Extracting resonance parameters from lattice data (Part I)

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PLAN

- Introduction: Stable/Unstable particles
- Non-interacting/Interacting particles
- Avoid Level Crossings
- Probability distribution method
- Test on the $O(4)$ model
- Numerical results
- Conclusions, outlook

PARTICLE MASSES

How we can determine the mass of particles on the lattice?

Stable particles (Partial Fourier Transform):

$$C(t,\vec{p}) = \int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\vec{p}\vec{x}} G(\vec{x},t;\vec{0},0) = \int \frac{d\omega}{2\pi} e^{i\omega t} \tilde{G}(\vec{p},\omega) \qquad \Rightarrow \qquad C(t,\vec{p}=0) \propto e^{-mt}$$

Unstable particles ($\phi \rightarrow 2\pi$, $m_\phi > 2m_\pi$)

- $C(t,\vec{p}=0) \propto e^{-2m_\pi t}$
- resonances do not correspond to isolated energy levels
(width related to a complex pole in the prop)
- on the lattice the energy spectrum is always REAL $\Rightarrow$ Isolated Levels!

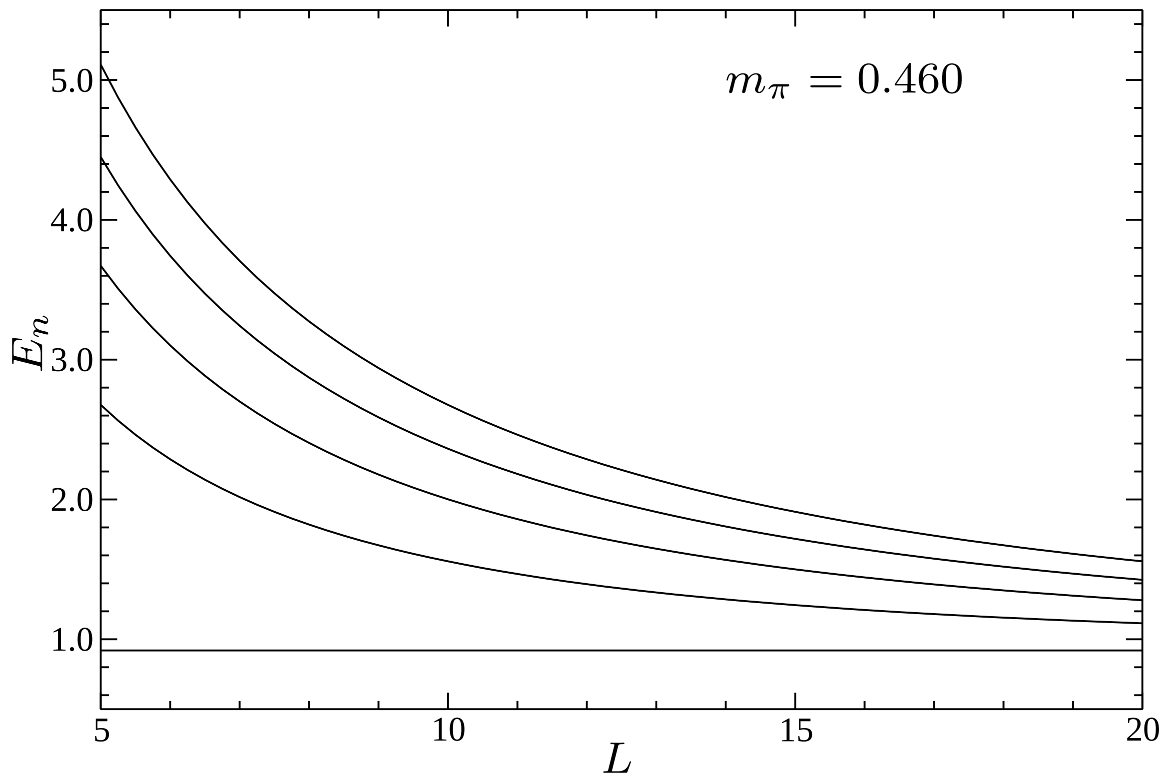
What we can see in the presence of a resonance on the lattice?

A rearrangement of the energy levels takes place (Avoid Level Crossing) related to a mixing between ϕ and 2π

NON INTERACTING PARTICLES

Consider a system of two non-interacting particles (opposite momentum) in a box

Two-particle energy spectrum ($V = L^3$):



$$E = 2\sqrt{m_\pi^2 + p^2}$$

$$p^2 = (2\pi)^2 \sum_{i=1}^{3} \frac{n_i^2}{L_i^2}$$

$$E_n = 2\sqrt{m_\pi^2 + (2\pi)^2 \sum_{i=1}^{3} \frac{n_i^2}{L_i^2}}$$

$\vec{n} = (0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0)$

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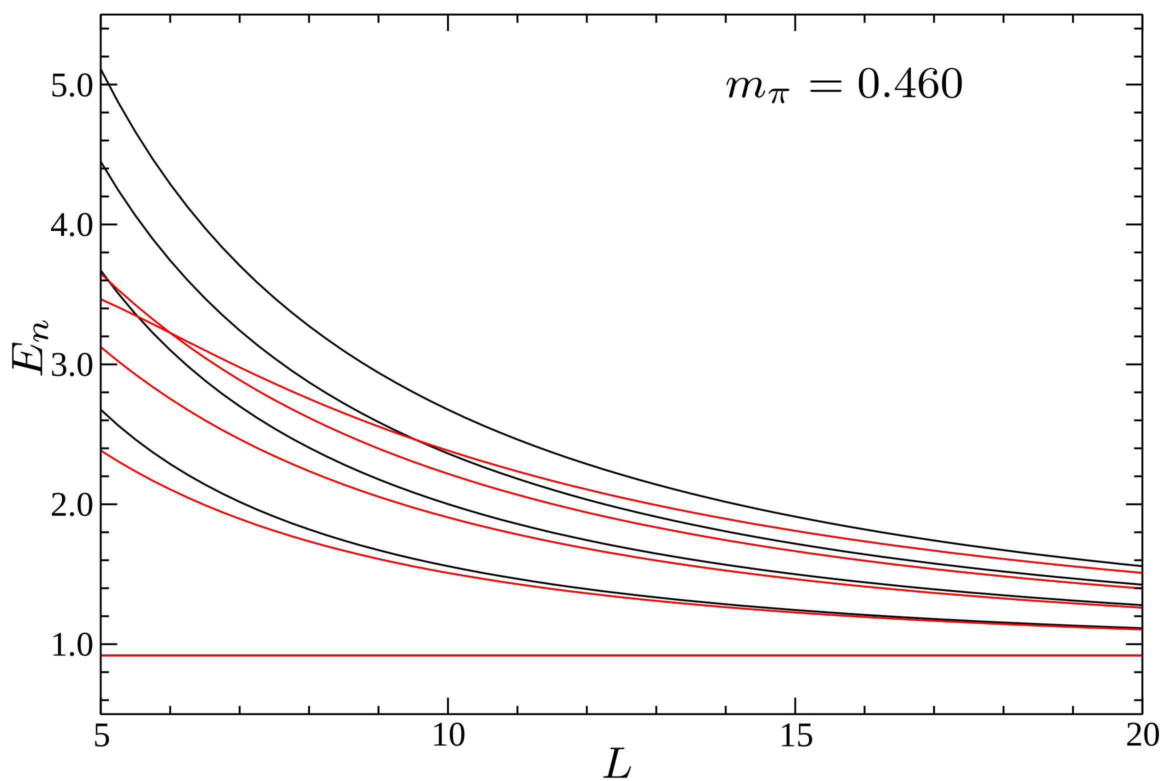


$$E = 4\sinh^{-1}\left[\frac{1}{2}\sqrt{m_\pi^2 + p^2}\right]$$

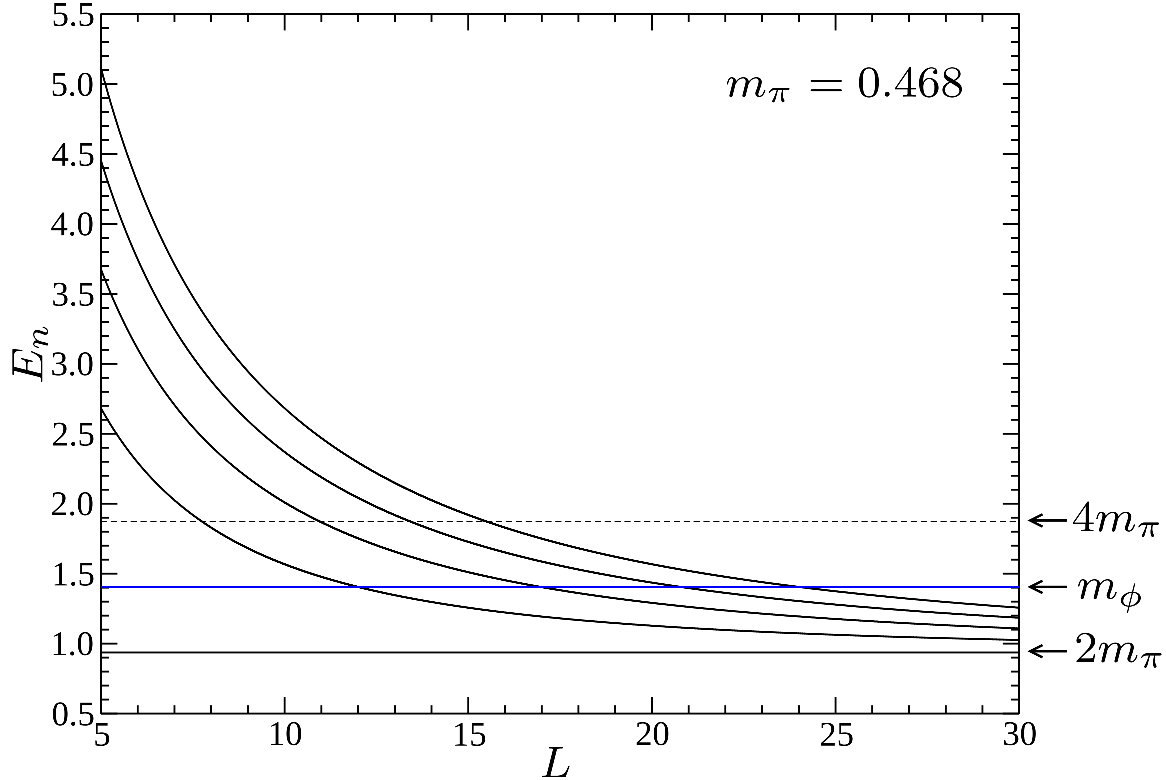
$$p^2 = 4\sum_{i=1}^{3}\sin\frac{\pi}{L_i}n_i$$

