Theory of the muon $g-2$: some recent developments

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Anomalous magnetic moment a_l

1. Electron a_e
   - Precision test of Quantum Electrodynamics (QED)
   - Most precise determination of fine structure constant α
2. Muon a_μ
   - Precision test of Standard Model (SM)
     Interplay of all sectors: QED, Weak and QCD (hadronic)
   - Sensitive to New Physics beyond the SM
   - Problem: Hadronic contributions (vacuum polarization, light-by-light scattering)
     $\longrightarrow$ largest sources of uncertainty

Jegerlehner + Nyffeler, The Muon $g-2$, Phys. Rept. 477, 1 (2009), arXiv:0902.3360 [hep-ph]

Nyffeler, Hadronic light-by-light scattering in the muon $g-2$: a new short distance constraint on pion exchange, Phys. Rev. D 79, 073012 (2009), arXiv:0901.1172 [hep-ph]

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Introduction: Basics of the anomalous magnetic moment

Electrostatic properties of charged particles:

Charge Q, Magnetic moment $\vec{\mu}$, Electric dipole moment $\vec{d}$

For a spin 1/2 particle:

$$\vec{\mu} = g \frac{e}{2m} \vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1 + a), \qquad a = \frac{1}{2}(g - 2) : \text{anomalous magnetic moment}$$

Long interplay between experiment and theory $\longrightarrow$ structure of fundamental forces

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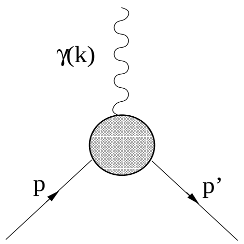
In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{u}(p')\left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}}\right]u(p)$$

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = a$$

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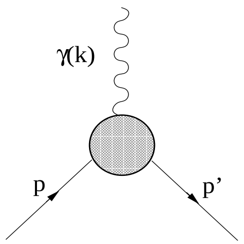

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a_e: Most precise determination of $\alpha = e^2/4\pi$.
a_μ: Less precisely measured than a_e, but all sectors of Standard Model (SM), i.e. QED, Weak and QCD (hadronic), contribute significantly. Sensitive to possible contributions from New Physics:

$$a_l \sim \left(\frac{m_l}{m_{\text{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 40000 \text{ more sensitive than } a_e \text{ [exp. precision} \longrightarrow \text{"only" factor 19]}$$

Milestones in measurements of a_μ

Authors	Lab	Muon Anomaly
Garwin et al. '60	CERN	0.001 13(14)
Charpak et al. '61	CERN	0.001 145(22)
Charpak et al. '62	CERN	0.001 162(5)
Farley et al. '66	CERN	0.001 165(3)
Bailey et al. '68	CERN	0.001 166 16(31)
Bailey et al. '79	CERN	0.001 165 923 0(84)
Brown et al. '00	BNL	0.001 165 919 1(59) (μ^+)
Brown et al. '01	BNL	0.001 165 920 2(14)(6) (μ^+)
Bennet et al. '02	BNL	0.001 165 920 4(7)(5) (μ^+)
Bennet et al. '04	BNL	0.001 165 921 4(8)(3) (μ^-)

World average experimental value ($g-2$ Collaboration at BNL, '06):

$$a_\mu^{\text{exp}} = (11\,659\,2080 \pm 63) \times 10^{-11} \quad [0.5\text{ppm}] \quad [\text{“old”}]$$

Actually, the BNL experiments measures two frequencies and needs ratio of muon to proton magnetic moment $\lambda = \mu_\mu/\mu_p$ as input to get a_μ. New CODATA 2008 value of λ from muonium hyperfine splitting leads to shift of $+9.2 \times 10^{-11}$:

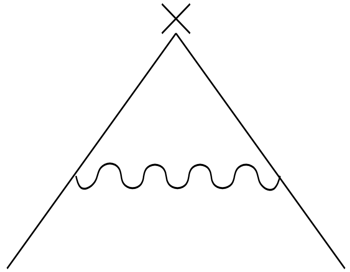
$$a_\mu^{\text{exp}} = (11\,659\,2089 \pm 63) \times 10^{-11} \quad [0.5\text{ppm}] \quad [\text{new !}]$$

Experimental value for a_e (Hanneke et al. '08):

$$a_e^{\text{exp}} = (11\,596\,521\,80.73 \pm 0.28) \times 10^{-12} \quad [0.24\text{ppb}]$$

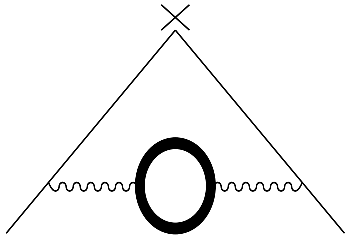
Some theoretical comments

- Anomalous magnetic moments are dimensionless
  To lowest order in QED perturbation theory:



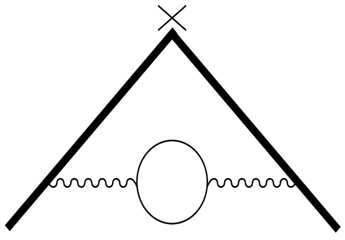
$$= a_e = a_\mu = \frac{\alpha}{2\pi}$$ [Schwinger '48]

- Loops with different masses $\Rightarrow a_e \neq a_\mu$
  - Internal large masses decouple:



$$= \left[\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - Internal small masses give rise to large log's of mass ratios:



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Anomalous magnetic moments from two to five loops

- Two loop diagrams with common fermion lines (7 Feynman diagrams)

$$a_l^{(4)} = \left[\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  [Petermann '57, Sommerfield '57]

- Three loop diagrams with common fermion lines (72 diagrams)

$$\begin{aligned} a_l^{(6)} &= \left[\frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3)\right. \\ &\quad + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right\} \\ &\quad \left. - \frac{239}{2160}\pi^4 + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5)\right]\left(\frac{\alpha}{\pi}\right)^3 \end{aligned}$$

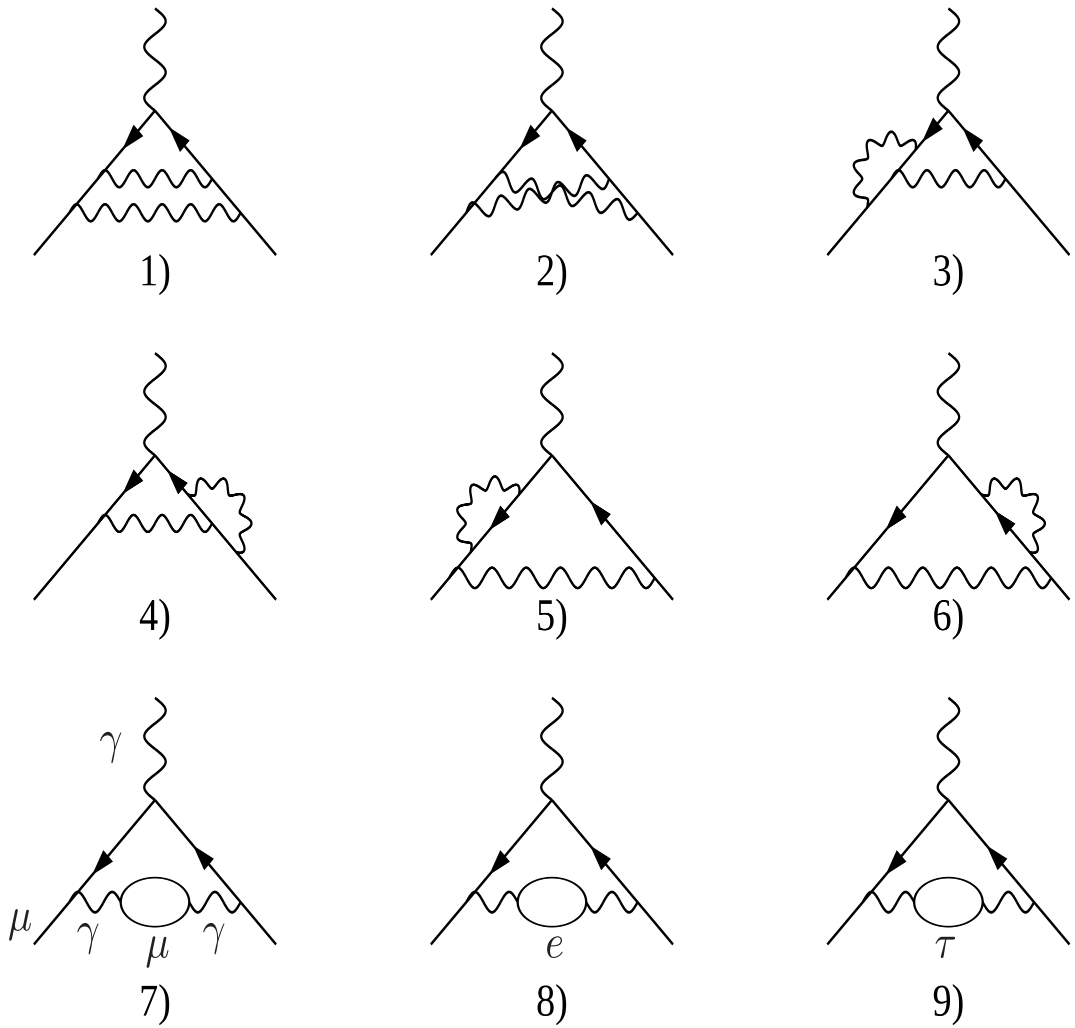
  [Laporta + Remiddi '96]

- Four loop diagrams with common fermion lines (891 diagrams)
  [Kinoshita et al. '99, . . . , '07, '08 ! (numerically)]

- Five loop diagrams with common fermion lines (12672 diagrams !)
  Only a bound exists. Largest uncertainty in theoretical prediction of a_e ! Numerical evaluation in progress, Kinoshita et al. '05, . . .

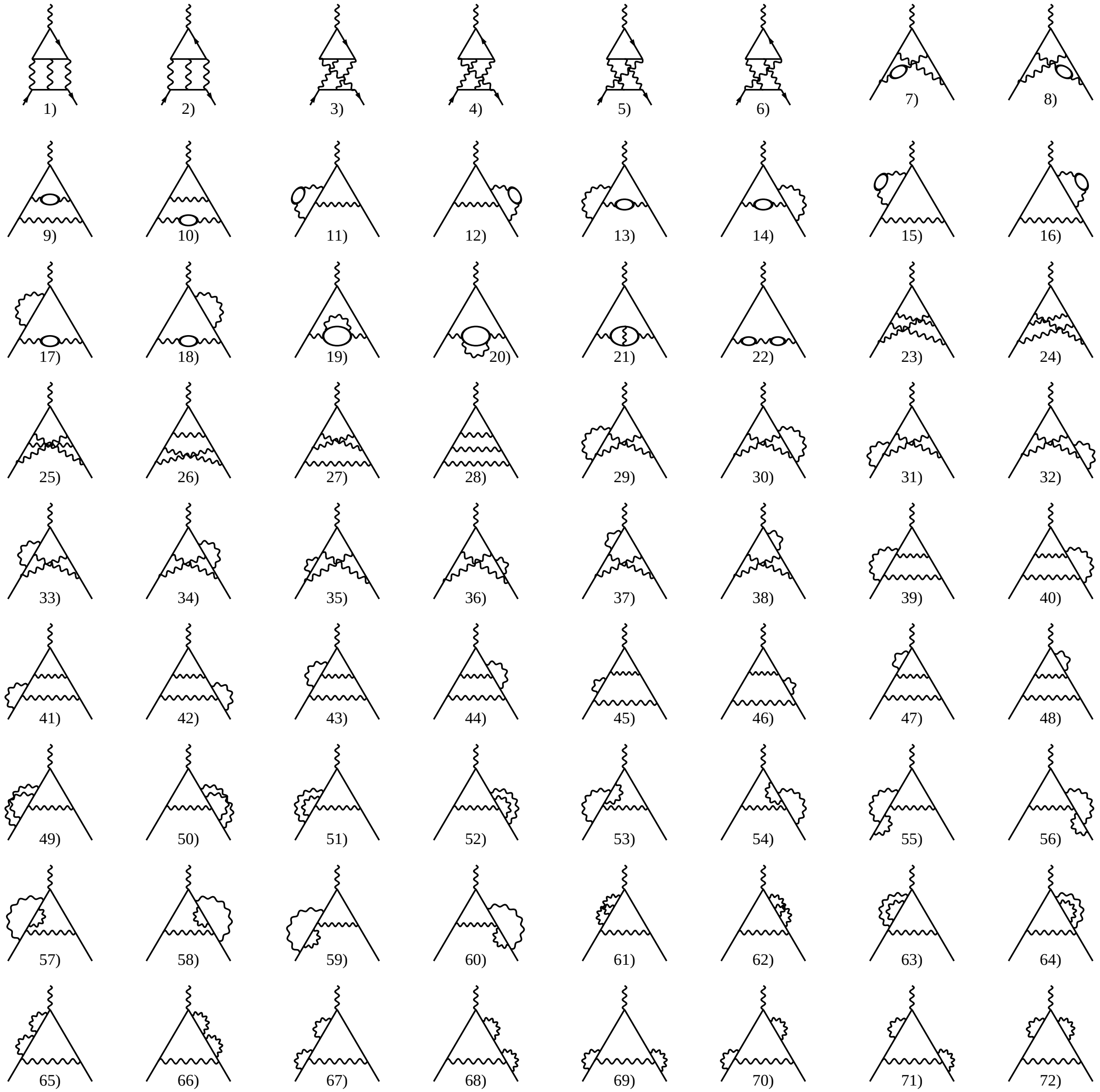
$$\begin{aligned} a_l &= 0.5\left(\frac{\alpha}{\pi}\right) - 0.328478965579193 78\ldots\left(\frac{\alpha}{\pi}\right)^2 \\ &\quad + 1.181241456587\ldots\left(\frac{\alpha}{\pi}\right)^3 - 1.9144(35)\left(\frac{\alpha}{\pi}\right)^4 + 0.0(4.6)\left(\frac{\alpha}{\pi}\right)^5 \end{aligned}$$

2-loop diagrams for a_μ



Diagrams 8) and 9) will lead to mass dependent contributions in a_μ.

