Exact results for two-color QCD at low and high density

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Outline

1. Patterns of chiral symmetry breaking in two-color QCD at low and high density
2. Effective Lagrangian and Leutwyer-Smilga-type sum rules
3. Random matrix model
4. Universal spectral correlations from RMT

Introduction

- we study the following theory:
  - two-color QCD
  - fundamental fermions, even N_f
  - nonzero quark chemical potential μ, zero (or very low) temperature
  - Euclidean space

$$\mathscr{L} = \overline{\psi}[iD(\mu) + M]\psi + \tfrac{1}{4}F^2$$

$$D(\mu) = \gamma_\nu D_\nu + \mu\gamma^0$$

M is the quark mass matrix (dimension N_f)

- main differences to QCD:
  - two quarks can form a baryon (color-neutral)
  - no sign problem for pairwise degenerate quark masses
- the phenomenology (pattern of chiral symmetry breaking, condensates, Nambu-Goldstone bosons, etc.) depends on the value of μ
- what happens at small μ is well known Kogut et al. 1999-2001
  our work: mainly large μ

Chiral symmetry

- for $M = 0$ we have $\{\gamma_5, D(\mu)\} = 0$ (chiral symmetry)
- because of the pseudo-reality of SU(2), the Dirac operator has an anti-unitary symmetry Halasz-Osborn-Verbaarschot 1997

$$[A, D(\mu)] = 0 \quad \text{with} \quad A = \tau_2 C \gamma_5 K$$
$$A^2 = \mathbb{1}$$

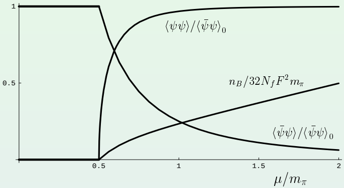
→ chiral orthogonal symmetry class (Dyson index $\beta = 1$)

- this enlarges the flavor symmetry group to $\mathrm{U}(2N_f)$ (at $\mu = 0$)
  $\mu \neq 0$ breaks this to $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A$
- depending on the value of μ, the dynamics of the theory breaks these symmetries in various ways, leading to chiral or diquark condensates

Patterns of chiral symmetry breaking

symmetries realized in various phases of two-color QCD:

Peskin 1980, Kogut et al. 1999-2000



1. $\mu = 0$, $\langle\bar\psi\psi\rangle = 0$, $\langle\psi\psi\rangle = 0$: SU$(2N_f)$
2. $\mu = 0$, $\langle\bar\psi\psi\rangle \neq 0$, $\langle\psi\psi\rangle = 0$: Sp$(2N_f)$
3. $\mu \neq 0$, $\langle\bar\psi\psi\rangle = 0$, $\langle\psi\psi\rangle = 0$: SU$(N_f)_L \times$ SU$(N_f)_R \times$ U$(1)_B$
4. $\mu \neq 0$, $\langle\bar\psi\psi\rangle \neq 0$, $\langle\psi\psi\rangle = 0$: SU$(N_f)_V \times$ U$(1)_B$
5. $\mu \neq 0$, $\langle\bar\psi\psi\rangle = 0$, $\langle\psi\psi\rangle \neq 0$: Sp$(N_f)_L \times$ Sp$(N_f)_R$
6. $\mu \neq 0$, $\langle\bar\psi\psi\rangle \neq 0$, $\langle\psi\psi\rangle \neq 0$: Sp$(N_f)_V$

we are mainly interested in asymptotically large μ → case 5

The physics at large μ

- for $\mu \gg \Lambda_{\mathrm{SU}(2)}$ a BCS superfluid of diquark pairs is formed (since there is an attractive channel between quarks near the Fermi surface)
- BCS gap for $\mathrm{SU}(2)_{\text{color}}$: Son 1998, T. Schäfer 1999

$$\Delta \sim \mu g^{-5} \exp\left(-\frac{2\pi^2}{g}\right) \quad \rightarrow \quad \Lambda_{\mathrm{SU}(2)} \ll \Delta \ll \mu$$