$\vec{n} = (0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0)$

NON INTERACTING PARTICLES

Introducing a new particle ϕ (at rest) interacting with π

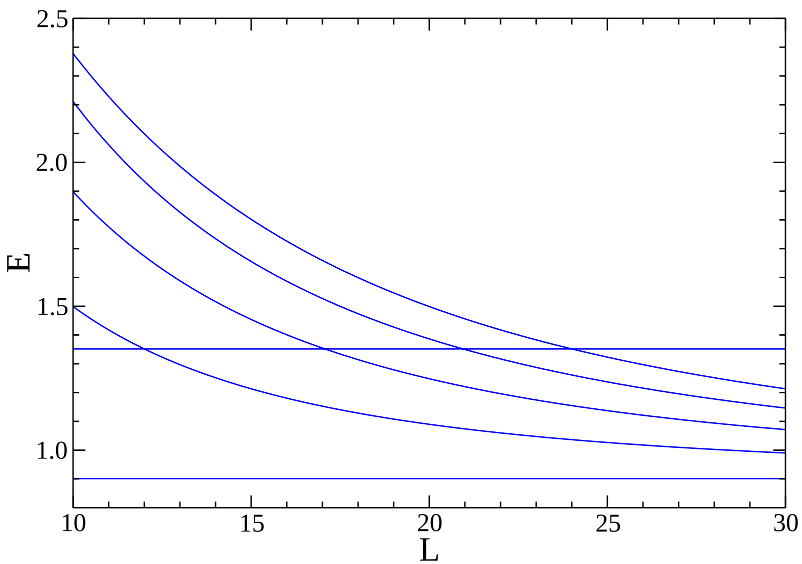
$2m_\pi < m_\phi < 4m_\pi \Rightarrow$ Elastic two-particle scattering



INTERACTING PARTICLES

Two-particle energy spectrum: $E = 2\sqrt{m_\pi^2 + p^2}$, but now $p_i \neq \frac{2\pi}{L_i} n$

Avoid Level Crossings (ALC)



Lüscher's formula:

$$\tan \delta_{l=0}(p) = \frac{\pi^{3/2} q}{Z_{00}(1; q^2)}$$

$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n} \in Z^3} (\vec{n}^2 - q^2)^{-s}$$

- $q = \frac{pL}{2\pi}$

- l angular momentum of scattering channel

It relates the infinite-volume elastic phase shift δ_0 to the finite-volume two-particle energy spectrum

$\delta_l(p) \Rightarrow$ the resonance parameters: $\tan \delta_l(p) = \frac{\Gamma/2}{m_\phi - E(p)}$

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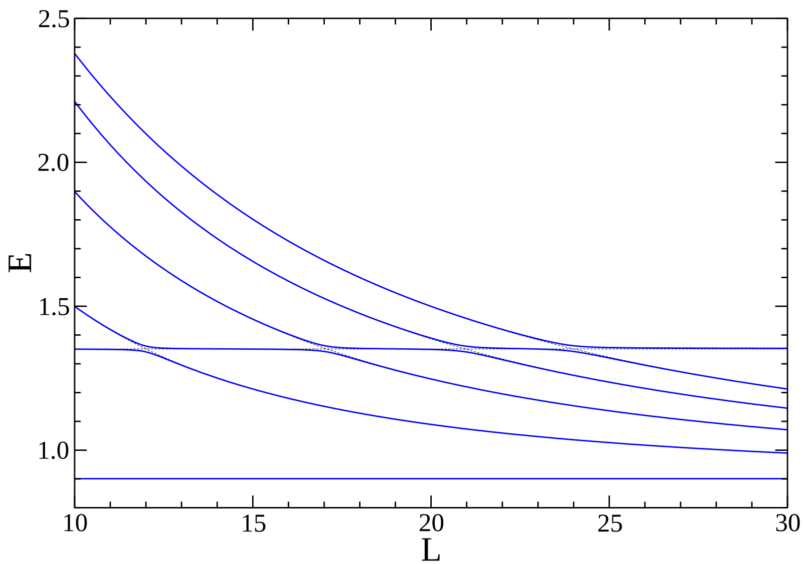

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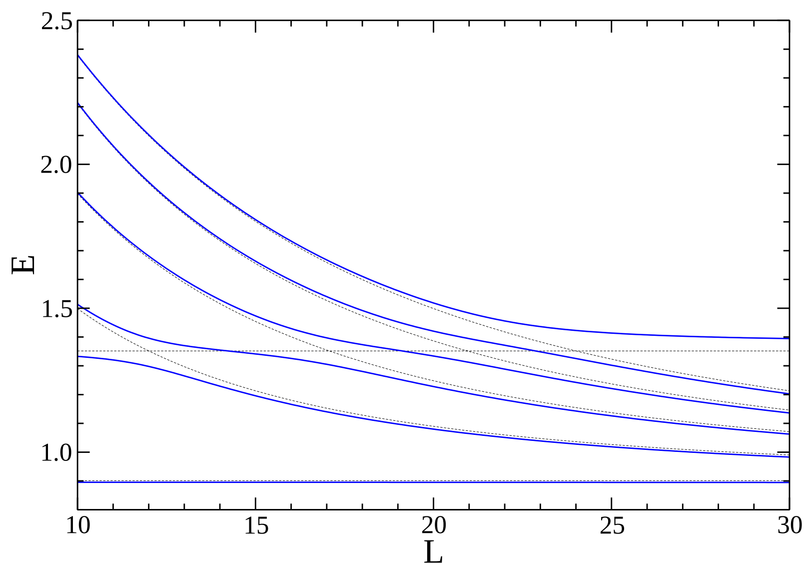

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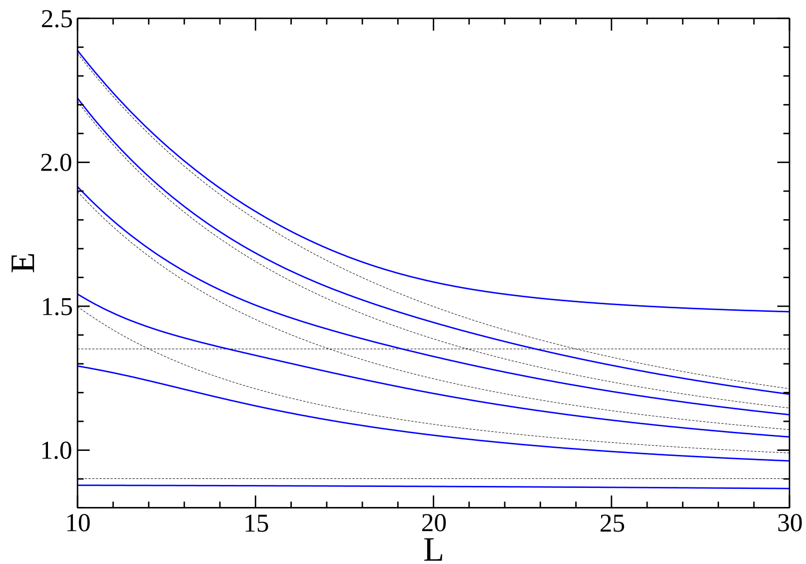

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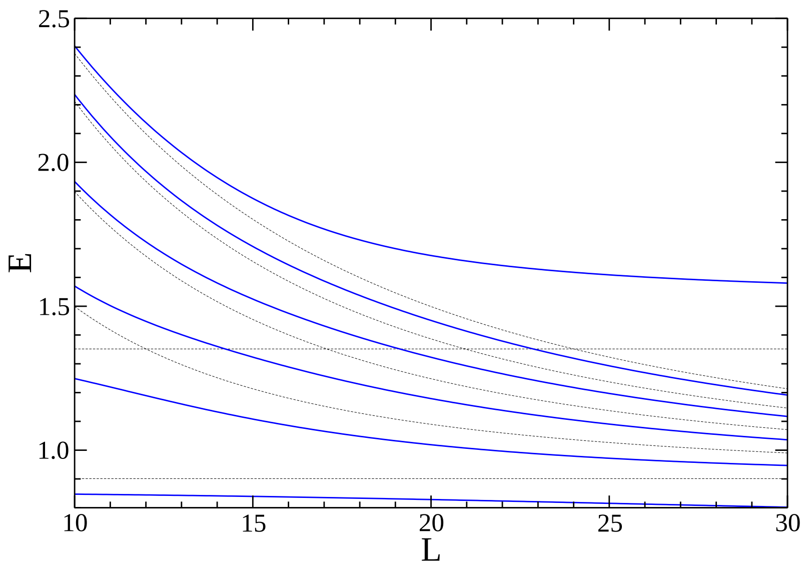