3-loop diagrams with common fermion lines



Electron anomalous magnetic moment a_e

With α from atomic interferometry: $\alpha^{-1}(\text{Rb}) = 137.035\,998\,84(91)$ [6.7ppb]

$$
\begin{aligned}
[a_e]_{\text{univ}} &= 11\,614\,097\,34.95 \times 10^{-12} && \text{[1-loop]} \\
&\quad -17\,723\,05.07 \times 10^{-12} && \text{[2-loops]} \\
&\quad +148\,04.20 \times 10^{-12} && \text{[3-loops]} \\
&\quad -55.73 \times 10^{-12} && \text{[4-loops]} \\
&= 11\,596\,521\,78.36\ \underbrace{(7.69)}_{\alpha\,(\text{Rb})}\ \underbrace{(10)}_{C_4}\ \underbrace{(31)}_{C_5} \times 10^{-12}
\end{aligned}
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Internal μ, τ -lines:
$$
\begin{aligned}
a_e(\mu,\tau) &= (\underbrace{5.197 \times 10^{-7}}_{\mu\ \text{VP}} + \underbrace{1.838 \times 10^{-9}}_{\tau\ \text{VP}}) \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + (\underbrace{-2.1768 \times 10^{-5}}_{\mu\ \text{HO VP}} + \underbrace{1.4394 \times 10^{-5}}_{\mu\ \text{LbyL}}) \times \left(\frac{\alpha}{\pi}\right)^3 \\
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Hadronic and weak corrections: $a_e^{\text{had}} = 1.676(18) \times 10^{-12}$, $a_e^{\text{weak}} = 0.039 \times 10^{-12}$

Collecting all contributions: $a_e^{\text{SM}}(\text{Rb}) = 11\,596\,521\,82.79(7.70) \times 10^{-12}$ $[6.6\text{ppb}]$

Comparison with exp. value (Hanneke et al. '08): $a_e^{\text{exp.}} - a_e^{\text{SM}}(\text{Rb}) = -2.06(7.70) \times 10^{-12}$

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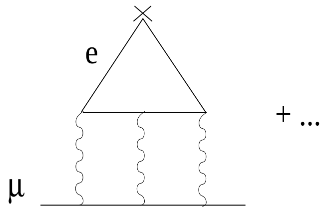
Had. and weak corrections under control ! This is why a_e is a good observable to extract α !
Inverting the series of a_e in $\alpha \Rightarrow \alpha^{-1}(a_e) = 137.035999084(51)$ $[0.37\text{ppb}]$

Note: shift in universal coefficient C_4 of $\mathcal{O}(\alpha^4)$ from $-1.7283(35)$ to $-1.9144(35)$ (Kinoshita et al. '07) has led to 7σ shift in $\alpha(a_e)$!

Muon anomalous magnetic moment $a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm had}$

QED contributions

- Vacuum polarization from electron loops: enhanced by QED short-distance log's
- Light-by-light scattering from electron loops: enhanced by QED infrared logarithms [Aldins et al. '69, '70; Laporta + Remiddi '93]



$$a_\mu^{(3)}\Big|_{\rm lbyl} = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \ldots\right]\left(\frac{\alpha}{\pi}\right)^3 = 20.947\ldots\left(\frac{\alpha}{\pi}\right)^3$$

- Loops with tau's: suppressed (decoupling)

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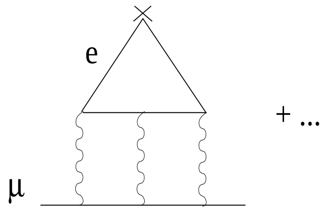

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- Loops with tau's: suppressed (decoupling)

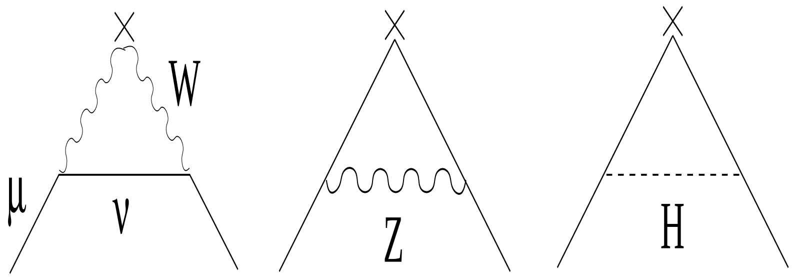
Result up to 5-loops (include contributions from all leptons):

$$\begin{aligned} a_\mu^{\rm QED} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,410 \underbrace{(27)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\ &\quad + 24.050\,509\,64 \underbrace{(46)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8105 \underbrace{(85)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\ &\quad + 663.0 \underbrace{(20.0)}_{\text{num. int.}} \underbrace{(4.6)}_{C_5\,\text{(univ)}} \times \left(\frac{\alpha}{\pi}\right)^5 \\ &= 116\,584\,718.104 \underbrace{(0.044)}_{\alpha(a_e)} \underbrace{(0.015)}_{m_\mu/m_{e,\tau}} \underbrace{(0.025)}_{C_4} \underbrace{(0.139)}_{C_5} [0.148] \times 10^{-11} \end{aligned}$$

Note: the 5-loop result is now no longer based on a renormalization group estimate, but on a numerical evaluation of all the diagrams which are known or likely to be enhanced (Kinoshita + Nio '05, '06), the first error on C_5 should cover all subleading contributions.

Electroweak contribution to a_μ

1-loop contribution:



$$a_\mu^{\text{EW},(1)} = (194.82 \pm 0.02) \times 10^{-11}$$

2-loop contribution:

$$a_\mu^{\text{EW},(2)} = (-42.08 \pm 1.80) \times 10^{-11}, \qquad \text{large since} \sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$$

Total EW contribution:

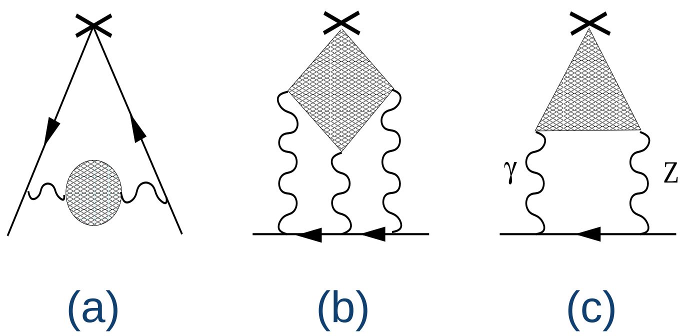
$$a_\mu^{\text{EW}} = (153.2 \pm 1.8) \times 10^{-11}$$

[Brodsky + Sullivan '67; . . . ; Knecht et al. '02; Czarnecki, Marciano, Vainshtein '02; . . .]

Hadronic contributions to a_μ

- QCD: quarks bound by strong gluonic interactions into hadronic states
- In particular for the light quarks $u, d, s \longrightarrow$ cannot use perturbation theory !
- Largest source of error in a_μ

Different types of contributions:



Light quark loop not well defined
$\longrightarrow$ Hadronic “blob”

(a) Hadronic vacuum polarization $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3)$

(b) Hadronic light-by-light scattering $\mathcal{O}(\alpha^3)$

(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

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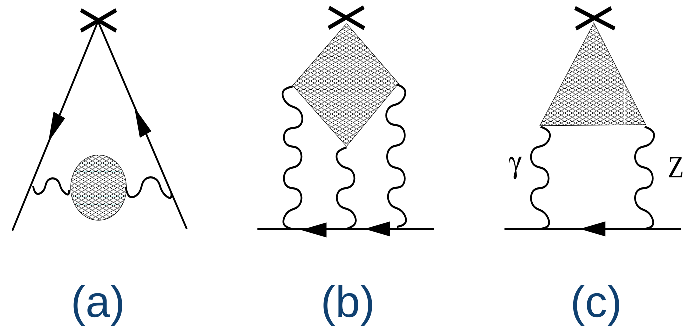

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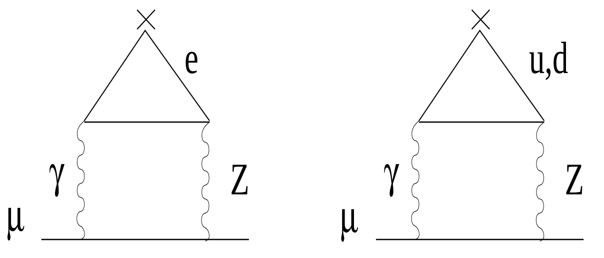
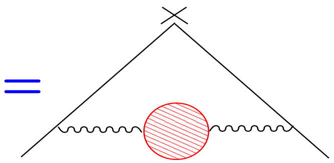
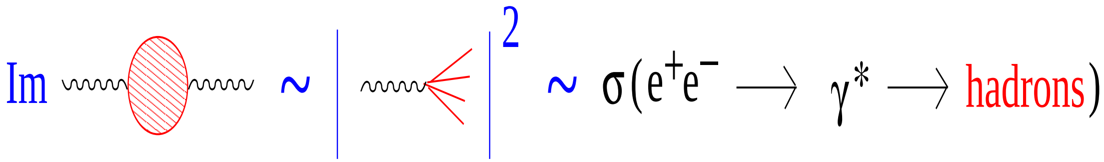
2-Loop EW
Small hadronic uncertainty $\sim 1 \times 10^{-11}$
from triangle diagrams
Anomaly cancellation within each generation !
Cannot separate leptons and quarks !



Hadronic vacuum polarization at $\mathcal{O}(\alpha^2)$

At lowest order: can use optical theorem (unitarity)

$a_\mu^{\text{had. v.p.}} =$ 



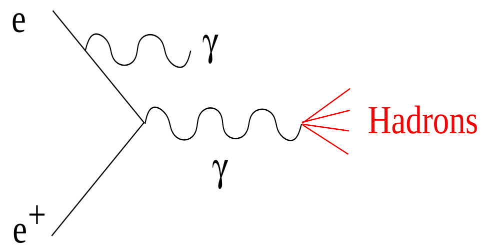
Starting from spectral representation for 2-point function:

$$a_\mu^{\text{had. v.p.}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s} K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

[Bouchiat + Michel ’61; Durand ’62; Brodsky + de Rafael ’68; Gourdin + de Rafael ’69]

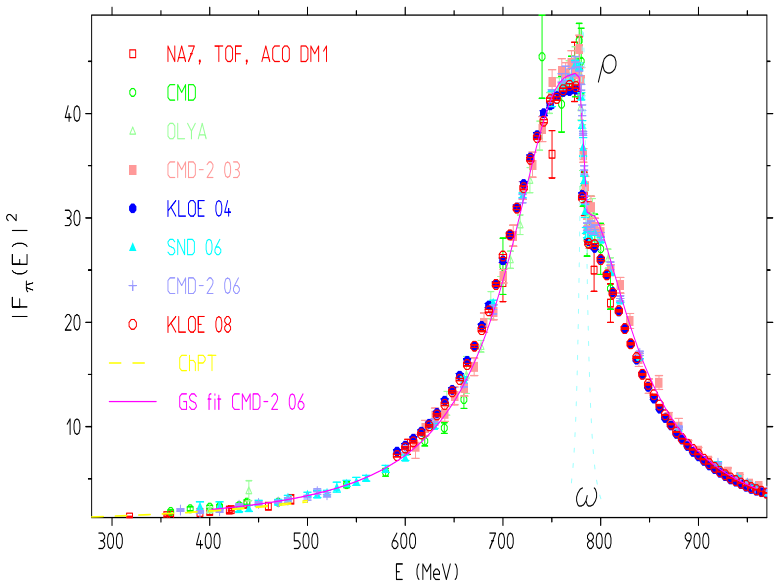
$K(s)$ slowly varying and positive; low-energy data very important due to factor $1/s$
$\sim 90\%$ from energy region below 1.8 GeV, $\sim 70\%$ from $\pi\pi[\rho(770)]$ channel

Other method instead of energy scan: “Radiative return” at colliders with fixed center of mass energy (DAΦNE, B-Factories, . . .)
[Binner, Kühn, Melnikov ’99; Czyż et al. ’00-’03]

