCD confirm the level structure of the fermion number and the associated spikes in the Polyakov line at each level transition. The curious smooth $\rightarrow$ sharp feature of the observables at the transitions needs study to determine if it is a result of larger coupling, or perhaps a result of working on the 4-torus.

# Conclusions

- QCD at non-zero chemical potential on $S^1 \times S^3$ has a complex action which results in the stationary solution lying in the configuration space of complexified gauge fields.
- The fermion number exhibits a level structure as a function of the chemical potential and spatial extent in the low temperature limit.
- The level transitions in the fermion number correspond to spikes in the Polyakov lines such that deconfinement takes place only during a level transition.

# Outlook

- Consider higher loop corrections to obtain effects from increased coupling strength and go beyond the Gaussian approximation in the $1/N$ expansion.
- Formulate a related theory from the gravity side (eg. $\mathcal{N} = 4$ SYM + fundamental flavor branes and chemical potential).
- Perform a lattice calculation of Yang-Mills theory / QCD on $S^1 \times S^3$ to determine the phase diagram as a function of the spatial volume??