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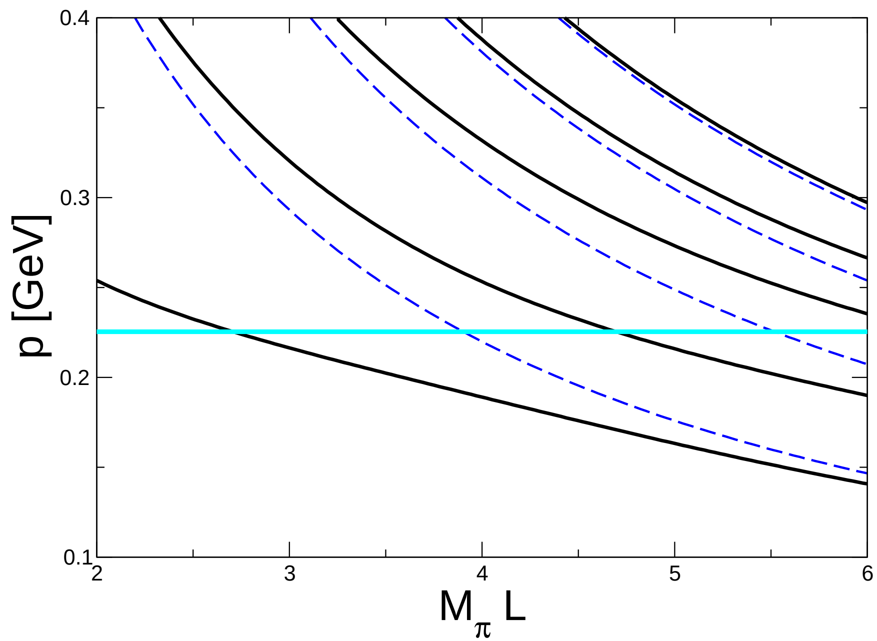

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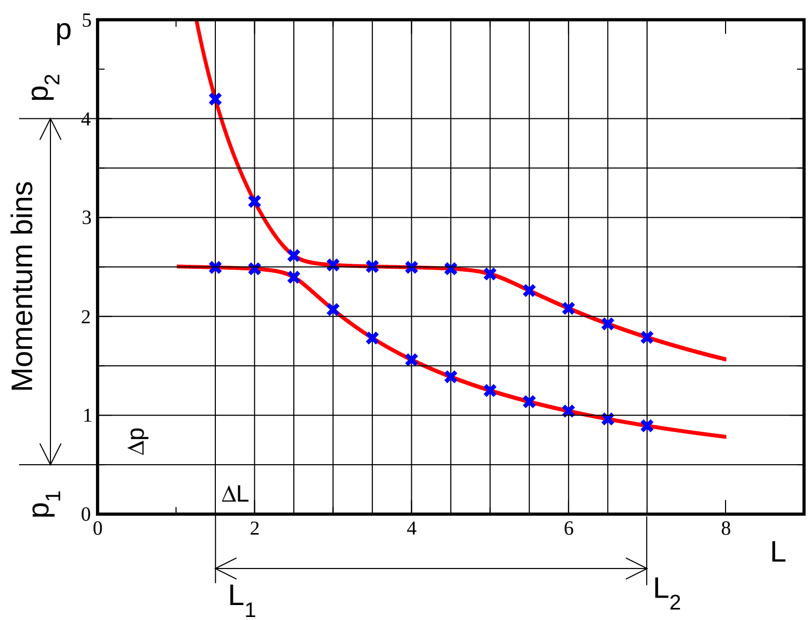
If the width is large the ALC is washed out! $\Rightarrow$ VERY HIGH PRECISION DATA!!! (Δ-resonance) [Eur. Phys. J. A **35** (2008) 281]

A NEW METHOD

[V. Bernard, M. Lage, A. Rusetsky and U. G. Meissner, Eur. Phys. J. A **35** (2008) 281]

They define a probability distribution $W(p) \Rightarrow \lim_{\Delta L \to 0} \frac{\Delta L}{\Delta p} = \frac{1}{p'(L)}$



Prescription:

- Choose the first N levels
- Slice the interval $[L_1, L_2]$ into equal parts ΔL
- Introduce momentum bins Δp in the interval $[p1, p2]$
- For each L identify the i-bin $p_i < p^n(L) < p_i + \Delta p$, $n = 1, \cdots, N$ in which $p^n(L)$ is contained
- Plot the probability distribution $W(p)$ for $p_1 < p < p_2$

A NEW METHOD

Differentiating the Lüscher formula we have:

$$W(p) = C \sum_{n=1}^{N} \left(\frac{L_n(p)}{p} + \frac{2\pi\delta'(p)}{p\phi'(\kappa_n(p))} \right)$$

Using approximations in the Lüscher formula and $N \to \infty$

$$W(p) - W_0(p) \propto \frac{1}{p} \left(\frac{\delta(p)}{p} - \delta'(p) \right)$$

$W_0(p)$ is the background when $\delta = 0$ (it comes out from the first term)

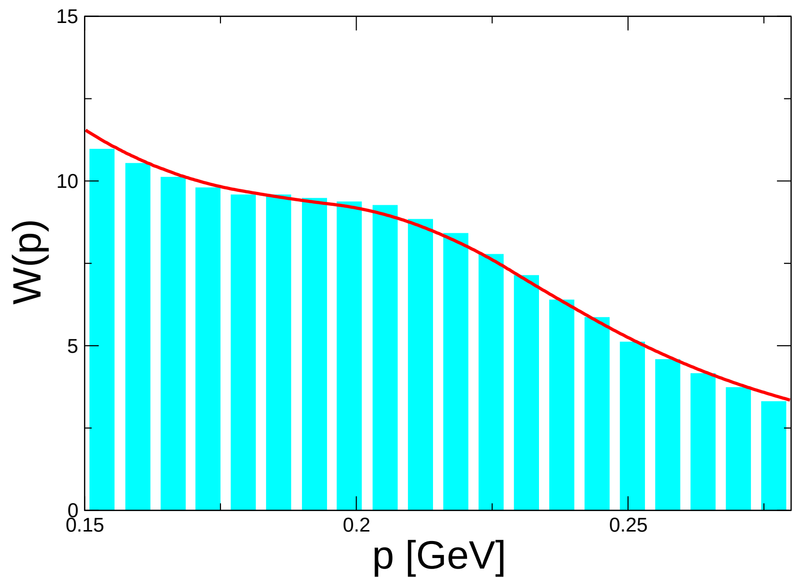
Assuming a smooth dependence on the momentum p, it follows the Breit-Wigner form of the scattering cross section with the SAME WIDTH

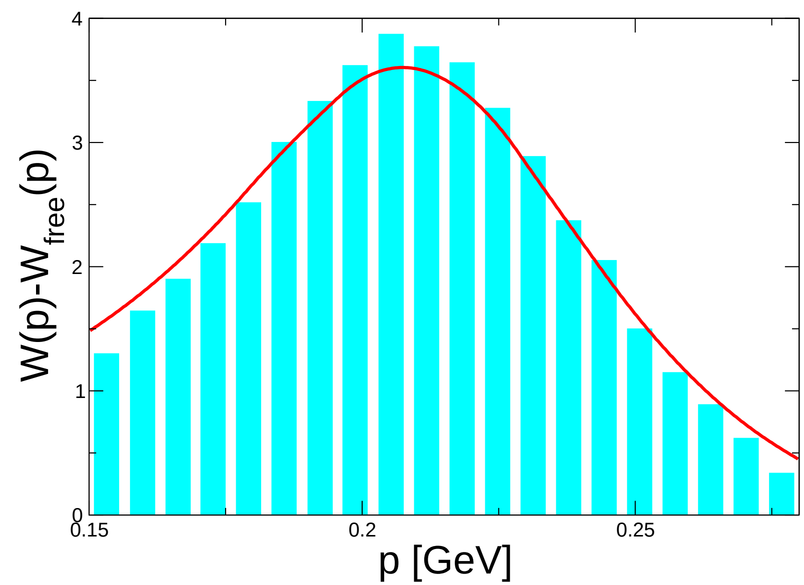
$$W(p) - W_0(p) \propto \frac{1}{[E^2 - m^2]^2 + m^2\Gamma^2}$$

A NEW METHOD

[V. Bernard, M. Lage, A. Rusetsky and U. G. Meissner, Eur. Phys. J. A **35** (2008) 281]

Using experimental phase shifts as input they produced synthetic data and showed it is possible to determine the parameters of a resonance even if the ALCs are washed out





N=1 (!!!!)

They do use the Lüscher formula!