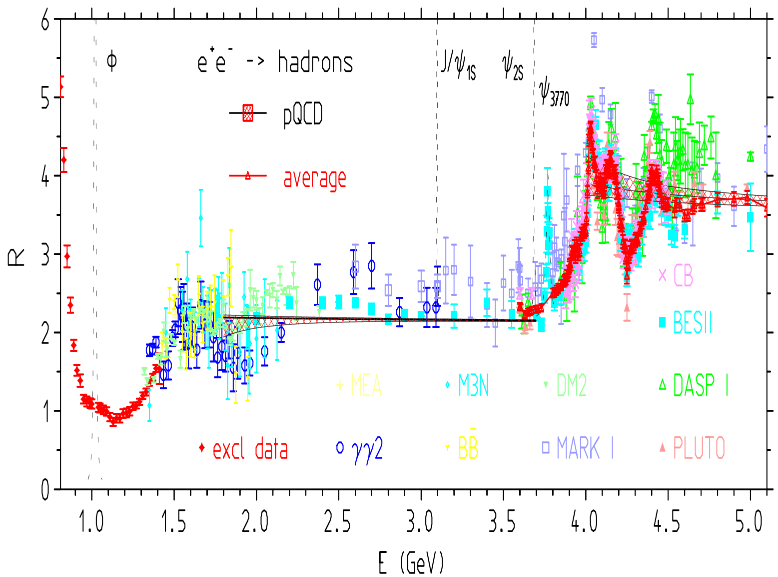


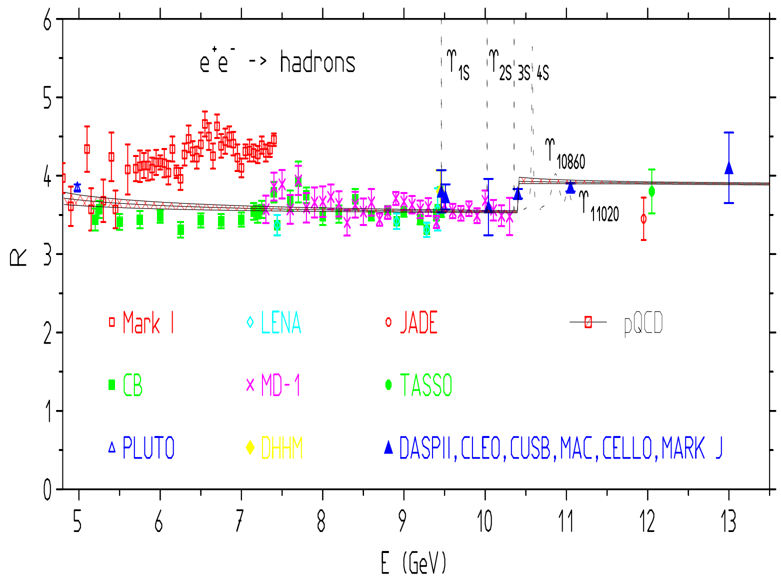
Measured hadronic cross-section

Pion form factor $|F_\pi(E)|^2$
($\pi\pi$-channel)



R-ratio:

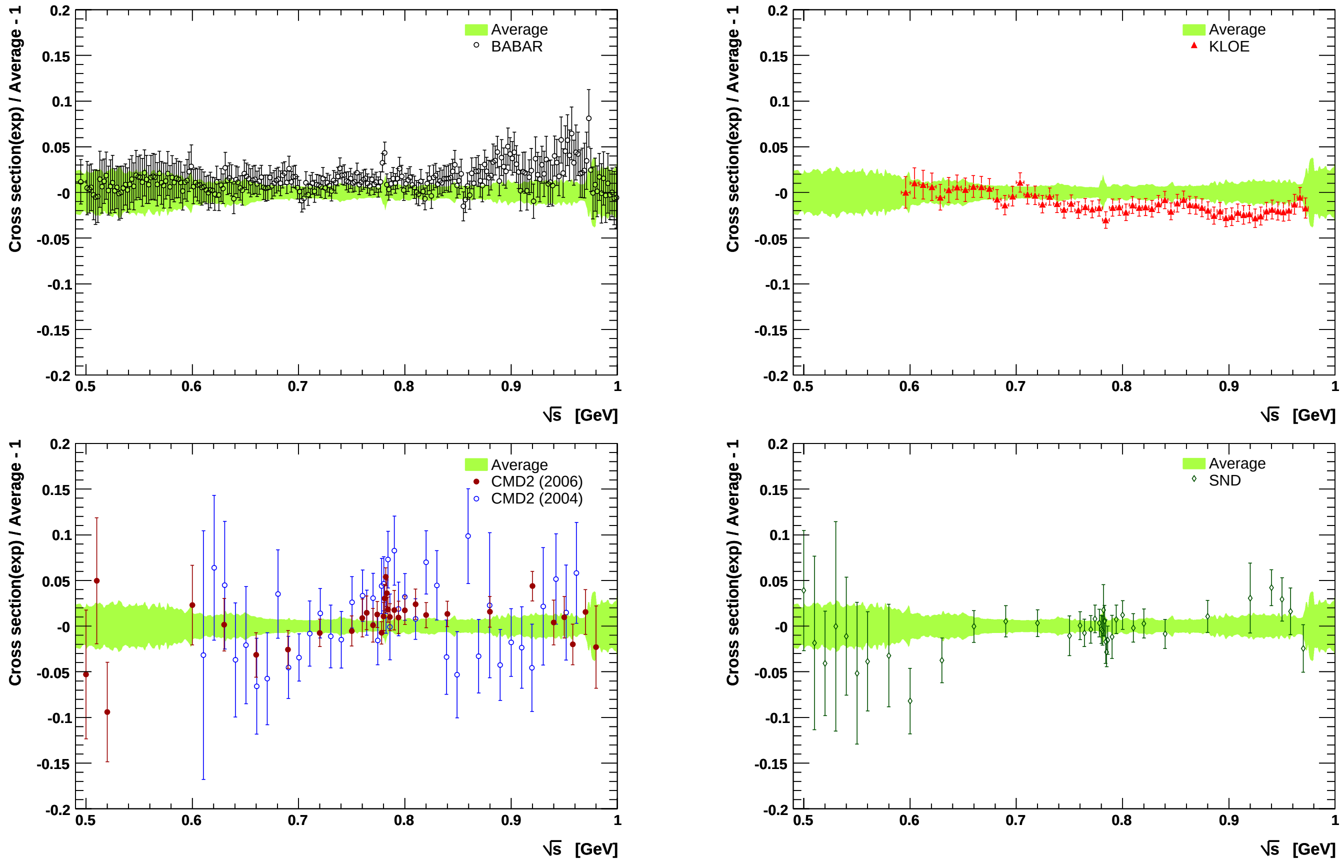




(from Jegerlehner + Nyffeler '09)

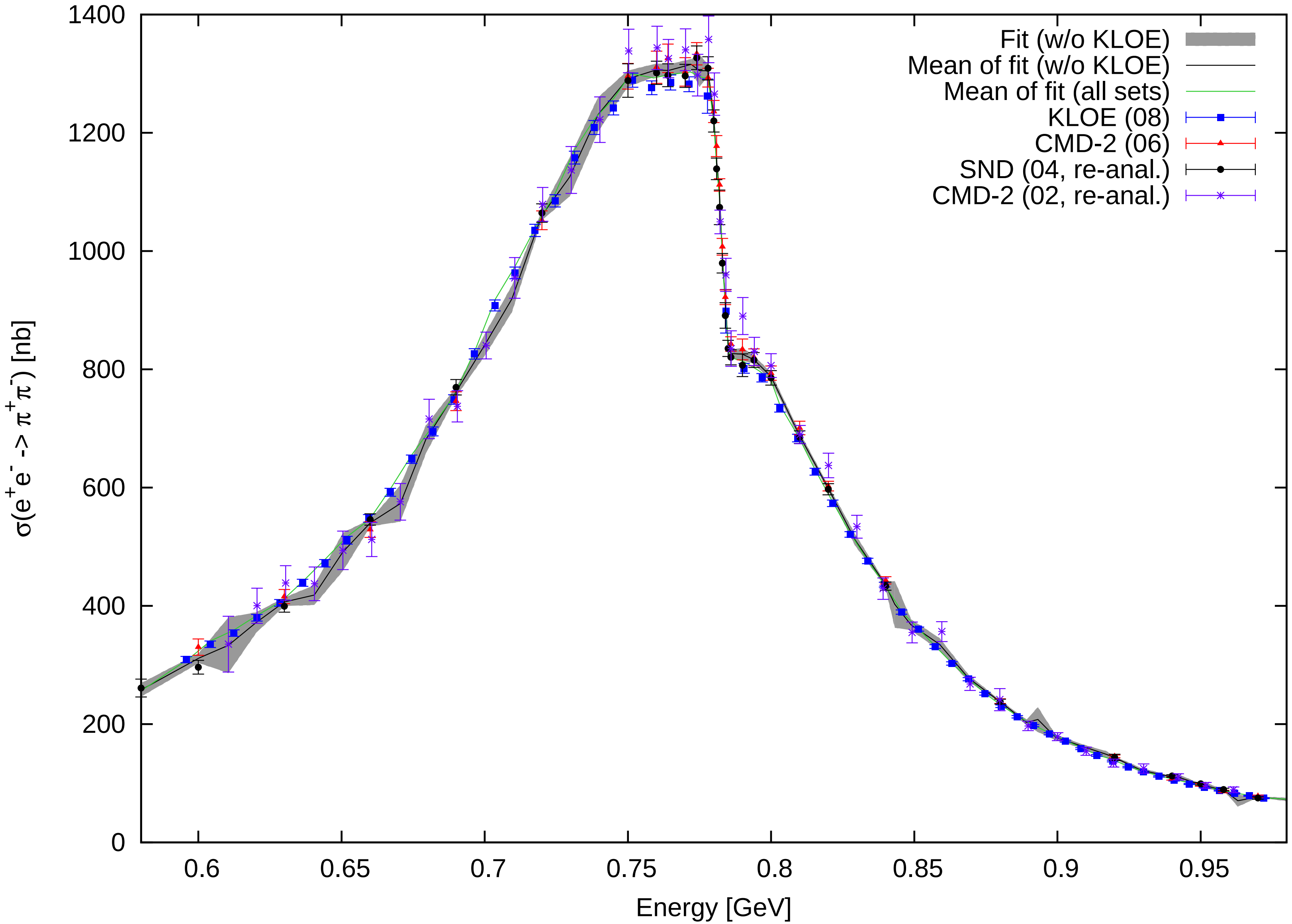
Compilation by Davier et al. ’09

Relative deviation from average (green shaded band) for several experiments:



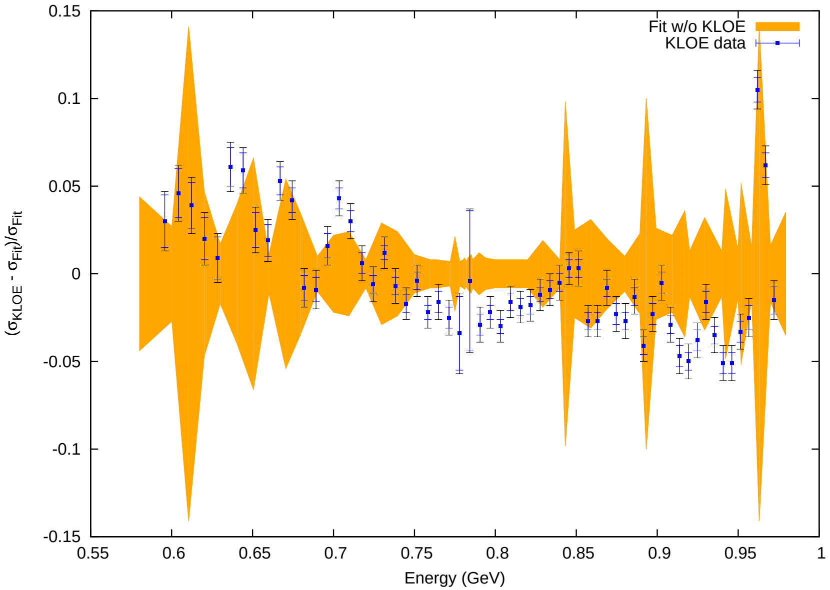
- Include new data from BABAR, which simultaneously measured $e^+e^- \longrightarrow \pi^+\pi^-\gamma(\gamma)$ and $e^+e^- \longrightarrow \mu^+\mu^-\gamma(\gamma)$. New BABAR data increase central value significantly.
- Error not reduced much, mainly because of incompatibility between BABAR and KLOE ’08, 3% deviation at ρ-peak (bigger than systematic errors of 0.5% of BABAR and 1.1% of KLOE) $\longrightarrow$ 2σ discrepancy in $a_\mu^{\text{had. v.p.}}$ from interval $0.63 - 0.958$ GeV.
- Davier et al.: Near ρ-peak CMD-2 agrees with BABAR, SND between BABAR and KLOE.

Compilation by Teubner et al. ’09



Most important data in the 2π channel and two fits with / without KLOE ’08 data.
(from arXiv:1001:5401 [hep-ph], Proceedings of PhiPsi Workshop, October ’09, Beijing)

Compilation by Teubner et al. ’09 (cont.)



Normalized difference of KLOE ’08 data and compilation without KLOE (from arXiv:1001:5401)

- Teubner et al.: Differences in shape do not allow to include KLOE data in fit of point-by-point combination. Fit procedure only allows readjustment of overall normalization of data sets within their syst. errors. (Although for most points the difference seems to be not much more than 1σ).
- Including KLOE data in fit: would lead to bad χ^2_{min}/d.o.f. and pull fit artificially upwards (up by 2% $\Rightarrow +50 \times 10^{-11}$ in a_μ, Hagiwara et al. ’07, based on KLOE ’04 data).
- Instead Teubner et al. take weighted average of results after integration of

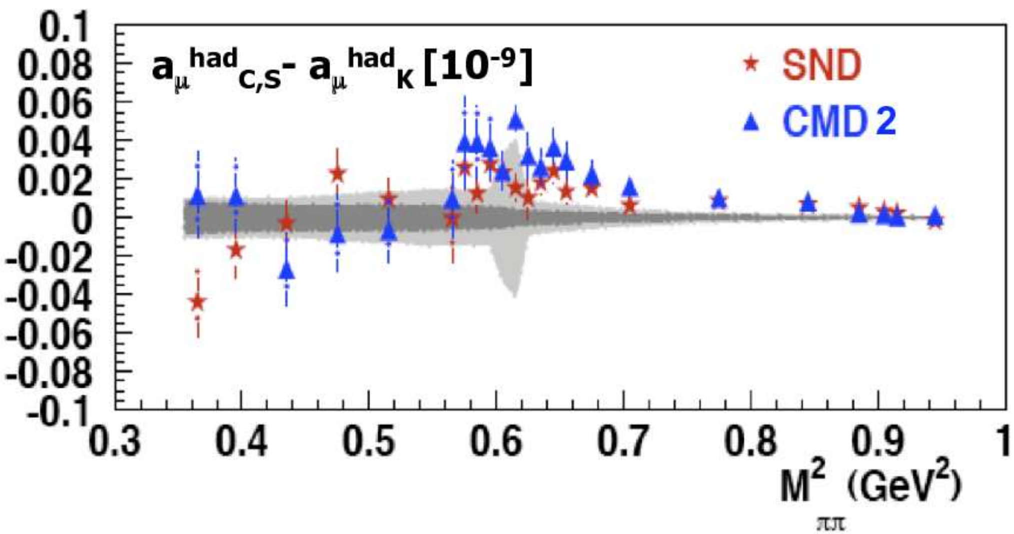
  Fit (without KLOE): $a_\mu^{\pi\pi,\text{w/out KLOE}} = (3841.2 \pm 25.1) \times 10^{-11}$

  KLOE data: $a_\mu^{\pi\pi,\text{KLOE}} = (3841.6 \pm 34.7) \times 10^{-11}$
- New BABAR ’09 data for $\pi\pi$ not yet included.