- $\mathrm{U}(1)_B$ is broken spontaneously, with a massless NG boson H
- $\mathrm{U}(1)_A$:
  - broken explicitly by the anomaly/instantons, but this effect disappears as $\mu \to \infty$ because instantons are screened in this limit
  - broken spontaneously by the diquark condensate

→ treat η' as NG boson

- $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ is broken spontaneously to $\mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$
  → $N_f^2 - N_f - 2$ NG bosons = "pions" (none for $N_f = 2$)

Chiral Lagrangian at large μ

- low-energy effective Lagrangian for NG bosons (valid for $p, M \ll \Delta$):

  Kanazawa-Wettig-Yamamoto 2009

$$\mathscr{L} = \frac{f_H^2}{2}\left\{|\partial_0 V|^2 - v_H^2 |\partial_i V|^2\right\} + \frac{N_f f_{\eta'}^2}{2}\left\{|\partial_0 A|^2 - v_{\eta'}^2 |\partial_i A|^2\right\}$$
$$+ \frac{f_\pi^2}{2}\,\mathrm{tr}\left\{|\partial_0 \Sigma_L|^2 - v_\pi^2 |\partial_i \Sigma_L|^2 + (L \longleftrightarrow R)\right\} - c\Delta^2\left\{A^2\,\mathrm{tr}(M\Sigma_R M^T \Sigma_L^\dagger) + \text{c.c.}\right\}$$

  - V corresponds to the H boson → decouples
  - A corresponds to the η'
  - the other NG bosons (“pions”) are in Σ_L and Σ_R
  - $c\Delta^2$ with $c = 3/4\pi^2$ from shift of vacuum energy due to quark mass
- mass formulas for the NG bosons (assuming equal quark masses):

$$m_{\Pi^a} = 0\,, \qquad f_\pi^2 m_{\tilde{\Pi}^a}^2 = 4c\Delta^2 m^2 = f_{\eta'}^2 m_{\eta'}^2$$

with $\Sigma_i = U_i I U_i^T$, $U_i = \exp(\#i\pi^a X^a)$, $I = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

$\Pi^a = \frac{1}{\sqrt{2}}(\pi_L^a + \pi_R^a)$ and $\tilde{\Pi}^a = \frac{1}{\sqrt{2}}(\pi_L^a - \pi_R^a)$

Finite-volume analysis ⟶ epsilon regime

- consider finite Euclidean box $V = L^4$
- zero-momentum modes of the NG bosons dominate if

$$\frac{1}{\Delta} \ll L \ll \frac{1}{m_{\Pi,\tilde{\Pi},\eta'}} \qquad \text{(epsilon-regime of QCD}_2\text{ at large } \mu\text{)}$$

- in this regime, partition function reduces to zero-mode group integrals:

$$Z(M) = \int_{\mathrm{U}(1)_A} dA \int_{\mathrm{SU}(N_f)_L} dU_L \int_{\mathrm{SU}(N_f)_R} dU_R \;\; \exp\left[-Vc\Delta^2\left\{A^2\,\mathrm{tr}(MU_RIU_R^TM^TU_L^*IU_L^\dagger) + \text{c.c.}\right\}\right]$$

- similar to QCD analysis by Leutwyler-Smilga 1992:
  expansion of $Z(M)$ in powers of M and comparison with the fundamental theory (QCD_2) yields sum rules for the Dirac eigenvalues