O(4) MODEL in the broken phase

$$\mathcal{L} = \frac{1}{2}\partial\phi_i\partial\phi_i + \lambda(\phi_i^2 - \nu^2)^2 - m_\pi^2\nu\phi_4 \ ,$$ with i=1,2,3,4

We expand it around the classical minimum $\phi_i\phi_i = \nu^2$ (introducing σ and ρ)

$\phi_i = (\nu + \sigma)\rho_i \ ,$ with i=1,2,3,4; (constraint $\rho_i\rho_i = 1$)

$$\mathcal{L} = \frac{1}{2}\nu^2\partial\rho_i\partial\rho_i + \frac{1}{2}\sigma^2\partial\rho_i\partial\rho_i + \frac{1}{2}\partial\sigma\partial\sigma + \nu\sigma\partial\rho_i\partial\rho_i + \lambda\sigma^4 + 4\nu^2\lambda\sigma^2 + 4\nu\lambda\sigma^3 - m_\pi^2\nu^2\rho_4 - m_\pi^2\nu\sigma\rho_4$$

$$U = \exp\left(\frac{i}{f}\pi_i\sigma_i\right) \Rightarrow \frac{1}{2}\mathrm{Tr}\left(\partial_\mu U\partial_\mu U^\dagger\right) \overset{f\to\infty}{\Rightarrow} \frac{1}{f^2}\sum_{i=1}^{3}\partial_\mu\pi^i\partial_\mu\pi^i$$

$$U = \rho_4 + i\sigma_i\tilde{\rho}_i \Rightarrow \frac{1}{2}\mathrm{Tr}\left(\partial_\mu U\partial_\mu U^\dagger\right) = \partial\rho_4\partial\rho_4 + \sum_{i=1}^{3}\partial\tilde{\rho}_i\partial\tilde{\rho}_i = \sum_{i=1}^{4}\partial\rho_i\partial\rho_i \qquad [\rho_4^2 + \tilde{\rho}_i^2 = 1]$$

$$\sum_{i=1}^{4}\partial\rho_i\partial\rho_i \simeq \frac{1}{f^2}\sum_{i=1}^{3}\partial\pi_i\partial\pi_i$$

O(4) MODEL in the broken phase

Redefining the fields π_i: $\tilde{\pi}_i = \pi_i \frac{\nu}{f}$

$$\mathcal{L} = \frac{1}{2}\partial\tilde{\pi}_i\partial\tilde{\pi}_i + \frac{1}{2\nu^2}\sigma^2\partial\tilde{\pi}_i\partial\tilde{\pi}_i + \frac{1}{2}\partial\sigma\partial\sigma + \frac{1}{\nu}\sigma\partial\tilde{\pi}_i\partial\tilde{\pi}_i$$
$$+ \lambda\sigma^4 + 4\nu^2\lambda\sigma^2 + 4\nu\lambda\sigma^3 + \frac{1}{2}m_\pi^2\tilde{\pi}_i\tilde{\pi}_i + \frac{m_\pi^2}{2\nu}\sigma\tilde{\pi}_i\tilde{\pi}_i$$

$$m_\sigma = 2\nu\sqrt{2\lambda}$$

Three-point interaction terms:

$$\frac{1}{\nu}\sigma\partial\tilde{\pi}_i\partial\tilde{\pi}_i$$
$$\frac{m_\pi^2}{2\nu}\sigma\tilde{\pi}_i\tilde{\pi}_i$$

When $\lambda \to \infty$ (and $m_\pi = 0$) $\nu_c \simeq 0.78$

TWO PARTICLE SPECTRUM

$$\tilde{\pi}(\vec{n},t) = \frac{1}{V}\sum_x \pi(\vec{x},t)e^{i\vec{x}\vec{p}} \qquad p_i = \frac{2\pi}{L_i}n_i \qquad n_i = 0,\cdots,L_i-1$$

We introduce operators with zero total momentum and zero isospin:

$$O_{\vec{n}}(t) = \sum_{i=1}^{3}\tilde{\pi}^i(\vec{n},t)\tilde{\pi}^i(-\vec{n},t) \qquad \text{NOTE: } \tilde{\pi}^i(-\vec{n},t) = \left[\tilde{\pi}^i(\vec{n},t)\right]^*$$

In particular (in a cubic lattice):

$$O_{n^2=0}(t) = \sum_{i=1}^{3}\tilde{\pi}^i(0,0,0,t)\tilde{\pi}^i(0,0,0,t)$$

$$\begin{aligned} O_{n^2=1}(t) &= \frac{1}{3}\left[\sum_{i=1}^{3}\tilde{\pi}^i(1,0,0,t)\tilde{\pi}^i(-1,0,0,t)+\tilde{\pi}^i(0,1,0,t)\tilde{\pi}^i(0,-1,0,t)+\right. \\ &+ \left.\tilde{\pi}^i(0,0,1,t)\tilde{\pi}^i(0,0,-1,t)\right] \end{aligned}$$

We consider $n^2 = 0,1,2,3,4$:
$(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, $(2,0,0)$ (all non equivalent permutations and signs)

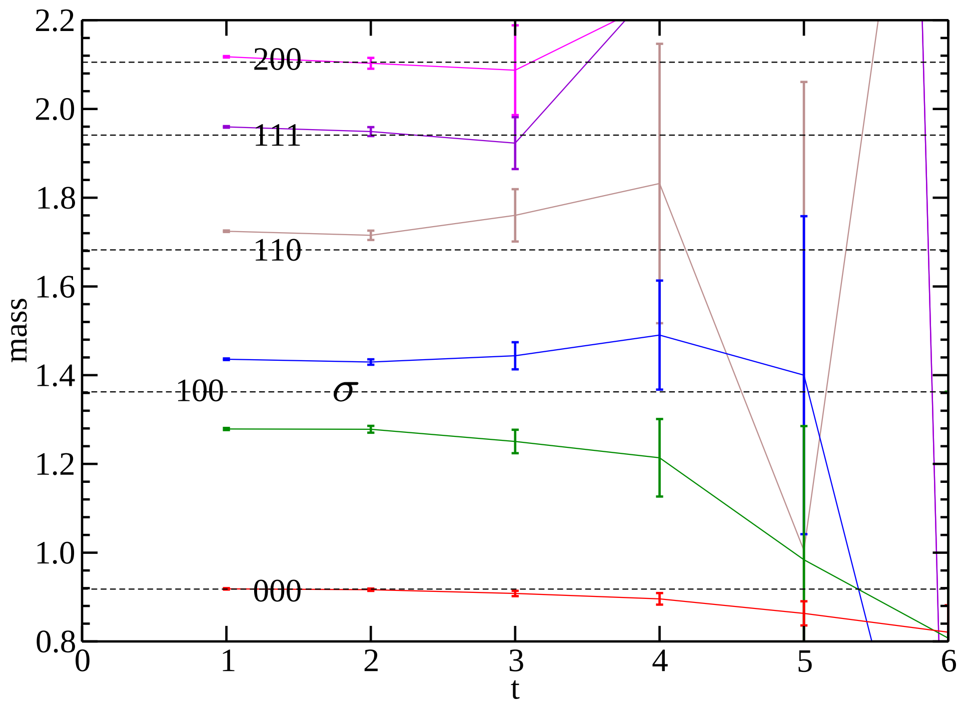
$$O^*(t) = \tilde{\sigma}(t) = \frac{1}{V}\sum_x \sigma(\vec{x},t)e^{i\vec{x}\vec{0}}$$

TWO PARTICLE SPECTRUM

Correlation matrix function $(i, j = 1, \cdots, R)$: $C_{ij}(t) = \langle O_i O_j \rangle$

Generalized eigenvalues problem:

$$C(t)\psi_\alpha = \lambda_\alpha(t, t_0) C(t_0)\psi_\alpha \qquad \lambda_\alpha(t, t_0) = e^{-(t-t_0)m_\alpha}$$



$12^3 \times 48$
$\nu = 1.0$
$\lambda = 1.4$
$m_\pi = 0.36$
data 7×10^5

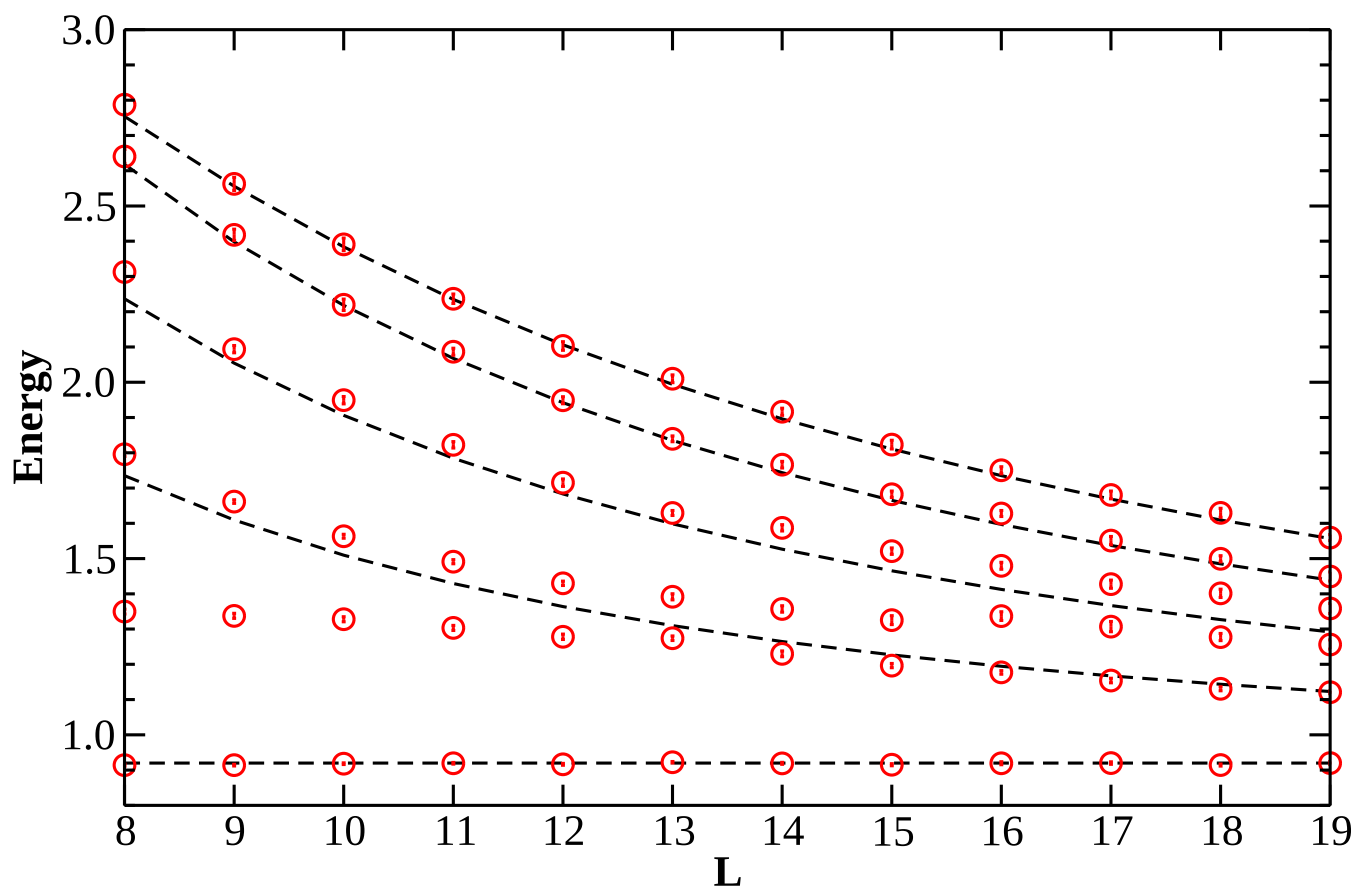
The pions are in a stationary scattering state; if they were free:

$$E_n = 4\sinh^{-1}\left[\frac{1}{2}\sqrt{m_\pi^2 + 4\sin^2(\frac{\pi}{L_x}n_x) + 4\sin^2(\frac{\pi}{L_y}n_y) + 4\sin^2(\frac{\pi}{L_z}n_z)}\right]$$

SPECTRUM EXAMPLE (I)

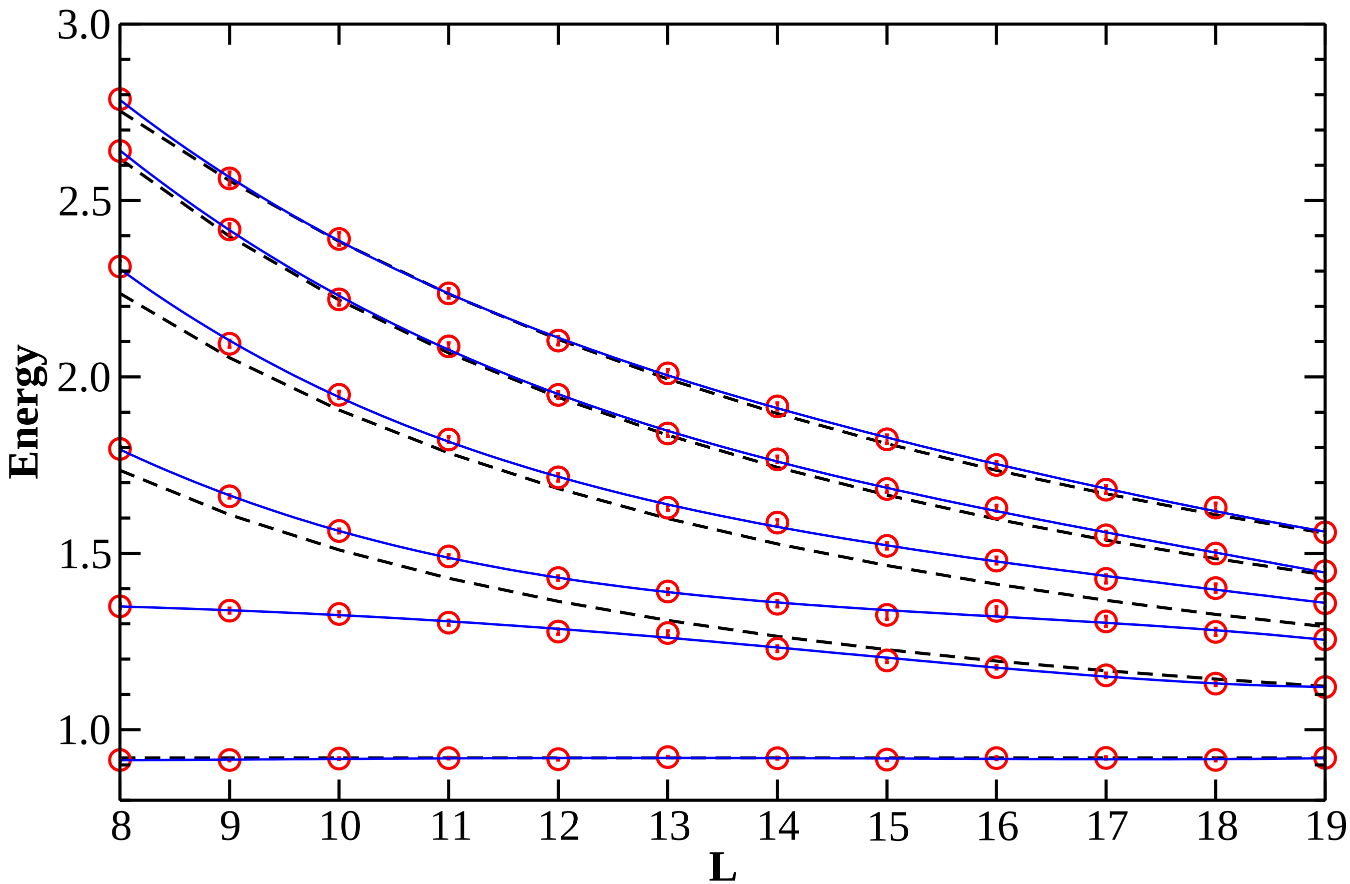
$\nu = 1.0, \lambda = 1.4, m_\pi = 0.36 \Rightarrow m_\pi^R = 0.460(2)$