Comparison of $a_\mu^{\pi\pi}$ between BABAR, CMD-2, KLOE, SND

E (GeV)	Exp.	$a_\mu^{\pi\pi} \times 10^{11}$	Ref.
$0.630 - 0.958$	CMD-2 '06	3615 ± 34	[1]
	SND '06	3610 ± 51	[1]
	KLOE '08	3567 ± 31	[1]
	BABAR '09	3652 ± 27	[2]
$0.592 - 0.923$	KLOE '08	3796 ± 33	[3]
	KLOE '09	3766 ± 33	[3]

[1]: KLOE Collaboration, PLB670, 285 (2009)
[2]: Davier et al., arXiv:0908.4300 [hep-ph]
[3]: S.E. Müller for the KLOE Collaboration, arXiv:0912.2205 [hep-ex] (PhiPsi '09 Proc.). New analysis with ISR photon emitted at large polar angles.

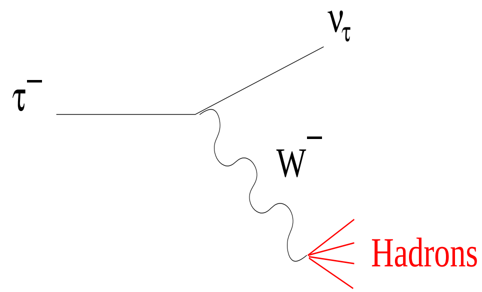


Absolute difference between dispersion integral value (in each energy bin) evaluated by CMD-2 or SND with respect to KLOE. The dark (light) band represents KLOE statistical (statistical⊕systematic) errors. Good agreement at the 1σ level.

G. Venanzoni for the KLOE Collaboration, arXiv:0906.4331 [hep-ex] (CIPANP '09 Proc.)

Spectral functions from hadronic τ-decays

Hadronic τ-decays e.g. $\tau^- \longrightarrow \nu_\tau \pi^- \pi^0$



Problem: Corrections due to isospin breaking (IB): $m_u \neq m_d$, electromagnetic radiative corrections

[Alemany et al. '98; Cirigliano et al. '01, '02; Davier et al. '03; Flores-Tlalpa et al. '06, '07]

$$\sigma_{X^0}^{I=1}(s) = \frac{4\pi\alpha^2}{s} v_{1,\,X^-}(s)$$

$$v_{1,\,X^-}(s) = \frac{m_\tau^2}{6\,|V_{ud}|^2} \frac{\mathrm{BR}_{X^-}}{\mathrm{BR}_e} \frac{1}{N_X} \frac{dN_X}{ds} \times \left(1 - \frac{s}{m_\tau^2}\right)^{-2} \left(1 + \frac{2s}{m_\tau^2}\right)^{-1} \frac{R_{\mathrm{IB}}(s)}{S_{EW}}$$

$$R_{\mathrm{IB}}(s) = \frac{\mathrm{FSR}(s)}{G_{EM}(s)} \frac{\beta_0^3(s)}{\beta_-^3(s)} \left| \frac{F_0(s)}{F_-(s)} \right|^2$$

- $S_{EW} = 1.0235 \pm 0.0003$: short-distance EW rad. corr. [Sirlin '78, ...]
- $G_{EM}(s)$: long-distance rad. corr. of $\mathcal{O}(\alpha)$ to photon inclusive τ spectrum (virtual + real photons, computed using some resonance Lagrangian $\longrightarrow$ model dependent !)
- Davier et al. '09: isospin breaking corr. $\Delta a_\mu^{\text{had. v.p.}}[\pi\pi, \tau] = (-160.7 \pm 18.5) \times 10^{-11}$ bigger than used in Davier et al. '03: $(-138 \pm 24) \times 10^{-11}$.
- Bring τ-data closer to e^+e^- data, although still about 2σ deviation in $a_\mu^{\text{had. v.p.}}$.

Values for $a_\mu^{\text{had v.p.}}$

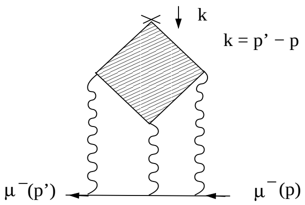
Selection of some evaluations:

Authors	Contribution to $a_\mu^{\text{had v.p.}} \times 10^{11}$
Davier et al. ’03 (e^+e^-)	$6963 \pm 62_{\text{exp}} \pm 36_{\text{rad}}$ [± 72]
Davier et al. ’03 ($e^+e^- + \tau$)	$7110 \pm 50_{\text{exp}} \pm 8_{\text{rad}} \pm 28_{\text{SU(2)}}$ [± 58]
de Troconiz, Yndurain ’05 (e^+e^-)	6935 ± 59
de Troconiz, Yndurain ’05 ($e^+e^- + \tau$)	7018 ± 58
Davier et al. ’06 (e^+e^-)	6909 ± 44
Hagiwara et al. ’07 (e^+e^-, inclusive)	6894 ± 46
Jegerlehner ’08; JN ’09 (e^+e^-)	6903.0 ± 52.6
Davier et al. ’09 (e^+e^-)	$6955 \pm 40_{\text{exp}} \pm 7_{\text{pQCD}}$ [± 41]
Davier et al. ’09 (τ)	$7053 \pm 39_{\text{exp}} \pm 7_{\text{rad}} \pm 7_{\text{pQCD}} \pm 21_{\text{IB}}$ [± 45]
Teubner et al. ’09 (e^+e^-)	$6894 \pm 36_{\text{exp}} \pm 18_{\text{rad}}$ [± 40]

- Even if values for $a_\mu^{\text{had v.p.}}$ after integration agree quite well, the systematic differences of a few % in the shape of the spectral functions from different experiments (BABAR, CMD-2, KLOE, SND) indicate that we do not yet have a complete understanding. Additional rad. corr. ?
- Use of τ data: maybe there are additional sources of isospin violation (Ghozzi + Jegerlehner ’04; Benayoun (et al.) ’08, ’09; Wolfe + Maltman ’09). Because of these open problems, most groups only use e^+e^- data.

Had. light-by-light scattering in the muon $g-2$: before Jan. '09

Classification of contributions (de Rafael '94)



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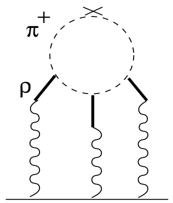

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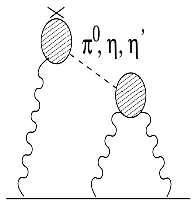
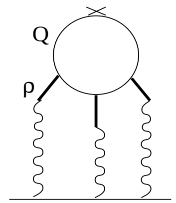


+ --- + Exchange of other resonances $(f_0, a_1, \ldots)$ +



+ ---

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Relevant scales in $\langle VVVV \rangle$ (off-shell !): $\sim m_\mu - 2$ GeV. No direct relation to exp. data, in contrast to hadronic vacuum polarization in $g-2 \longrightarrow$ need hadronic (resonance) model

de Rafael '94: last term can be interpreted as irreducible contribution to 4-point function $\langle VVVV \rangle$. Appears as short-distance complement of low-energy hadronic models.

Reduce model dependence by imposing exp. and theor. constraints on form factors, e.g. from QCD short-distances (OPE) to get better matching with perturbative QCD for high momenta

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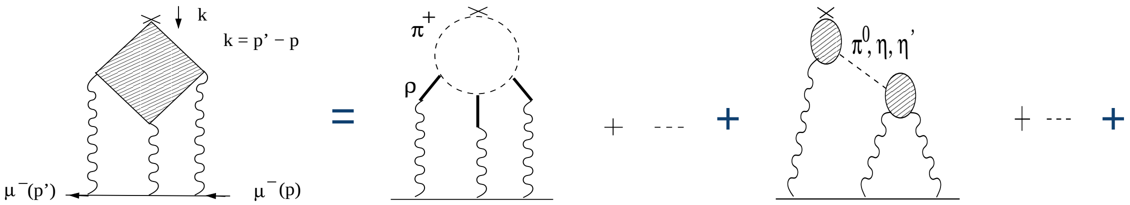

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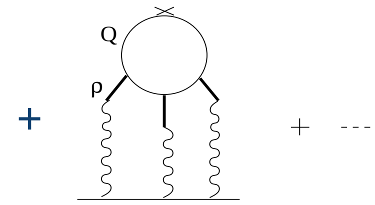

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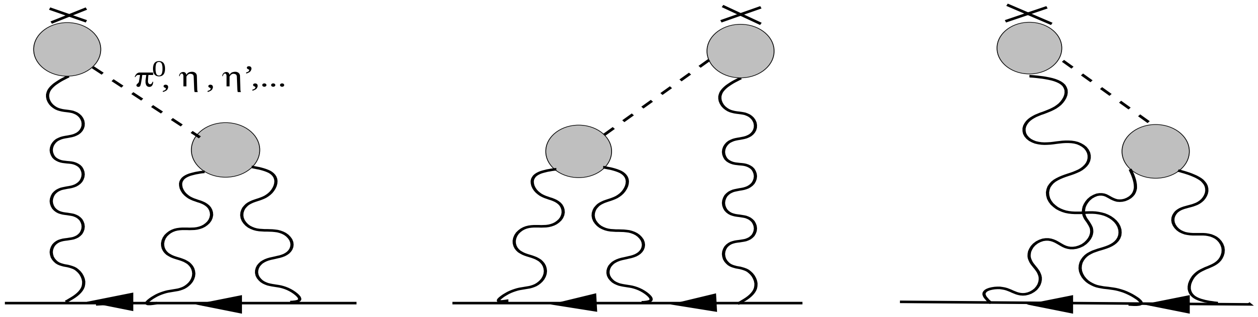
Contribution to $a_\mu \times 10^{11}$

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) $[a_1]$	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007:	+110 (40)				
		ud.: -45	ud.: $+\infty$		ud.: +60

BPP = Bijnens, Pallante, Prades ’96, ’02; HKS = Hayakawa, Kinoshita, Sanda ’06, ’98, ’02; KN = Knecht, Nyffeler ’02

MV = Melnikov, Vainshtein ’04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; ud. = undressed, i.e. point vertices without form factors

Pseudoscalar-exchange contribution to had. LbyL scattering



- Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$
- Numerically dominant contribution to had. LbyL scattering
- Exchange of lightest state π^0 yields largest contribution $\longrightarrow$ warrants special attention
- Following Bijnens, Pallante, Prades '95, '96; Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, we can define off-shell form-factor for π^0 as follows:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2\cdot y)}\, \langle 0|T\{j_\mu(x) j_\nu(y) P^3(0)\}|0\rangle$$

$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \frac{i\langle\overline{\psi}\psi\rangle}{F_\pi}\, \frac{i}{(q_1+q_2)^2 - m_\pi^2}\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

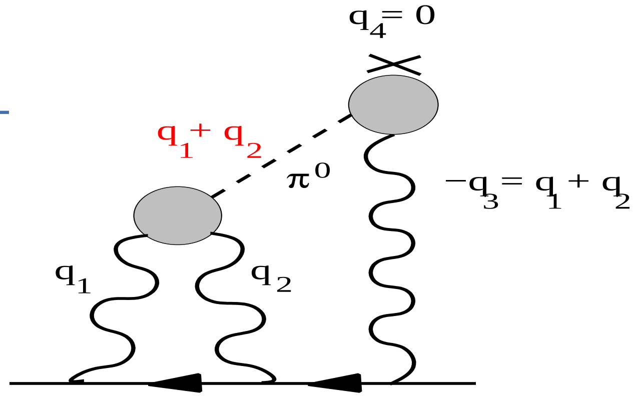
Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

j_μ = light quark part of the electromagnetic current: $j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$, $\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $\hat{Q} = \mathrm{diag}(2, -1, -1)/3$

$P^3 = \overline{\psi} i\gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d\right)/2$, $\langle\overline{\psi}\psi\rangle$ = single flavor quark condensate

Note: for off-shell pions, instead of $P^3(x)$, we could use any other suitable interpolating field, like $(\partial^\mu A_\mu^3)(x)$ or even an elementary pion field $\pi^3(x)$!