Leutwyler-Smilga-type sum rules

- symmetries of the Dirac spectrum: eigenvalues come in quadruples $\lambda, -\lambda, \lambda^*, -\lambda^*$ (or in pairs if λ is real or purely imaginary)
- Leutwyler-Smilga-type sum rules:

$$\left\langle \sum_n{}' \frac{1}{\lambda_n^2} \right\rangle = \left\langle \sum_{m<n}{}' \frac{1}{\lambda_m^2 \lambda_n^2} \right\rangle = \left\langle \sum_n{}' \frac{1}{\lambda_n^6} \right\rangle = 0$$

$$\left\langle \sum_n{}' \frac{1}{\lambda_n^4} \right\rangle = (4c\Delta^2 V_4)^2 \frac{1}{4(N_f-1)^2}$$

  - ${}'$ means $\mathrm{Re}\,\lambda_n > 0$
  - sum rules have also been generalized to massive quarks
  - scale determined by BCS gap Δ: $\lambda_{\text{min}} \sim 1/\Delta\sqrt{V_4}$
    → sum rules can be used to extract Δ from lattice simulations
- $D(\mu) = \gamma_\nu D_\nu + \mu\gamma^0$ → naive expectation at large μ: $\lambda_n \sim \mu$
  but we have a Fermi surface → typical momentum ∼ Fermi momentum
  → delicate cancellations between $\gamma_\nu D_\nu$ and $\mu\gamma^0$

Microscopic spectral density

- the sum rules are a typical universal result (due to global symmetries)
  → can be derived from random matrix theory (RMT)
- spectral density of Dirac operator:

$$\rho(\lambda) = \left\langle \sum_n \delta^2(\lambda - \lambda_n) \right\rangle$$

- generating function of sum rules: microscopic spectral density

$$\rho_s(z) = \lim_{V_4 \to \infty} \frac{\pi^2}{3\Delta^2 V_4} \rho\left(\frac{\pi z}{\sqrt{3\Delta^2 V_4}}\right)$$

→ need to construct random matrix model and compute $\rho_s(z)$ from it

Random matrix models at small μ

- two-matrix model for three colors: Osborn 2004

$$D(\mu) = \begin{pmatrix} 0 & iA + \mu B \\ iA^\dagger + \mu B^\dagger & 0 \end{pmatrix}$$

  with complex random $(N + \nu) \times N$ matrices A and B
  (ν = topological charge)

- analogously for two colors: Akemann-Phillips-Sommers 2009

$$D(\mu) = \begin{pmatrix} 0 & A + \mu B \\ -A^T + \mu B^T & 0 \end{pmatrix}$$

  with real random $(N + \nu) \times N$ matrices A and B

- what is the random matrix model at large μ?
  the scale in the 0-dimensional effective Lagrangian is set by Δ, not μ
  → μ should not be an explicit parameter in the random matrix model

Random matrix model at large μ

- the model at large μ is given by Kanazawa-Wettig-Yamamoto 2010

$$D(\mu) = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

with real random $N \times N$ matrices A and B
(ν could be included, but topology is suppressed at large μ)

- we have

$$\det{}^{N_f} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \det{}^{N_f/2} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \det{}^{N_f/2} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$$

  - for each of the two factors on the RHS, chiral symmetry is broken according to $\mathrm{SU}(N_f) \to \mathrm{Sp}(N_f)$ Halasz-Verbaarschot 1995
    → for even N_f, we have the desired symmetry breaking pattern for case 5:
    $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$

Effective Lagrangian from random matrix model

- the complete random matrix model is

$$Z(M) = \int dA\,dB\, e^{-N\,\mathrm{tr}(AA^T + BB^T)} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f\,\mathbb{1} & A \\ B & m_f\,\mathbb{1} \end{pmatrix}$$

- now some standard manipulations:
  - $\det$ = Grassmann integral
  - integration over A and B
  - Hubbard-Stratonovich transformation
  - integration over Grassman variables
  - saddle-point approximation (large parameter = N)

this reproduces exactly the static limit of $\mathscr{L}_{\text{eff}}$ if we identify

$$\underset{\nearrow}{N m_{\text{RMT}}^2} = \frac{3}{4\pi^2} V_4 \underset{\nwarrow}{\Delta^2 M^2}$$

dimensionless RMT quantities physical quantities

Relation between the random matrix models at small and large μ

- our model is actually the $\mu \to 1$ limit of the model at small μ
- at small μ, the static limit of the relevant effective Lagrangian also follows correctly from RMT

