Loop Gas Formulation of Supersymmetric Quantum Mechanics

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Outline

Supersymmetric Quantum Mechanics on the Lattice

- Consider the Lagrangian for supersymmetric quantum mechanics

$$\mathcal{L} = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \overline{\psi}\left(\frac{d}{dt} + P''(\phi)\right)\psi\,,$$

  - real commuting bosonic 'coordinate' ϕ,
  - complex anticommuting fermionic 'coordinate' ψ,
  - $P(\phi)$ superpotential and $P'(\phi) \doteq \frac{dP}{d\phi}$
- Two supersymmetries in terms of Majorana fields $\psi_{1,2}$:

$$\begin{aligned} \delta_A\phi &= \psi_1\varepsilon_A, & \delta_B\phi &= \psi_2\varepsilon_B, \\ \delta_A\psi_1 &= \tfrac{d\phi}{dt}\varepsilon_A, & \delta_B\psi_1 &= -iP'\varepsilon_B, \\ \delta_A\psi_2 &= iP'\varepsilon_A, & \delta_B\psi_2 &= \tfrac{d\phi}{dt}\varepsilon_B. \end{aligned}$$

Naive Discretisation of SUSY QM

- Define fields on lattice sites $x = na$, $n = 0, \ldots, L-1$.
- Eliminate fermion doubling using the Wilson operator which in $d = 1$ simplifies to the backward derivative

$$\nabla^- \psi_x = \psi_x - \psi_{x-a}$$

- Use the same derivative for the bosonic fields
- SUSY variation δ_A leads to

$$\delta_A S_L = i\varepsilon_A \sum_x \psi_2 \left(-\nabla^+ P' + P'' \nabla^- \phi \right)$$

  - remains because of the absence of the Leibniz rule on the lattice
  - term is $\mathcal{O}(a)$ and vanishes in the naive continuum limit

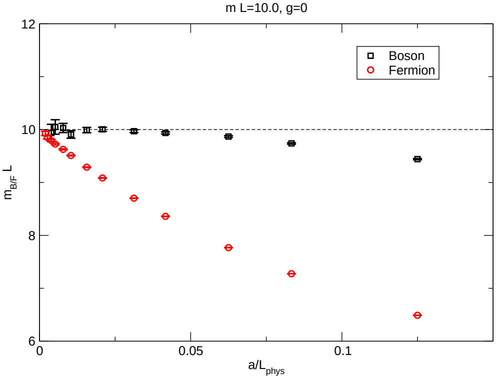
Unbroken Supersymmetry

- Superpotential $P(\phi)$ must be of even degree
- Consider the simplest case

$$P(\phi) = \frac{1}{2} m\phi^2 + \frac{1}{4} g\phi^4$$

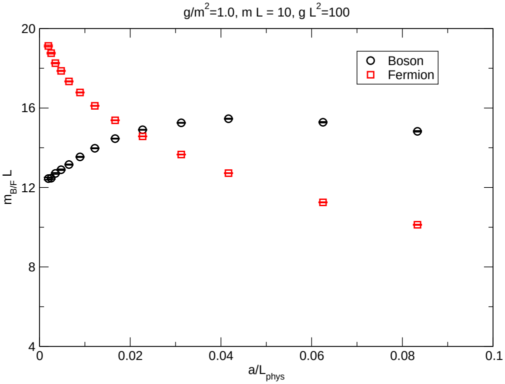
- Use loop gas formulation for the simulations
  - Expand both the bosonic and fermionic fields
  - Sample the two point function while updating the configurations

Naive Discretisation at $g = 0$



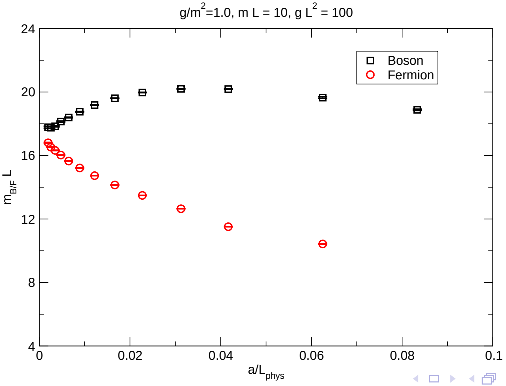
Naive Discretisation at $g \neq 0$

- The superpotential $P(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$ generates interaction term $\propto g\overline{\psi}\psi\phi^2$:



Naive Discretisation at $g \neq 0$ with Counterterm

- Radiative corrections spoil the continuum limit
  - Theory is superrenormalisable
  - Divergencies can be tuned away by adding counter term $\frac{1}{2}\sum P'' = \frac{1}{2}\sum 3g\phi^2$



Exact Twisted Lattice Supersymmetry

- Find a combination of supersymmetries which can be transferred to the lattice.
- Recall symmetry breaking of the lattice action:

$$\begin{aligned} \delta_A S_L &= i\varepsilon_A \sum_x \psi_2 \left(-\nabla^+ P' + P'' \nabla^- \phi\right) \\ &= -i\delta_B \sum_x P' \nabla^- \phi. \end{aligned}$$

- Note the similar term for δ_B,

$$\delta_B S_L = i\delta_A \sum_x P' \nabla^- \phi,$$

so linear the linear combination $\delta \equiv \delta_A + i\delta_B$ gives

$$\delta S_L = -\delta \sum_x P' \nabla^- \phi.$$

- Correction term $P' \nabla^- \phi$ is a surface term vanishing in the limit $a \to 0$.
- Corrected lattice action is invariant under the 'twisted' supersymmetry δ:

$$S_L^{\text{exact}} = \sum_x \frac{1}{2} \left(\nabla^- \phi\right)^2 + \frac{1}{2} P'^2 + \overline{\psi} \left(\nabla^- + P''\right) \psi + P' \nabla^- \phi$$