Off-shell versus on-shell form factors



- Off-shell form factors have been used to evaluate the pion-exchange contribution in BPP '96, HKS '96, HK '98, but this seems to have been forgotten later. “Rediscovered” by Jegerlehner in '07. Consider diagram:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

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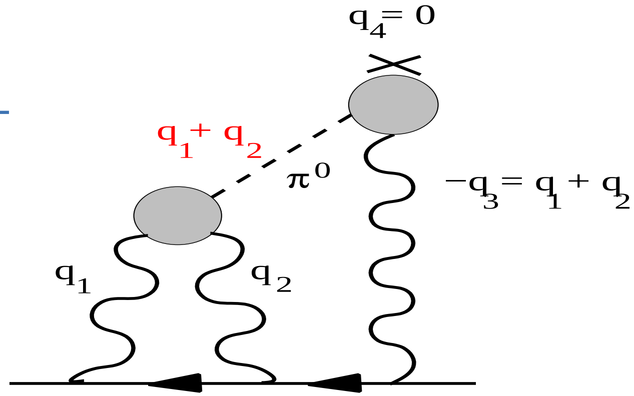

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- On the other hand, Bijnens, Persson ’01, Knecht, Nyffeler ’02 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$

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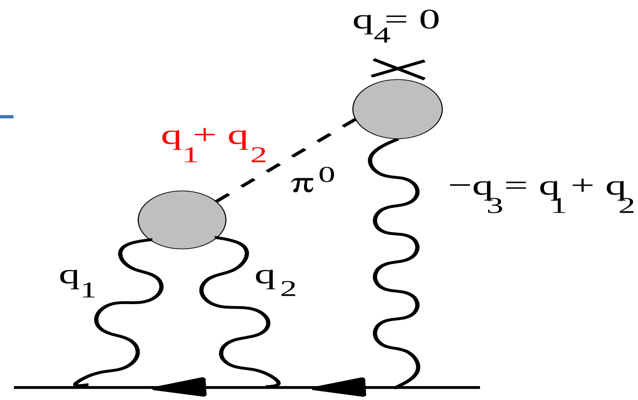

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- Melnikov, Vainshtein ’04 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \ \times \ \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the Wess-Zumino-Witten term
- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution ! In general, off-shell form factors will enter at both vertices.
- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell. Only in some specific model where pions appear as propagating fields can one identify the contribution from off-shell pions.

Experimental constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

1. Any hadronic model of the form factor has to reproduce the decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}\left[1 + \mathcal{O}\left(m_q\right)\right]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small), which fairly well reproduces the decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.6)$ eV for $F_\pi = 92.2$ MeV. This leads to the constraint

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}$$

Note: recently the PrimEx Collaboration presented the (preliminary) value $\Gamma(\pi^0 \to \gamma\gamma) = (7.82 \pm 0.23)$ eV (A.M. Bernstein, talk at Chiral Dynamics 2009). Effect on F_π not taken into account so far ! See also Kampf + Moussallam ’09.

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2. Information on the form factor with one on-shell and one off-shell photon from the process $e^+e^- \to e^+e^-\pi^0$

Experimental data (CELLO '90, CLEO '98) fairly well confirm the Brodsky-Lepage behavior:

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

Maybe with slightly different prefactor !

Note: recent data from BABAR '09 do not show this fall-off !

QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht + Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

- If the space-time arguments of all three currents approach each other one obtains (up to corrections $\mathcal{O}(\alpha_s)$):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

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- When the space-time arguments of the two vector currents in $\langle VVP \rangle$ approach each other (OPE leads to Green's function $\langle AP \rangle$):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, (\lambda q_1)^2, (q_2 - \lambda q_1)^2) = \frac{2F_0}{3}\frac{1}{\lambda^2}\frac{1}{q_1^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

Higher twist corrections have been worked out in Shuryak + Vainshtein '82, Novikov et al. '84 (in chiral limit):

$$\lim_{\lambda\to\infty} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, (\lambda q_1)^2, (\lambda q_1)^2)}{\mathcal{F}_{\pi^0\gamma\gamma}(0,0,0)} = -\frac{8}{3}\pi^2 F_0^2 \left\{ \frac{1}{\lambda^2 q_1^2} + \frac{8}{9}\frac{\delta^2}{\lambda^4 q_1^4} + \mathcal{O}\left(\frac{1}{\lambda^6}\right)\right\}$$

δ^2 parametrizes the relevant higher-twist matrix element
The sum-rule estimate in Novikov et al. '84 yielded $\delta^2 = (0.2 \pm 0.02)$ GeV^2

New short-distance constraint on form factor at external vertex

- When the space-time argument of one of the vector currents approaches the argument of the pseudoscalar density in $\langle VVP \rangle$ one obtains (Knecht + Nyffeler, EPJC '01):

$$\langle V \underbrace{VP}_{\text{OPE}} \rangle \to \langle VT \rangle \qquad \text{Vector-Tensor two-point function}$$

$$\delta^{ab} (\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = \int d^4x \, e^{ip\cdot x} \, \langle 0 | T\{ V_\mu^a(x) \, (\overline{\psi} \, \sigma_{\rho\sigma} \, \frac{\lambda^b}{2} \, \psi)(0) \} | 0 \rangle, \qquad \sigma_{\rho\sigma} = \frac{i}{2} [\gamma_\rho, \gamma_\sigma]$$

$\Rightarrow$ New short-distance constraint on off-shell form factor at external vertex (Nyffeler '09):

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3} \chi + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad (*)$$

where χ is the quark condensate magnetic susceptibility of QCD in the presence of a constant external electromagnetic field (Ioffe, Smilga '84):

$$\langle 0 | \bar{q} \sigma_{\mu\nu} q | 0 \rangle_F = e \, e_q \, \chi \, \langle \overline{\psi} \psi \rangle_0 \, F_{\mu\nu}, \qquad e_u = 2/3, \; e_d = -1/3$$

- Note that there is no falloff in OPE in $(*)$, unless χ vanishes !
- Corrections of $\mathcal{O}(\alpha_s)$ in OPE $\Rightarrow$ χ depends on renormalization scale μ
- Unfortunately there is no agreement in the literature what the value of $\chi(\mu)$ should be ! Range of values from $\chi(\mu \sim 0.5 \text{ GeV}) \approx -9 \text{ GeV}^{-2}$ (Ioffe, Smilga '84; Vainshtein '03, . . . , Narison '08) to $\chi(\mu \sim 1 \text{ GeV}) \approx -3 \text{ GeV}^{-2}$ (Balitsky, Yung '83; Ball et al. '03; . . . ; Ioffe '09). Running with μ cannot explain such a difference.

New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD)

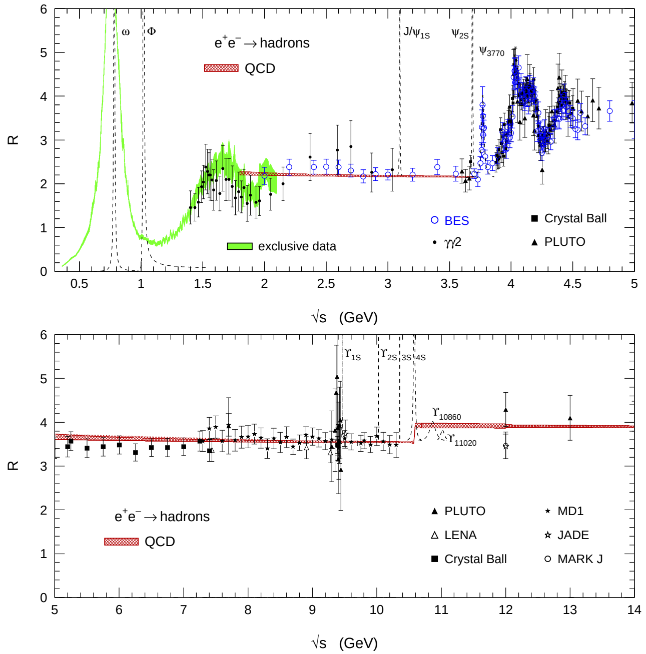
New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

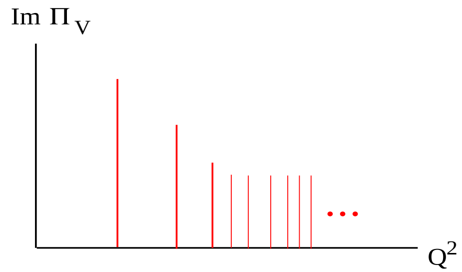
- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD)

Example: 2-point function $\langle VV \rangle \longrightarrow$ spectral function $\mathrm{Im}\Pi_V \sim \sigma(e^+e^- \longrightarrow \text{hadrons})$

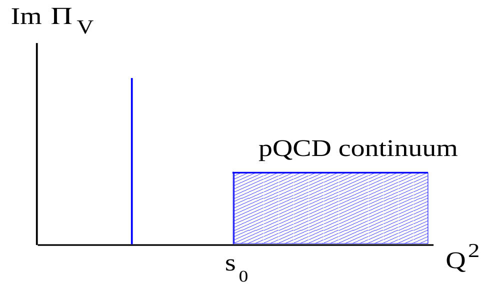
Real world (Davier et al., '03)



Large-N_C QCD ('t Hooft '74)



Minimal Hadronic Approximation (MHA)



Scale s_0 fixed by the OPE

Off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD

Knecht + Nyffeler, EPJC '01

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all QCD short-distance (OPE) constraints
- Reproduces Brodsky-Lepage behavior (confirmed by CLEO, but not by recent BABAR data):

$$\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$$

- Normalized to decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.6)$ eV

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Off-shell LMD+V form factor:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\rm LMD+V}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)}$$

$$\begin{aligned} P_H^V(q_1^2, q_2^2, q_3^2) &= h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4 \\ &\quad + h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7, \qquad q_3^2 = (q_1 + q_2)^2 \end{aligned}$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV

We view our evaluation as being a part of a full calculation of the hadronic light-by-light scattering contribution using a resonance Lagrangian along the lines of the Resonance Chiral Theory (Ecker et al. '89, ...), which also fulfills all the relevant QCD short-distance constraints.

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to recent BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2, $h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$
with χ^2/dof $= 15.0/15 = 1.0$

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h_3, h_4, h_6 are unknown / less constrained:

- New short-distance constraint $\Rightarrow h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$ $(*)$
  LMD ansatz for $\langle VT \rangle \Rightarrow \chi^{\mathrm{LMD}} = -2/M_V^2 = -3.3$ GeV^{-2} (Balitsky, Yung '83)
  Close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} (Ball et al. '03)
  Assume large-N_C (LMD/LMD+V) framework is self-consistent
  $\Rightarrow \chi = -(3.3 \pm 1.1)$ GeV^{-2}
  $\Rightarrow$ vary $h_3 = (0 \pm 10)$ GeV^2 and determine h_4 from relation $(*)$ and vice versa
- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to h_6
  Assume that LMD/LMD+V estimates of low-energy constants from chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$ are self-consistent. Assume 100% error on estimate for the relevant, presumably small low-energy constant $\Rightarrow h_6 = (5 \pm 5)$ GeV^4

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i

- The h_i enter the LMD+V form factor linearly in the numerator, therefore (Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i\, \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\, \tilde{h}_i\, \tilde{h}_j\right]$$

with dimensionless coefficients $c_i, c_{ij} \sim 10^{-4}$ (see Nyffeler '09 for the values), if we measure the h_i in appropriate units of GeV $\longrightarrow \tilde{h}_i$.

h_1, h_3, h_4 not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2\, M_{V_2}^2\, \chi$, because of the new short-distance constraint.

- h_1, h_2, h_5, h_7 are quite well known $\longrightarrow$ can write down a simplified expression with only h_3, h_4, h_6 as free parameters (up to constraint):

$$\begin{aligned} a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \Big[\, & 503.3764 - 6.5223\, \tilde{h}_3 - 5.0962\, \tilde{h}_4 + 7.8557\, \tilde{h}_6 \\ & + 0.3017\, \tilde{h}_3^2 + 0.5683\, \tilde{h}_3\, \tilde{h}_4 - 0.1747\, \tilde{h}_3\, \tilde{h}_6 \\ & + 0.2672\, \tilde{h}_4^2 - 0.1411\, \tilde{h}_4\, \tilde{h}_6 + 0.0642\, \tilde{h}_6^2 \Big] \times 10^{-4} \end{aligned}$$

New estimate for pseudoscalar-exchange contribution

- π^0
  - Our new estimate (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

    With off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ which obeys new short-distance constraint.

  - Largest uncertainty from $h_6 = (5 \pm 5)$ GeV^4 $\Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

    If we would vary $h_6 = (0 \pm 10)$ GeV^4 $\Rightarrow \pm 12 \times 10^{-11}$!

  - Varying $\chi = -(3.3 \pm 1.1)$ GeV^{-2} $\Rightarrow \pm 2.1 \times 10^{-11}$

    Exact value of χ not that important, but range does not include Vainshtein's estimate $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$ GeV^{-2}

  - Varying $h_3 = (0 \pm 10)$ GeV^2 $\Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$)
  - With h_1, h_5 from fit to recent BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11}$ $\longrightarrow$ result unchanged !

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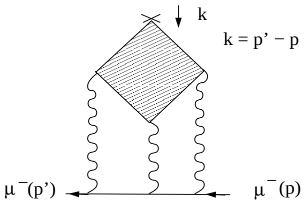
  - With h_1, h_5 from fit to recent BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11}$ $\longrightarrow$ result unchanged !
- η, η'
  - Short-distance analysis of LMD+V form factor in Knecht, Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
  - Simplified approach: VMD form factors normalized to decay width Γ(PS $\longrightarrow \gamma\gamma$).
  $$\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}^{\mathrm{VMD}}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \text{PS} = \eta, \eta'$$
  - $\Rightarrow a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$
  Not taking pole-approximation as done in Melnikov, Vainshtein '04 !
  Note: VMD form factor has too strong damping at large momenta $\longrightarrow$ values might be a bit too small !
- Our estimate for the sum of all light pseudoscalars (Nyffeler '09; Jegerlehner, Nyffeler '09):
$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

Summary of hadronic light-by-light scattering results



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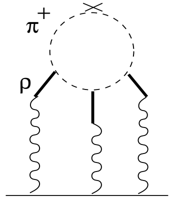

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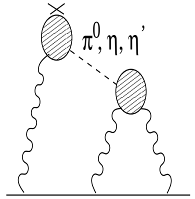
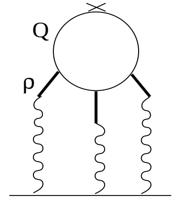


+ ··· + Exchange of other resonances $(f_0, a_1, \ldots)$ +



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Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

- Evaluations of full had. LbyL scattering contribution:
  - BPP = Bijnens, Pallante, Prades '95, '96, '02
    Use mainly ENJL (Extended Nambu-Jona-Lasinio) model; but for some contributions also other models
  - HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02
    Use mainly HLS (Hidden Local Symmetry) model; often HLS = VMD
- Partial evaluation:
  - MV = Melnikov, Vainshtein '04 (large-N_C QCD: LMD, LMD+V)
- Recent summaries on had. LbyL scattering:
  - BP = Bijnens, Prades '07
  - PdRV = Prades, de Rafael, Vainshtein '09 (arXiv:0901.0306)
    Analyzed results obtained by different groups with various models and suggested new estimates for some individual contributions (shifted central values, enlarged errors) to cover range of results.
  - JN = Jegerlehner, Nyffeler '09 (arXiv:0902.3360)
    Includes new evaluation of pion-exchange contribution (Nyffeler '09, arXiv:0901.1172).