→ the same matrix model describes two different cases ($\mu \ll \Lambda_{\mathrm{SU}(2)}$ and $\mu \gg \Lambda_{\mathrm{SU}(2)}$) with two completely different patterns of spontaneous chiral symmetry breaking:

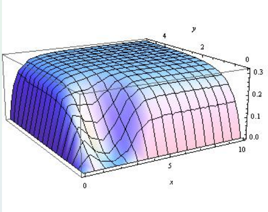
case 4:	$\mu \sim \frac{1}{\sqrt{V}}$	$N m_{\mathrm{RMT}} \sim V\Sigma M$
case 5:	$\mu \gg \Lambda_{\mathrm{SU}(2)}$	$N m_{\mathrm{RMT}}^2 \sim V\Delta^2 M^2$

Microscopic spectral density

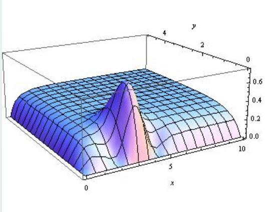
- in addition to the Dirac eigenvalues in the complex plane, there are eigenvalues that are exactly on the real/imaginary axis
  → there is a density of complex eigenvalues and a density of real/imaginary eigenvalues
- recently, we have computed the microscopic spectral densities
  - to be published soon
  - in collaboration with Akemann-Phillips
- results obtained:
  - $\rho_s^{\mathbb{C}}(z)$ and $\rho_s^{\mathbb{R}}(x)$ / $\rho_s^{i\mathbb{R}}(y)$
  - quenched and unquenched
  - dependence on topological charge ν
  - limits of weak and strong/maximum non-hermiticity

Some examples of ρ_s

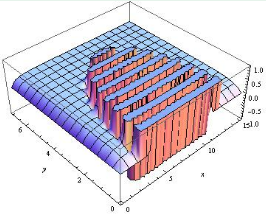
- $N_f = 2$, maximum non-hermiticity ($\mu \gg \Lambda_{\mathrm{SU(2)}}$)



$m_1 = m_2$
(no sign problem)



$m_1 \approx m_2$
(mild sign problem)



$m_1 \gg m_2$
(strong sign problem)

- can study sign problem by detuning the quark masses
  → rapid oscillations within an ellipse (as in QCD with three colors)
  Akemann-Osborn-Splittorff-Verbaarschot 2005

Summary and outlook

Summary:

- constructed effective Lagrangian and RMT for QCD$_2$ at large μ
- random matrix model is the “large-μ” limit of the previously known model at small μ, even though the physics is completely different
- new analytical results in the epsilon-regime of QCD$_2$ at large μ:
  - Leutwyler-Smilga-type sum rules
  - microscopic spectral density
- analytical results can be used to extract BCS gap Δ from lattice data

Outlook:

- add diquark source to effective Lagrangian and matrix model $\rightarrow \langle\psi\psi\rangle$
- study spectrum of $DD^\dagger$
- lattice simulations using staggered adjoint fermions at various μ

Backup slide

- page 5:

$$\langle\bar{\psi}\psi\rangle = -\frac{\partial E_{\text{vac}}}{\partial m} \qquad \langle\psi\psi\rangle = -\frac{\partial E_{\text{vac}}}{\partial j} \qquad n_B = -\frac{\partial E_{\text{vac}}}{\partial \mu}$$

- page 5:
  $\psi\psi \equiv \psi^T C\gamma_5\psi$ (scalar diquark, favored over pseudoscalar diquark by instanton-induced interaction)

- page 6:
  gluons are light at large μ (even lighter than η') T. Schäfer 2002
  but don't couple to NG bosons in leading order (weak coupling)