- Remark: The bosonic action can also be written as

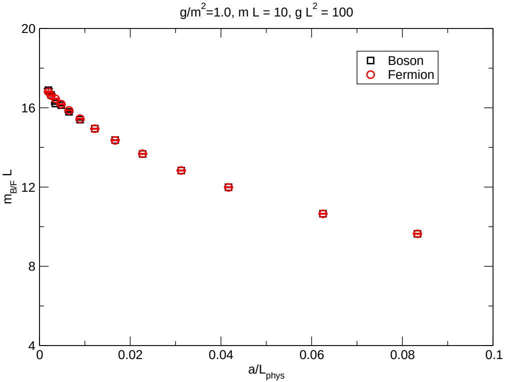
$$S_B^{\text{exact}} = \sum_x \frac{1}{2} \left(\nabla^- \phi + P'\right)^2$$

which exposes the relation to a (local) Nicolai map

  - variable transformation $\phi \to \mathcal{N} = \nabla^- \phi + P'(\phi)$,
  - action becomes Gaussian,
  - Jacobian cancels exactly the fermion determinant.

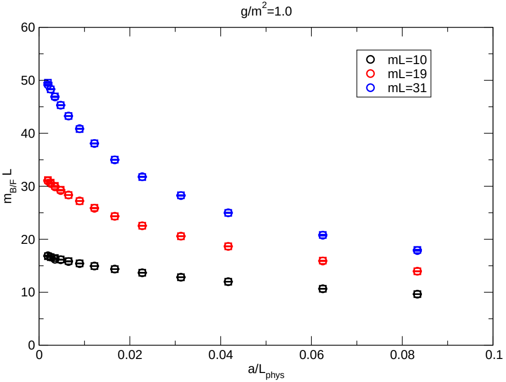
Q-exact Discretisation at $g \neq 0$

- Now simulate this SUSY-exact (or Q-exact) action:



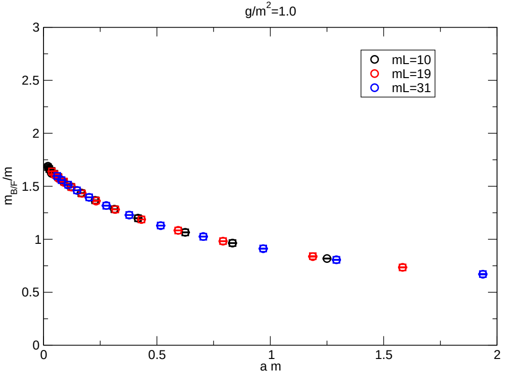
SUSY Q-exact simulation in the continuum limit $L \to \infty$

- Simulate different situations for mL



SUSY Q-exact simulation in the continuum limit $a \cdot m \to 0$

- Reprint the same graph in terms of $m_{B/F}/m$ versus am



Broken Supersymmetry

- Witten index provides a necessary but not sufficient condition for SSB:

$$W \equiv \lim_{\beta\to\infty} \mathrm{Tr}(-1)^F \exp(-\beta H) \quad \Rightarrow \begin{cases} = 0 & \text{SSB may occur} \\ \neq 0 & \text{no SSB} \end{cases}$$

- In SUSY QM: Index equivalent to partition function with periodic boundary conditions e.g.

$$W = Z_{\text{per}} = Z_0 - Z_1$$

  - Z_0 configurations without fermion
  - Z_1 configurations with fermion

Naive Discretisation with Counterterm

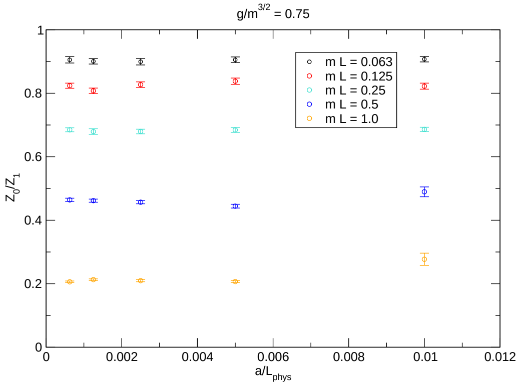
- Superpotential $P(\phi)$ must be of odd degree
- Consider the simplest case

$$P(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3$$

- Use loop gas formulation for the simulations
  - Add counterterm $\frac{1}{2}\sum P''(\phi) = \frac{1}{2}\sum 2g\phi$ to reach the continuum limit
  - Loop gas formulation does not suffer from sign problem $\rightarrow$ simulations of broken SUSY possible!

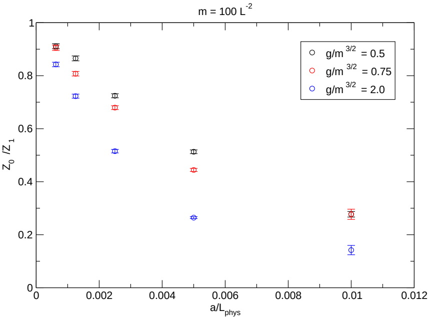
Continuum Limit with $mL = \text{constant}$

- Simulate Z_0/Z_1 for $m \cdot L = \text{const}$
- Expect $Z_0/Z_1 = 1$ in the continuum limit



Continuum Limit

- Take the continuum limit by scaling $a \cdot m \to 0$ faster than $L \to \infty$



Conclusion and Ongoing Work

- Loop gas formulation works for all tested unbroken supersymmetric systems
- We are able to simulate to an arbitrary precision
- Loop gas formulation also allows for studies of broken supersymmetry
- Careful treatment of the continuum limit in the broken case necessary

- Further investigation of the broken case needed (correlation functions, masses, etc...)