Pseudoscalar exchanges

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
modified ENJL (off-shell) [BPP]	59(9)	85(13)
VMD / HLS (off-shell) [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	—
LMD+V (off-shell) [N]	72(12)	99(16)
AdS/QCD (off-shell ?) [HoK]	69	107
[PdRV]	—	114(13)
[JN]	72(12)	99(16)

BPP = Bijnens, Pallante, Prades '95, '96, '02 (ENJL = Extended Nambu-Jona-Lasinio model); HK(S) = Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02 (HLS = Hidden Local Symmetry model); KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; DB = Dorokhov, Broniowski '08 (χQM = Chiral Quark Model); N = Nyffeler '09; HoK = Hong, Kim '09; PdRV = Prades, de Rafael, Vainshtein '09; JN = Jegerlehner, Nyffeler '09

- BPP use rescaled VMD result for η, η'. Also all LMD+V evaluations use VMD for η, η' !
- Off-shell form factors used in BPP, HKS presumably do not fulfill new short-distance constraint at external vertex and might have too strong damping $\longrightarrow$ smaller values.
- Our result for pion with off-shell form factors at both vertices is not too far from value given by MV '04, but this is pure coincidence ! Approaches not comparable ! MV '04 evaluate pion-pole contribution and use on-shell form factors (constant form factor at external vertex).
  Note: Following MV '04 and using $h_2 = -10$ GeV^2 we obtain 79.8×10^{-11} for the pion-pole contribution, close to 79.6×10^{-11} given in Bijnens, Prades '07 and 79.7×10^{-11} in DB '08
- Nonlocal χQM: strong damping for off-shell pions. AdS/QCD: error estimated to be $< 30\%$.

Axial-vector exchanges

Model for $\mathcal{F}_{A^*\gamma^*\gamma^*}$	$a_\mu(a_1) \times 10^{11}$	$a_\mu(a_1, f_1, f_1') \times 10^{11}$
ENJL-VMD [BPP] (nonet symmetry)	2.5(1.0)	—
ENJL-like [HKS,HK] (nonet symmetry)	1.7(1.7)	—
LMD [MV] (f_1 pure octet, f_1' pure singlet)	5.7	17
LMD [MV] (ideal mixing)	5.7	22(5)
[PdRV]	—	15(10)
[JN]	—	22(5)

- MV '04: derived QCD short-distance constraint for axial-vector pole contribution with on-shell form factor $\mathcal{F}_{A\gamma^*\gamma^*}$ at both vertices
- Simple VMD ansatz: short-distance constraints forbids form factor at external vertex. Assuming all axial-vectors in the nonet have same mass M leads to

$$a_\mu^{\mathrm{AV}} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{M^2} N_C \,\mathrm{Tr}\left[\hat{Q}^4\right] \left(\frac{71}{192} + \frac{81}{16} S_2 - \frac{7\pi^2}{144}\right) + \ldots \approx 1010 \frac{m_\mu^2}{M^2} \times 10^{-11}$$

($\hat{Q} = \mathrm{diag}(2/3, -1/3, -1/3), \quad S_2 = 0.26043$)

Strong dependence on mass M:

$M = 1300$ MeV: $a_\mu^{\mathrm{AV}} = 7 \times 10^{-11}$, $\qquad M = M_\rho$: $a_\mu^{\mathrm{AV}} = 28 \times 10^{-11}$ (with $+\ldots$)

- More sophisticated LMD ansatz (Czarnecki, Marciano, Vainshtein '03): see Table. Now there is form factor at external vertex. Dressing leads to lower effective mass M. Furthermore f_1, f_1' have large coupling to photons $\longrightarrow$ huge enhancement compared to BPP, HKS !

Scalar exchanges

Model for $\mathcal{F}_{S^*\gamma^*\gamma^*}$	a_μ(scalars) $\times\ 10^{11}$
Point coupling	$-\infty$
ENJL [BPP]	−7(2)
[PdRV]	−7(7)
[JN]	−7(2)

- Within ENJL model: scalar exchange contribution related by Ward identities to (constituent) quark loop $\longrightarrow$ HK argued that effect of (broad) scalar resonances below several hundred MeV might already be included in sum of (dressed) quark loops and (dressed) $\pi + K$ loops !
- Potential double-counting is definitely an issue for the broad sigma meson $f_0(600)$ ($\leftrightarrow \pi^+\pi^-; \pi^0\pi^0$). Ongoing debate whether the scalar resonances $f_0(980), a_0(980)$ are two-quark or four-quark states.
- It is not clear which scalar resonances are described by ENJL model. Model parameters fixed by fitting various low-energy observables and resonance parameters, among them $M_S = 980$ MeV. However, model then yields $M_S^{\mathrm{ENJL}} = 620$ MeV.
- Can the usually broad scalar resonances be described by a simple resonance Lagrangian which works best in large-N_C limit, i.e. for very narrow states ?

Charged pion and kaon loops

Model $\pi^+\pi^-\gamma^*(\gamma^*)$	$a_\mu(\pi^\pm) \times 10^{11}$	$a_\mu(\pi^\pm, K^\pm) \times 10^{11}$
Point coupling (scalar QED)	-45.3	-49.8
VMD [KNO, HKS]	-16	$-$
full VMD [BPP]	$-18(13)$	$-19(13)$
HLS [HKS,HK]	-4.45	$-4.5(8.1)$
[MV] (all N_C^0 terms !)	$-$	$0(10)$
[PdRV]	$-$	$-19(19)$
[JN]	$-$	$-19(13)$

- Dressing leads to a rather huge suppression compared to scalar QED ! Very model dependent.
- MV '04 studied HLS model via expansion in $(m_\pi/M_\rho)^2$ and $(m_\mu - m_\pi)/m_\pi$:

$$a_{\mu;\mathrm{HLS}}^{\mathrm{LbL};\pi^\pm} = \left(\frac{\alpha}{\pi}\right)^3 \sum_{i=0}^{\infty} f_i\left[\frac{m_\mu - m_\pi}{m_\pi}, \ln\left(\frac{M_\rho}{m_\pi}\right)\right]\left(\frac{m_\pi^2}{M_\rho^2}\right)^i = \left(\frac{\alpha}{\pi}\right)^3 (-0.0058)$$
$$= (-46.37 + 35.46 + 10.98 - 4.70 - 0.3 + \ldots) \times 10^{-11} = -4.9(3) \times 10^{-11}$$

- Large cancellation between first three terms in series. Expansion converges only very slowly. Main reason: typical momenta in the loop integral are of order $\mu = 4m_\pi \approx 550$ MeV and the effective expansion parameter is μ/M_ρ, not m_π/M_ρ.
- MV '04: Final result is very likely suppressed, but also very model dependent $\longrightarrow$ chiral expansion looses predictive power $\longrightarrow$ lumped together all terms subleading in N_C.

Dressed quark loops

Model	$a_\mu(\text{quarks}) \times 10^{11}$
Point coupling	$\mathbf{62(3)}$
ENJL + bare heavy quark [BPP]	$\mathbf{21(3)}$
VMD [HKS, HK]	$\mathbf{9.7(11.1)}$
[PdRV] (Bare c-quark only !)	$\mathbf{2.3}$
[JN]	$\mathbf{21(3)}$

- de Rafael '94: dressed quark loops can be interpreted as irreducible contribution to the 4-point function $\langle VVVV \rangle$. They also appear as short-distance complement of low-energy hadronic models.
- Quark-hadron duality: the quark loops also model contributions from exchanges and loops of heavier hadronic states, like $\pi', a_0', f_0', p, n, \ldots$.
- Again very large model-dependent effect of the dressing (form factors).
- Recently, PdRV '09 argued that the dressed light-quark loops should not be included as separate contribution. They assume them to be already covered by using the short-distance constraint from MV '04 for the pseudoscalar-pole contribution. Why should this be the case ?

Hadronic light-by-light scattering in the muon $g-2$: anno 2010

Some results for the various contributions to $a_\mu^{\mathrm{LbyL;had}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	$-$	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	$-$	22 ± 5	$-$	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	$-$	$-$	$-$	$-$	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	$-$	$-$	$-$	-19 ± 19	-19 ± 13
π, K loops $+$subl. N_C	$-$	$-$	$-$	0 ± 10	$-$	$-$	$-$
quark loops	21 ± 3	9.7 ± 11.1	$-$	$-$	$-$	2.3	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades ’95, ’96, ’02; HKS = Hayakawa, Kinoshita, Sanda ’95, ’96; HK = Hayakawa, Kinoshita ’98, ’02; KN = Knecht, Nyffeler ’02; MV = Melnikov, Vainshtein ’04; BP = Bijnens, Prades ’07; MdRR = Miller, de Rafael, Roberts ’07; PdRV = Prades, de Rafael, Vainshtein ’09; N = Nyffeler ’09, JN = Jegerlehner, Nyffeler ’09

- Pseudoscalar-exchange contribution dominates numerically. But other contributions are not negligible. Note cancellation between π, K-loops and quark loops !
- PdRV: Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV ’04 on pseudoscalar-pole contribution. Why should this be the case ?
  Added all errors in quadrature ! Like HK(S). Too optimistic ?
- N, JN: Evaluation of the axial vectors by MV ’04 is definitely some improvement over earlier calculations. It seems, however, again to be only the axial-vector pole contribution.
  Added all errors linearly. Like BPP, MV, BP, MdRR. Too pessimistic ?

Hadronic light-by-light scattering in the electron $g-2$

Using the same procedure and models as for the muon we obtain (Nyffeler ’09):

$$
\begin{aligned}
a_e^{\mathrm{LbyL};\pi^0} &= (2.98 \pm 0.34) \times 10^{-14} \\
a_e^{\mathrm{LbyL};\eta} &= 0.49 \times 10^{-14} \\
a_e^{\mathrm{LbyL};\eta'} &= 0.39 \times 10^{-14} \\
a_e^{\mathrm{LbyL;PS}} &= (3.9 \pm 0.5) \times 10^{-14}
\end{aligned}
$$

Note: naive rescaling would yield a too small result

$$a_e^{\mathrm{LbyL};\pi^0}(\text{rescaled}) = (m_e/m_\mu)^2 \; a_\mu^{\mathrm{LbyL};\pi^0} = 1.7 \times 10^{-14}$$

Assuming that pseudoscalars give dominant contribution yields the “guesstimate” (Jegerlehner, Nyffeler ’09):

$$a_e^{\mathrm{LbyL;had}} = (3.9 \pm 1.3) \times 10^{-14}$$

Agrees with large-log estimate of pseudoscalar pole-contribution by PdRV ’09 (published version):

$$a_e^{\mathrm{LbyL;had}} = (3.5 \pm 1.0) \times 10^{-14}$$

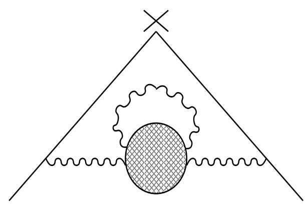
Outlook on had. LbyL scattering

- Most recent estimates:
  - PdRV '09: $a_\mu^{\text{LbyL}} = (105 \pm 26) \times 10^{-11}$
  - N, JN '09: $a_\mu^{\text{LbyL}} = (116 \pm 39) \times 10^{-11}$
- Compare with errors on
  - Had. vac. pol.: $40 - 53 \times 10^{-11}$
  - BNL $g-2$ exp. 63×10^{-11}
  - Future $g-2$ exp.: 15×10^{-11}
- Some progress made in recent years for pseudoscalars and axial-vector contributions, implementing many experimental and theoretical constraints. More work needed for η, η' !
- More uncertainty for exchanges of scalars (and heavier resonances) and for (dressed) pion + kaon loop and (dressed) quark loops. Furthermore, there are some cancellations.
- Soon results from Lattice QCD ? $\langle VVVV \rangle$ needs to be integrated over phase space of 3 off-shell photons $\longrightarrow$ much more complicated than hadronic vacuum polarization !

Outlook on had. LbyL scattering (cont.)