Relative error: 0.5% - 1.0%



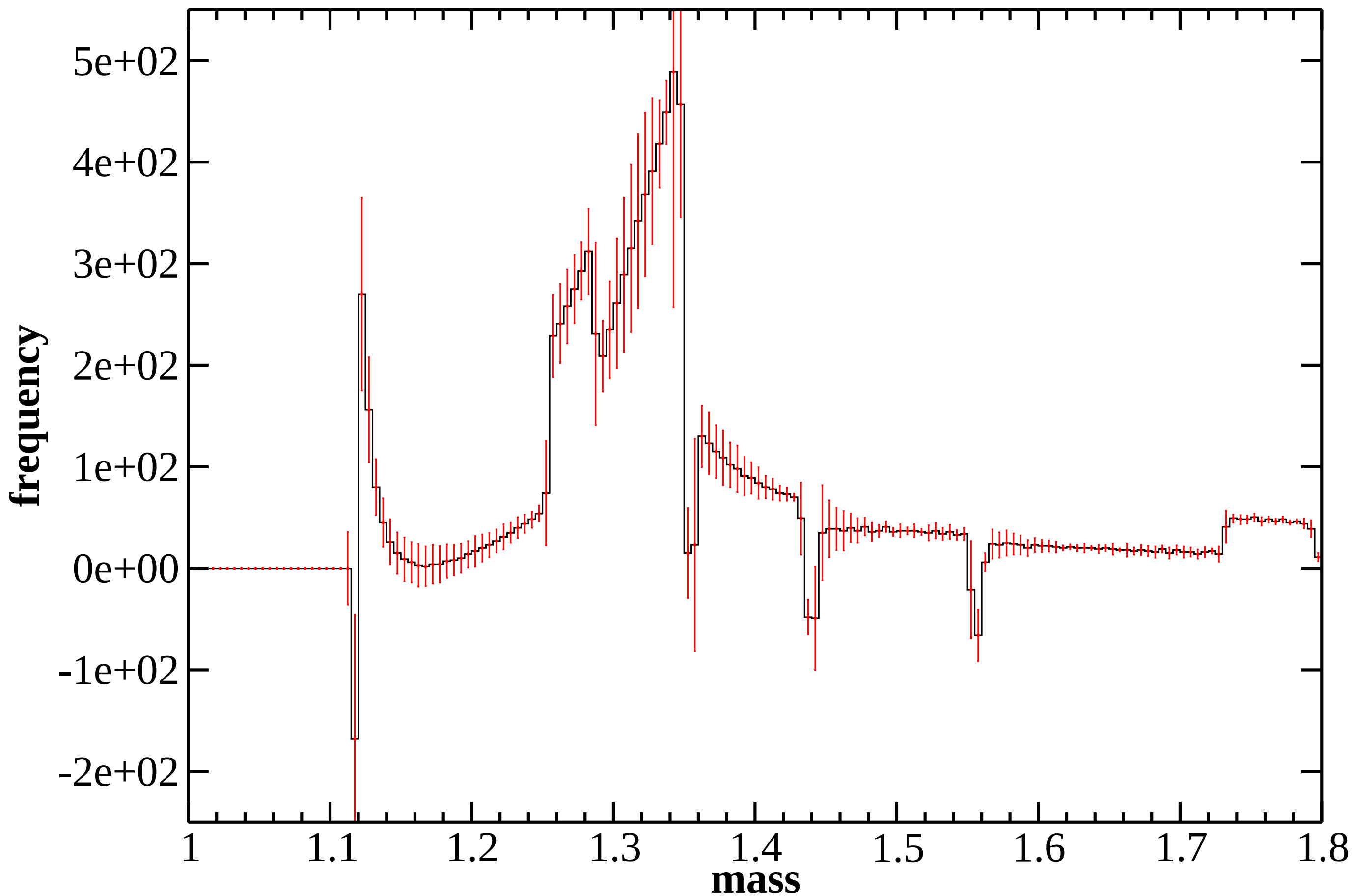
FIT

Order of the fitting polynomials: 3, 4, 5 $\Rightarrow$ systematic errors

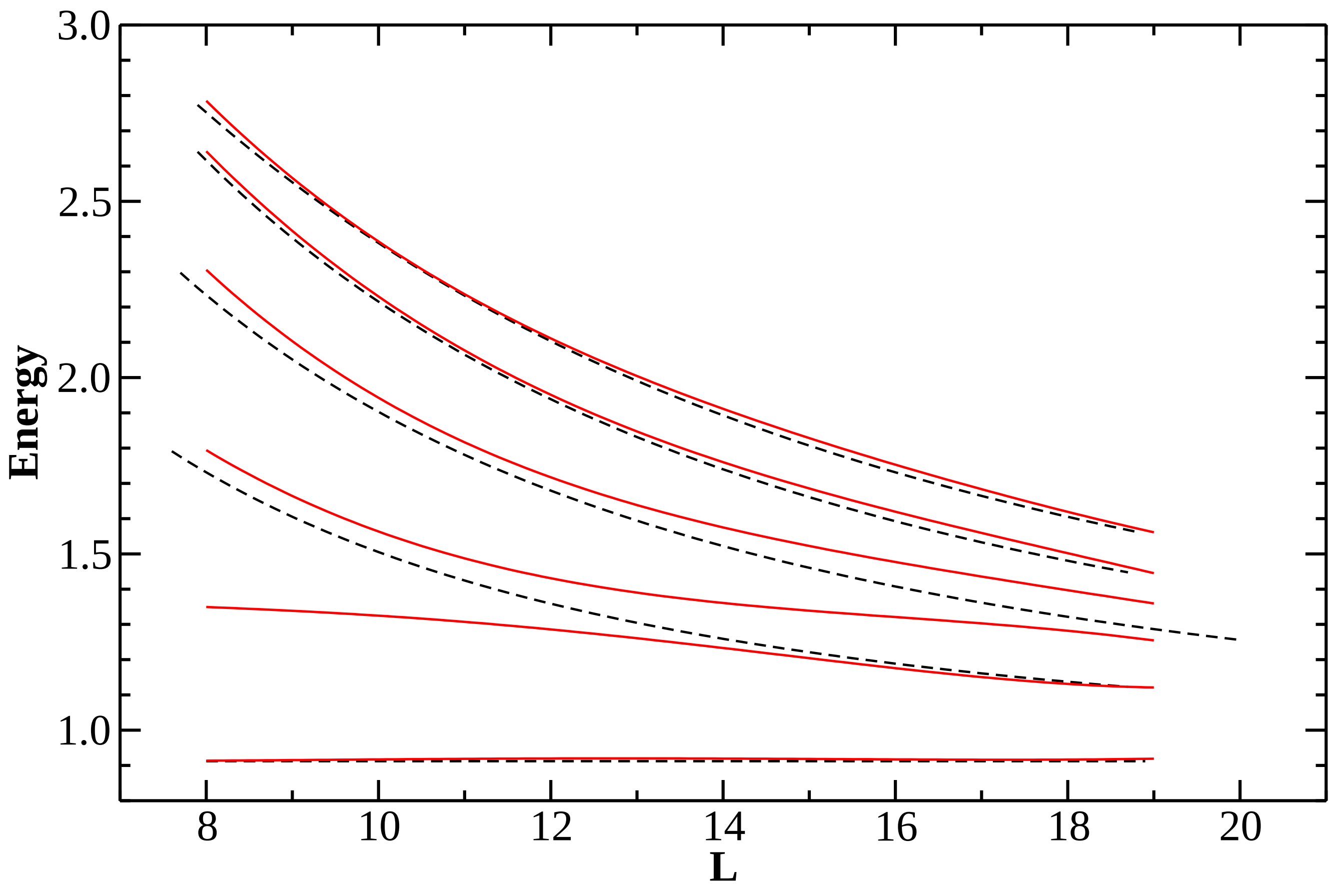


SUBTRACTED HISTOGRAM

A lot of jumps, spikes !!! (intersections with $L = 8$ and $L = 19$)

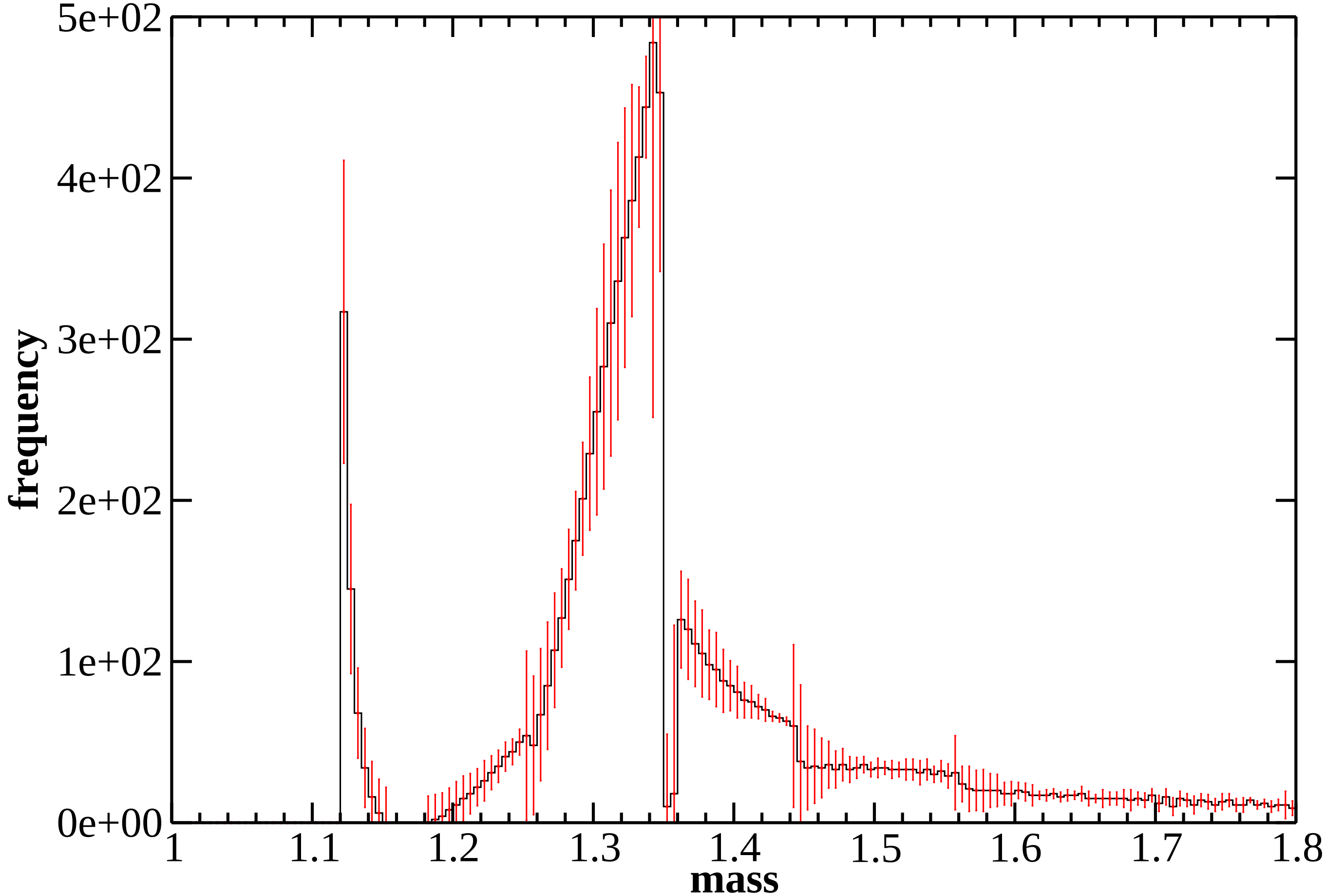


CORRECT SUBTRACTION



HISTOGRAM (I)