Suggested way forward in the meantime:

- Important to have unified consistent framework (model) which deals with all contributions.
- Purely phenomenological approach: resonance Lagrangian where all couplings are fixed from experiment. Non-renormalizable Lagrangian: how to achieve matching with pQCD ?
- Large-N_C framework: matching Green's functions with QCD short-distance constraints. (e.g. using Resonance Chiral Theory $\longrightarrow$ many unknown couplings enter).
- In both approaches: experimental information on various hadronic form factors with on-shell and off-shell photons would be very helpful, e.g. on $\mathcal{F}_{P\gamma^*\gamma^*}$, $\mathcal{F}_{S\gamma^*\gamma^*}$, $\mathcal{F}_{A\gamma^*\gamma^*}$ and $\mathcal{F}_{\pi^+\pi^-\gamma^*\gamma^*}$. e^+e^- colliders running around $1-3$ GeV, like DAΦNE-2, could help to measure some of these hadronic form factors for $0 < q_\gamma^2 < 2$ GeV.
- One can maybe get additional informations / cross-checks on form factors from lattice QCD. Recently a first lattice study of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ was presented: E. Shintani et al., arXiv:0912.0253 [hep-lat], Proceedings of Lattice 2009.
- Supplement these approaches with model-independent low-energy theorems in some particular limits (e.g. $m_\mu \longrightarrow 0, m_\pi \longrightarrow 0$ with m_μ / m_π fixed).
- Test models for had. LbyL scattering by comparison with exp. results for higher order contributions to had. vacuum polarization:



Conclusions

Summary of contributions to a_μ (Jegerlehner + Nyffeler '09):

- Leptonic QED contributions: $a_\mu^{\text{QED}} = (116\,584\,718.10 \pm 0.15) \times 10^{-11}$
- Electroweak contributions: $a_\mu^{\text{EW}} = (\underbrace{153.2}_{?} \pm \underbrace{1.8}_{?}) \times 10^{-11}$
- Hadronic contributions:
  - Vacuum Polarization: $a_\mu^{\text{had. v.p.}}(e^+e^-) = (\underbrace{6903.0}_{??} \pm \underbrace{52.6}_{??} - (100.3 \pm 2.2)) \times 10^{-11}$
  - Light-by-Light scattering: $a_\mu^{\text{LbyL}} = \underbrace{(116 \pm 39)}_{??} \times 10^{-11}$
- Total SM contribution:
  $a_\mu^{\text{SM}}(e^+e^-) = (116\,591\,790.0 \pm \underbrace{52.6}_{\text{v.p.}} \pm \underbrace{39}_{\text{LbyL}} \pm \underbrace{1.8}_{\text{QED + EW}} \quad [\pm 66.2]) \times 10^{-11}$
- "New" experimental value (shifted $+9.2 \times 10^{-11}$): $a_\mu^{\text{exp}} = (116\,592\,089 \pm 63) \times 10^{-11}$

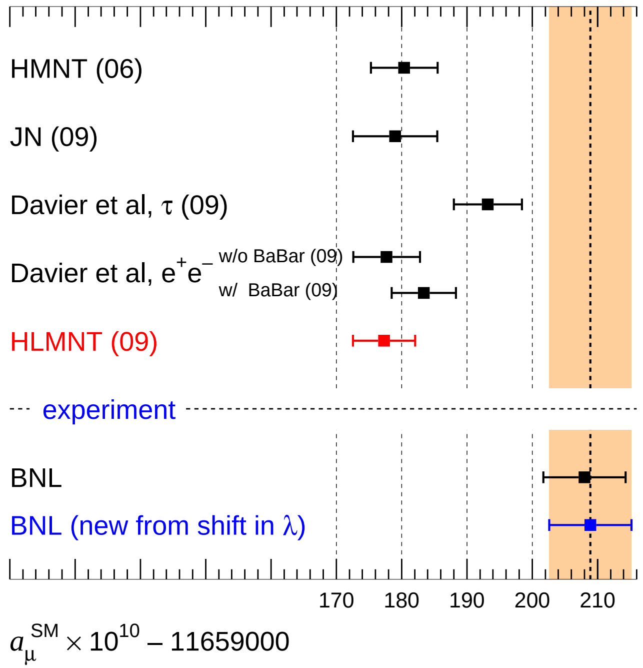
$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) = (299 \pm 92) \times 10^{-11} \qquad [3.3\,\sigma]$$

For comparison:

Davier et al. '09:	$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) = (255 \pm 80) \times 10^{-11}$	$[3.2\,\sigma]$
Davier et al. '09:	$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(\tau) = (157 \pm 82) \times 10^{-11}$	$[1.9\,\sigma]$
Teubner et al. '09:	$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) = (316 \pm 79) \times 10^{-11}$	$[4.0\,\sigma]$

Note: Davier et al. '09; Teubner et al. '09 use $a_\mu^{\text{LbyL}} = (105 \pm 26) \times 10^{-11}$ by PdRV '09.

Conclusions (cont.)



(Teubner et al., arXiv:1001:5401 [hep-ph])

Had. vacuum polarization:

- Using e^+e^- data: $3 - 4\sigma$ discrepancy between experiment and theory. Sign of New Physics ?
- Although evaluations using e^+e^- are all consistent, there are systematic deviations in spectral functions from different experiments, in particular around ρ-peak.
- Error from Had. VP underestimated ?
- τ-data have come closer to $e + e^-$ data, because of larger isospin breaking corrections.

Had. LbyL scattering:

- PdRV '09: $a_\mu^{\text{LbyL}} = (105 \pm 26) \times 10^{-11}$
- N, JN '09: $a_\mu^{\text{LbyL}} = (116 \pm 39) \times 10^{-11}$
  New evaluation of pion-exchange contribution using off-shell form factors in large-N_C QCD (new short-distance constraint at external vertex). We do not use pole-approximation.
- In view of the many still unresolved problems, we advocate conservative error estimate.

Note: whereas Had. VP can in principle be improved by better measurements of hadronic cross sections, Had. LbyL could be the show-stopper, if we really want to fully profit from a future $g - 2$ experiment at Fermilab or JPARC with a targeted precision of 15×10^{-11} !

Backup slides

Integral representation for pion-exchange contribution

Projection onto the muon $g-2$ leads to (Knecht, Nyffeler '02):

$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} &= -e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
&\times \left[\frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2)\ \mathcal{F}_{\pi^0*\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2 - m_\pi^2} T_1(q_1, q_2; p) \right. \\
&+ \left. \frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)\ \mathcal{F}_{\pi^0*\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} T_2(q_1, q_2; p) \right] \\
T_1(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2 \\
&- \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + 8(p\cdot q_2) q_1^2 q_2^2 - \frac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2 \\
&+ \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1\cdot q_2)^2 \\
T_2(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_1)^2 q_2^2 \\
&+ \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + \frac{8}{3}(p\cdot q_1) q_1^2 q_2^2 \\
&+ \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1\cdot q_2)^2
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

Jegerlehner, Nyffeler '09: could perform non-trivial integrations over angles $P\cdot Q_1, P\cdot Q_2$ (in Euclidean space) $\longrightarrow$ 3-dimensional integral representation for general form factors !
Integration variables: Q_1^2, Q_2^2 and angle θ between Q_1 and Q_2: $Q_1\cdot Q_2 = |Q_1||Q_2|\cos\theta$

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$

- 2-loop integral $\longrightarrow$ 8-dim. integral. Integration over 3 angles can be done easily
- 5 non-trivial integrations: 2 moduli: $|q_1|$, $|q_2|$, 3 angles: $p \cdot q_1$, $p \cdot q_2$, $q_1 \cdot q_2$ (recall $p^2 = m_\mu^2$).
- Observation: $p \cdot q_1$, $p \cdot q_2$ do not appear in the model-dependent form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$
- Can perform those two angular integrations by averaging expression for $a_\mu^{\mathrm{LbyL};\pi^0}$ over the direction of p (Jegerlehner + Nyffeler '09)

Method of Gegenbauer polynomials (hyperspherical approach)

(Baker, Johnson, Willey '64, '67; Rosner '67; Levine, Roskies '74; Levine, Remiddi, Roskies '79)

Denote by $\hat{K}$ unit vector of four-momentum vector K in Euclidean space

Propagators in Euclidean space:

$$\frac{1}{(K-L)^2 + M^2} = \frac{Z_{KL}^M}{|K||L|} \sum_{n=0}^{\infty} \left(Z_{KL}^M\right)^n C_n(\hat{K} \cdot \hat{L})$$

$$Z_{KL}^M = \frac{K^2 + L^2 + M^2 - \sqrt{(K^2 + L^2 + M^2)^2 - 4K^2L^2}}{2|K||L|}$$

Use orthogonality conditions of Gegenbauer polynomials:

$$\int d\Omega(\hat{K})\, C_n(\hat{Q}_1 \cdot \hat{K})\, C_m(\hat{K} \cdot \hat{Q}_2) = 2\pi^2 \frac{\delta_{nm}}{n+1} C_n(\hat{Q}_1 \cdot \hat{Q}_2)$$

$$\int d\Omega(\hat{K})\, C_n(\hat{Q} \cdot \hat{K})\, C_m(\hat{K} \cdot \hat{Q}) = 2\pi^2\, \delta_{nm}$$

$\hat{Q}_1 \cdot \hat{K}$ = Cosine of angle between the four-dimensional vectors Q_1 and K

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

Average over direction $\hat{P}$ (note: $P^2 = -m_\mu^2$):

$$\langle \cdots \rangle = \frac{1}{2\pi^2} \int d\Omega(\hat{P}) \cdots$$

After reducing numerators in the functions T_i in $a_\mu^{\mathrm{LbyL};\pi^0}$ against denominators of propagators, one is left with the following integrals, denoting propagators by $(4) \equiv (P + Q_1)^2 + m_\mu^2, \quad (5) \equiv (P - Q_2)^2 + m_\mu^2$:

$$\begin{aligned}
\langle \frac{1}{(4)} \frac{1}{(5)} \rangle &= \frac{1}{m_\mu^2 R_{12}} \arctan\left(\frac{zx}{1 - zt}\right) \\
\langle (P \cdot Q_1) \frac{1}{(5)} \rangle &= -(Q_1 \cdot Q_2) \frac{(1 - R_{m2})^2}{8 m_\mu^2} \\
\langle (P \cdot Q_2) \frac{1}{(4)} \rangle &= (Q_1 \cdot Q_2) \frac{(1 - R_{m1})^2}{8 m_\mu^2} \\
\langle \frac{1}{(4)} \rangle &= -\frac{1 - R_{m1}}{2 m_\mu^2} \\
\langle \frac{1}{(5)} \rangle &= -\frac{1 - R_{m2}}{2 m_\mu^2}
\end{aligned}$$

$Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta$, $\quad t = \cos\theta \quad$ (θ = angle between Q_1 and Q_2), $\quad Q_i \equiv |Q_i|$

$R_{mi} = \sqrt{1 + 4m_\mu^2/Q_i^2}$, $\quad x = \sqrt{1 - t^2}$, $\quad R_{12} = Q_1 Q_2 x$, $\quad z = \frac{Q_1 Q_2}{4 m_\mu^2} (1 - R_{m1})(1 - R_{m2})$

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

In this way one obtains (Jegerlehner + Nyffeler '09)

$$a_\mu^{\mathrm{LbL};\pi^0} = -\frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt\,\sqrt{1-t^2}\,Q_1^3\,Q_2^3$$
$$\times\left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_2^2,-Q_1^2,-Q_3^2)\,\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_2^2,-Q_2^2,0)}{(Q_2^2+m_\pi^2)}\,I_1(Q_1,Q_2,t)\right.$$
$$\left.+\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_3^2,-Q_1^2,-Q_2^2)\,\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_3^2,-Q_3^2,0)}{(Q_3^2+m_\pi^2)}\,I_2(Q_1,Q_2,t)\right]$$

where $Q_3^2=(Q_1+Q_2)^2$, $\quad Q_1\cdot Q_2 = Q_1 Q_2\cos\theta$, $\quad t=\cos\theta$

$$I_1(Q_1,Q_2,t) = X(Q_1,Q_2,t)\Big(8\,P_1\,P_2\,(Q_1\cdot Q_2) - 2\,P_1\,P_3\,(Q_2^4/m_\mu^2 - 2\,Q_2^2) - 2\,P_1\,(2-Q_2^2/m_\mu^2+2\,(Q_1\cdot Q_2)\,/m_\mu^2)$$
$$+4\,P_2\,P_3\,Q_1^2 - 4\,P_2 - 2\,P_3\,(4+Q_1^2/m_\mu^2 - 2\,Q_2^2/m_\mu^2) + 2/m_\mu^2\Big)$$
$$-2\,P_1\,P_2\,(1+(1-R_{m1})\,(Q_1\cdot Q_2)\,/m_\mu^2) + P_1\,P_3\,(2-(1-R_{m1})\,Q_2^2/m_\mu^2) + P_1\,(1-R_{m1})/m_\mu^2$$
$$+P_2\,P_3\,(2+(1-R_{m1})^2\,(Q_1\cdot Q_2)\,/m_\mu^2) + 3\,P_3\,(1-R_{m1})/m_\mu^2$$

$$I_2(Q_1,Q_2,t) = X(Q_1,Q_2,t)\Big(4\,P_1\,P_2\,(Q_1\cdot Q_2) + 2\,P_1\,P_3\,Q_2^2 - 2\,P_1 + 2\,P_2\,P_3\,Q_1^2 - 2\,P_2 - 4P_3 - 4/m_\mu^2\Big)$$
$$-2\,P_1\,P_2 - 3\,P_1\,(1-R_{m2})/(2m_\mu^2) - 3\,P_2\,(1-R_{m1})/(2m_\mu^2) - P_3\,(2-R_{m1}-R_{m2})/(2m_\mu^2)$$
$$+P_1\,P_3\,(2+3\,(1-R_{m2})\,Q_2^2/(2m_\mu^2) + (1-R_{m2})^2\,(Q_1\cdot Q_2)\,/(2m_\mu^2))$$
$$+P_2\,P_3\,(2+3\,(1-R_{m1})\,Q_1^2/(2m_\mu^2) + (1-R_{m1})^2\,(Q_1\cdot Q_2)\,/(2m_\mu^2))$$

where $P_1^2 = 1/Q_1^2$, $P_2^2 = 1/Q_2^2$, $P_3^2 = 1/Q_3^2$, $\quad X(Q_1,Q_2,t) = \frac{1}{Q_1 Q_2\,x}\arctan\left(\frac{zx}{1-zt}\right)$