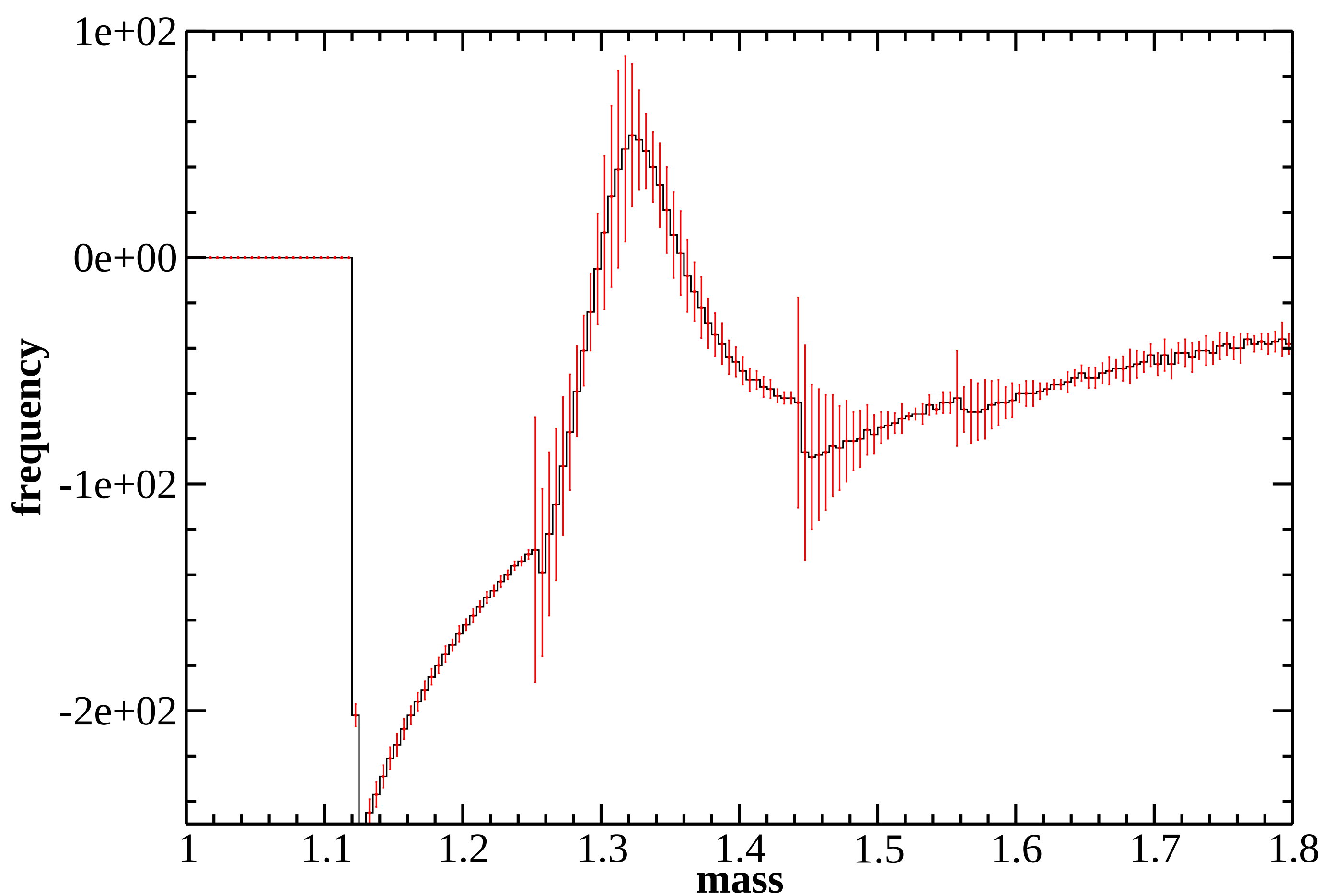
Big jump! no Breit-Wigner shape



HISTOGRAM (II)

If we exclude the 2 levels which are “without” background

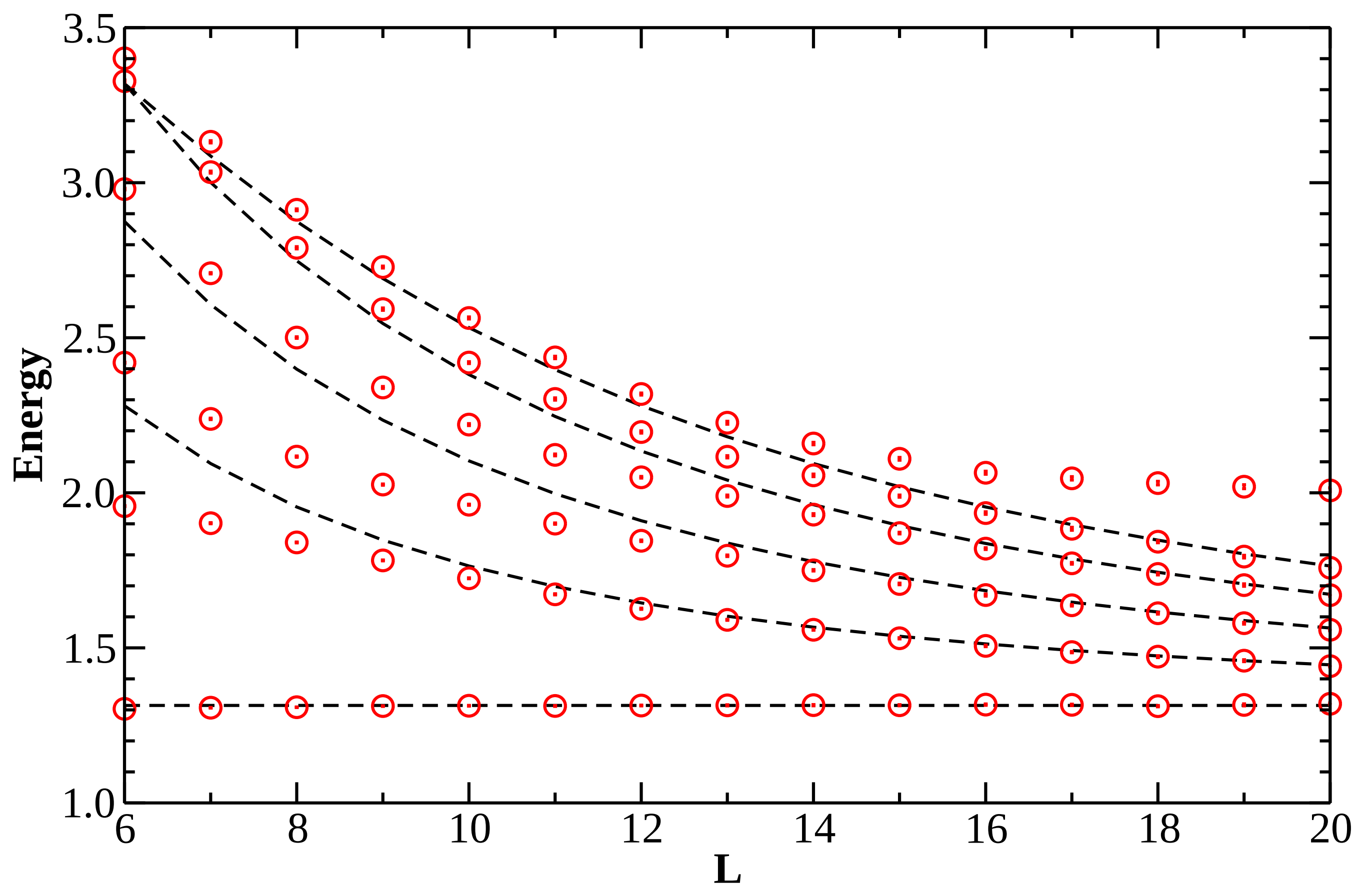
Sliding window fit around the peak: $m = 1.330(5)$, $\Gamma = 0.10(5)$, $\chi^2/dof < 1$



SPECTRUM EXAMPLE (II)

$\nu = 1.0, \lambda = 4.0, m_\pi = 0.56 \Rightarrow m_\pi^R = 0.657(3)$

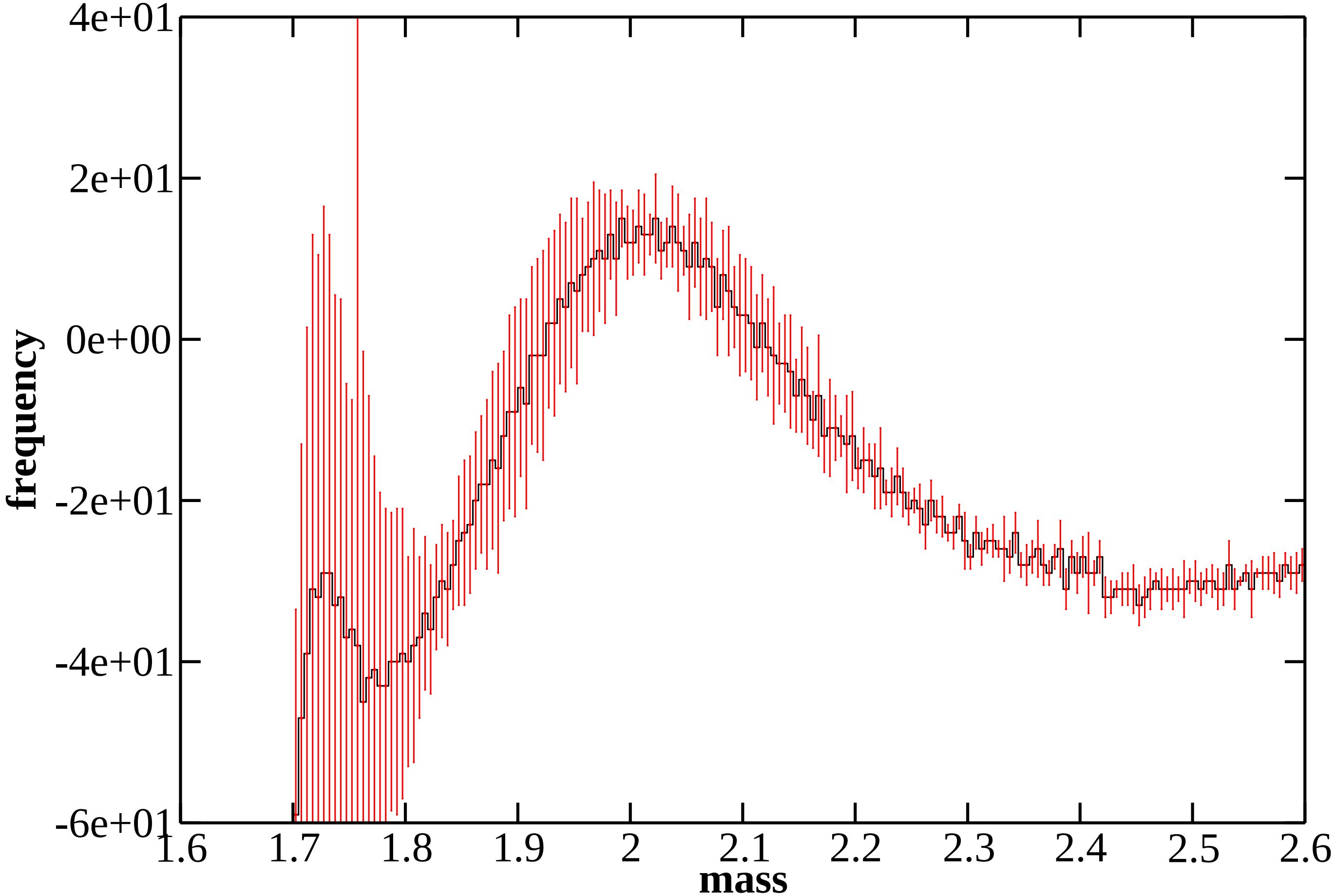
Relative error: 0.05% - 0.2%



HISTOGRAM

If we exclude the 2 levels which are “without” background

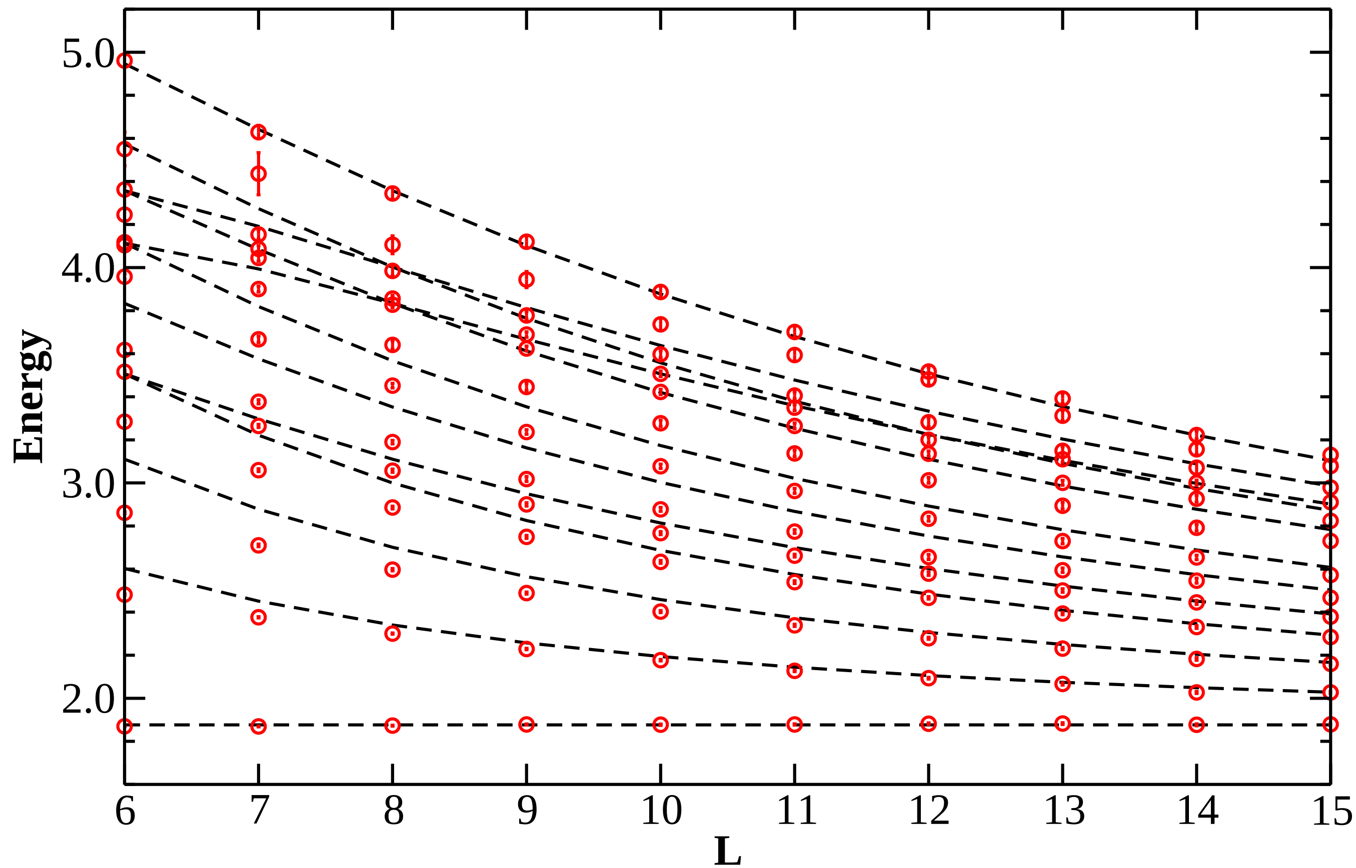
Sliding window fit around the peak: $m = 2.01(2)$, $\Gamma = 0.35(10)$, $\chi^2/dof < 1$



SPECTRUM EXAMPLE (III)

$\nu = 1.0, \lambda = 200.0, m_\pi = 0.86 \Rightarrow m_\pi^R = 0.938(3)$

Relative error: 0.15% - 0.4%



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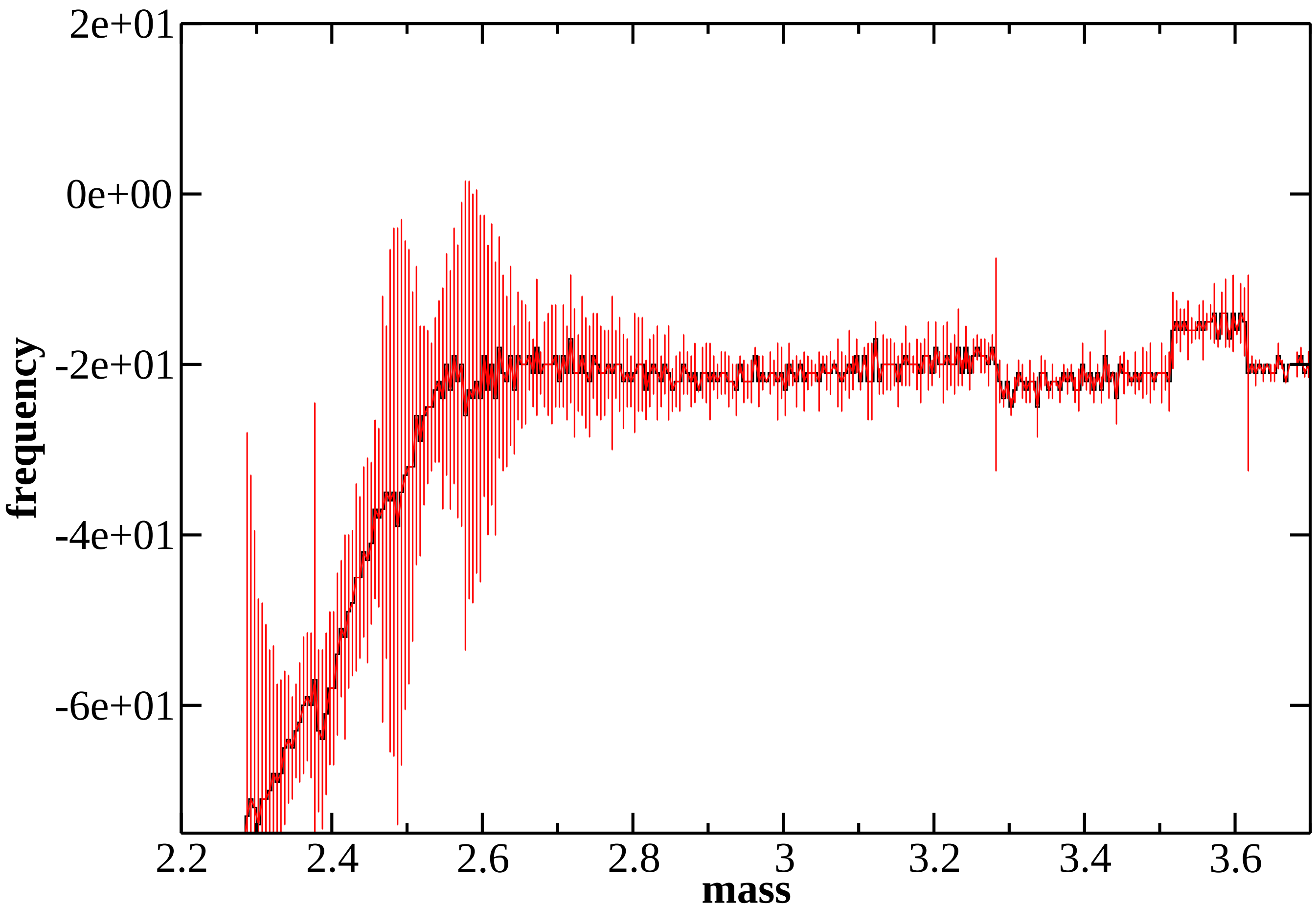

ERRORS

- From the background (statistical error from m_π):

$$E = 4\sinh^{-1}\left[\frac{1}{2}\sqrt{m_\pi^2 + p^2}\right]$$

- From the “experimental” spectrum:
systematic errors that come from the different polynomials we use

Relative error in E(L)	m	$\delta(m)/m$	Γ	$\delta(\Gamma)/\Gamma$
0.5%-1.0%	1.330(5)	0.4%	0.10(5)	50%
0.05%-0.2%	2.01(2)	1.0%	0.35(10)	28%
0.15%-0.4%	??	??	??	??

Conclusion, Outlook (1/2)

Is it only a method to visualize the resonance? or is it a method to determine the parameters?

- It is important that $N >> 1$
- Be careful with the background subtraction!
- We have to throw away the two levels related directly with the resonance (no background)
- It is important to determine correctly the systematic errors to have a trustworthy fit!

Conclusion, Outlook (2/2)

- If the resonance is narrow: m and Γ with low statistic!
- If the resonance is broad: high statistic!

Outlook:

- Can this method be used in the inelastic region ?