Estimates for the quark condensate magnetic susceptibility χ

Authors	Method	$\chi(\mu)$ [GeV]$^{-2}$	Footnote
Ioffe, Smilga '84	QCD sum rules	$\chi(\mu = 0.5\text{ GeV}) = -\left(8.16^{+2.95}_{-1.91}\right)$	[1]
Narison '08	QCD sum rules	$\chi = -(8.5 \pm 1.0)$	[2]
Vainshtein '03	OPE for $\langle VVA \rangle$	$\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$	[3]
Gorsky, Krikun '09	AdS/QCD	$\chi = -(2.15 N_C)/(8\pi^2 F_\pi^2) = -9.6$	[4]
Dorokhov '05	Instanton liquid model	$\chi(\mu \sim 0.5 - 0.6\text{ GeV}) = -4.32$	[5]
Ioffe '09	Zero-modes of Dirac operator	$\chi(\mu \sim 1\text{ GeV}) = -3.52\ (\pm 30 - 50\%)$	[6]
Buividovich et al. '09	Lattice	$\chi = -1.547(6)$	[7]
Balitsky, Yung '83	LMD for $\langle VT \rangle$	$\chi = -2/M_V^2 = -3.3$	[8]
Belyaev, Kogan '84	QCD sum rules for $\langle VT \rangle$	$\chi(0.5\text{ GeV}) = -(5.7 \pm 0.6)$	[9]
Balitsky et al. '85	QCD sum rules for $\langle VT \rangle$	$\chi(1\text{ GeV}) = -(4.4 \pm 0.4)$	[9]
Ball et al. '03	QCD sum rules for $\langle VT \rangle$	$\chi(1\text{ GeV}) = -(3.15 \pm 0.30)$	[9]

[1]: QCD sum rule evalation of nucleon magnetic moments.
[2]: Recent reanalysis of these sum rules for nucleon magnetic moments. At which scale μ ?
[3]: Probably at low scale $\mu \sim 0.5$ GeV, since pion dominance was assumed in derivation.
[4]: From derivation in holographic model it is not clear what is the relevant scale μ.
[5]: The scale is set by the inverse average instanton size ρ^{-1}.
[6]: Study of zero-mode solutions of Dirac equation in presence of arbitrary gluon fields (à la Banks-Casher).
[7]: Again à la Banks-Casher. Quenched lattice calculation for $SU(2)$. μ dependence is not taken into account. Lattice spacing corresponds to 2 GeV.
[8]: The leading short-distance behavior of Π_{VT} is given by (Craigie, Stern '81)

$$\lim_{\lambda \to \infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2} \frac{\langle \overline{\psi}\psi \rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

Assuming that the two-point function Π_{VT} is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies this OPE constraint leads to the ansatz (Balitsky, Yung '83, Belyaev, Kogan '84, Knecht, Nyffeler, EPJC '01)

$$\Pi_{VT}^{\mathrm{LMD}}(p^2) = -\langle \overline{\psi}\psi \rangle_0 \frac{1}{p^2 - M_V^2} \quad \Rightarrow \chi^{\mathrm{LMD}} = -\frac{2}{M_V^2} = -3.3\text{ GeV}^{-2}$$

Not obvious at which scale. Maybe $\mu = M_V$ as for low-energy constants in ChPT.
[9]: LMD estimate later improved by taking more resonance states $\rho', \rho'', \ldots$ in QCD sum rule analysis of $\langle VT \rangle$.
Note that the last value by Ball et al. is very close to original LMD estimate !

Constraining the LMD+V model parameter h_6

- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to value of h_6. We can get some indirect information on size and sign of h_6 as follows.
- Estimates of low-energy constants in chiral Lagrangians via exchange of resonances work quite well. However, we may get some corrections, if we consider the exchange of heavier resonances as well. Typically, a large-N_C error of 30% can be expected.
- In $\langle VVP \rangle$ appear 2 combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$, denoted by A_{V,p^2} and $A_{V,(p+q)^2}$ in Knecht, Nyffeler, EPJC '01.

$$A_{V,p^2}^{\mathrm{LMD}} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11 \, \frac{10^{-4}}{F_\pi^2}$$

$$A_{V,p^2}^{\mathrm{LMD+V}} = \frac{F_\pi^2}{8M_{V_1}^4} \frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2} \left(1 + \frac{M_{V_1}^2}{M_{V_2}^2}\right) = -1.36 \, \frac{10^{-4}}{F_\pi^2}$$

The relative change is only about 20%, well within expected large-N_C uncertainty !

$$A_{V,(p+q)^2}^{\mathrm{LMD}} = -\frac{F_\pi^2}{8M_V^4} = -0.26 \, \frac{10^{-4}}{F_\pi^2}, \qquad A_{V,(p+q)^2}^{\mathrm{LMD+V}} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4} h_6$$

Note that $A_{V,(p+q)^2}^{\mathrm{LMD}}$ is "small" compared to A_{V,p^2}^{LMD}. About same size as absolute value of the shift in A_{V,p^2} when going from LMD to LMD+V !

- Assuming that LMD/LMD+V framework is self-consistent, but allowing for a 100% uncertainty of $A_{V,(p+q)^2}^{\mathrm{LMD}}$, we get the range $h_6 = (5 \pm 5)$ GeV^4

Detailed results for the pion-exchange contribution (Nyffeler '09)

$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor:

	$h_6 = 0$ GeV^4	$h_6 = 5$ GeV^4	$h_6 = 10$ GeV^4
$h_3 = -10$ GeV^2	68.4	74.1	80.2
$h_3 = 0$ GeV^2	66.4	71.9	77.8
$h_3 = 10$ GeV^2	64.4	69.7	75.4
$h_4 = -10$ GeV^2	65.3	70.7	76.4
$h_4 = 0$ GeV^2	67.3	72.8	78.8
$h_4 = 10$ GeV^2	69.2	75.0	81.2

$\chi = -3.3$ GeV^{-2}, $h_1 = 0$ GeV^2, $h_2 = -10.63$ GeV^2 and $h_5 = 6.93$ GeV$^4 - h_3 m_\pi^2$

When varying h_3 (upper half of table), h_4 is fixed by constraint $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$.
In the lower half the procedure is reversed.

Within scanned region:
Minimal value: 63.2×10^{-11} $[\chi = -2.2$ GeV$^{-2}, h_3 = 10$ GeV$^2, h_6 = 0$ GeV$^4]$
Maximum value: 83.3×10^{-11} $[\chi = -4.4$ GeV$^{-2}, h_4 = 10$ GeV$^2, h_6 = 10$ GeV$^4]$

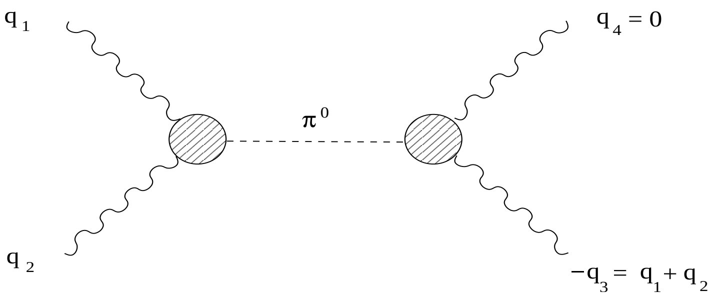
Take average of results for $h_6 = 5$ GeV^4 for $h_3 = 0$ GeV^2 and $h_4 = 0$ GeV^2 as estimate:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

Added errors from χ, h_3 (or h_4) and h_6 linearly. Do not follow Gaussian distribution !

The short-distance constraint by Melnikov and Vainshtein

- Melnikov, Vainshtein '04 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

- From this they deduced for the LbyL scattering amplitude (for finite q_1^2, q_2^2):

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu} \tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ = field strength tensors of internal photons with polarization vectors ϵ_i, for external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$.

- From the expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ it is again obvious that Melnikov and Vainshtein only consider the pion-pole contribution !
- No 2nd form factor at ext. vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by constant $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0)$!
- Overall $1/q_3^2$ behavior for large q_3^2 (apart from $f_3^{\sigma\rho}$). MV '04: agrees with quark-loop !
- For our off-shell LMD+V form factor at external vertex we get for large q_3^2:

$$\frac{3}{F_\pi} \mathcal{F}^{\text{LMD+V}}_{\pi^{0*}\gamma^*\gamma}(q_3^2, q_3^2, 0) \overset{q_3^2 \to \infty}{\to} \frac{h_1 + h_3 + h_4}{M_{V_1}^2 M_{V_2}^2} = \frac{2c_{VT}}{M_{V_1}^2 M_{V_2}^2} = x$$

With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with MV '04 !
Note: for large-N_C only the sum of all resonance exchanges has to match with quark-loop !

Hadronic light-by-light scattering: ChPT approach

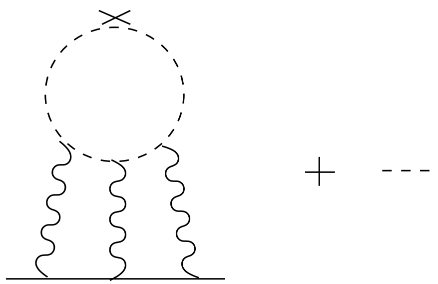
EFT for $E \ll 1$ GeV with pions, photons and muons

[de Rafael ’94; M. Knecht, A.N., M. Perrottet, E. de Rafael, ’02; Ramsey-Musolf + Wise ’02]

Note: chiral counting here refers to contribution to a_μ. Differs from counting in de Rafael ’94 !

Contributions to $a_\mu^{\text{LbyL;had}}$

$\mathcal{O}(p^6)$: charged pion loop
(finite, subleading in $1/N_C$)

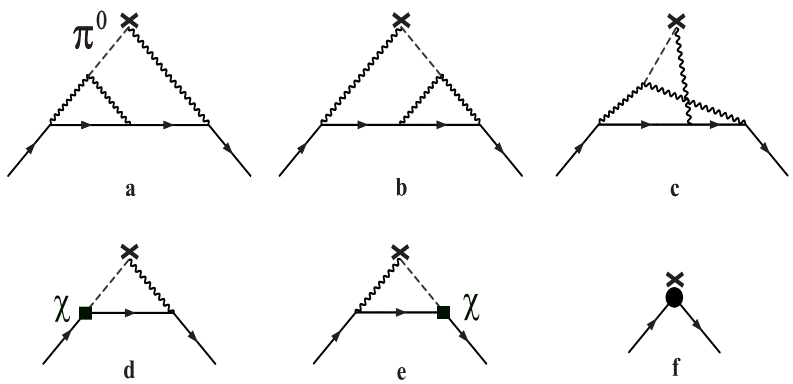


$\mathcal{O}(p^8)$: pion-pole (leading in $1/N_C$)

Divergent 2-loop contribution

$\longrightarrow$ need counterterms

1. One-loop graphs with insertion of χ (■) = coupling $\overline{\psi}\,\gamma_\mu\gamma_5\,\psi\,\partial^\mu\pi^0$
2. Local counterterm (●)



$\Rightarrow a_\mu^{\text{LbyL;had}}$ cannot be obtained in (pure) EFT framework
$\longrightarrow$ resonance models for form factors

Hadronic light-by-light scattering: Large log's

Renormalization group in EFT $\Rightarrow$ leading "large" logarithm $\ln^2(\mu_0/m_\mu)$

$$a_\mu^{\text{LbyL;had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f\left(\frac{m_{\pi^\pm}}{m_\mu}, \frac{m_{K^\pm}}{m_\mu}\right) \right.$$ (loops with pions and kaons)

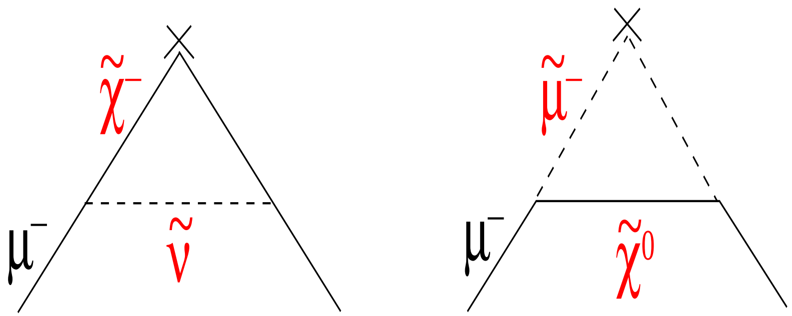
$$\left. + \overbrace{N_C \left(\frac{m_\mu^2}{16\pi^2 F_\pi^2}\frac{N_C}{3}\right)}^{\mathcal{C}\approx 0.025\ \text{(universal)}} \left[\ln^2\frac{\mu_0}{m_\mu} + \overbrace{c_1}^{\chi(\mu_0)\ (\pi^0\to e^+e^-)} \ln\frac{\mu_0}{m_\mu} + c_0\right] + \ldots \right\}$$

$f = -0.038$; $\mu_0 \sim M_\rho$: hadronic scale, $\ln\frac{M_\rho}{m_\mu} \sim 2$

Problem: π^0-exchange $\longrightarrow$ cancellation between $\ln^2$ and $\ln$:

$$\begin{aligned} a_\mu^{\text{LbyL};\pi^0}\Big|_{\text{VMD}} &= \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \left[\ln^2\frac{M_\rho}{m_\mu} + c_1 \ln\frac{M_\rho}{m_\mu} + c_0\right] \\ &\overset{\text{Fit}}{=} \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \ [3.94 - 3.30 + 1.08] \\ &= [12.3 - 10.3 + 3.4] \times 10^{-10} \\ &= 5.4 \times 10^{-10} \end{aligned}$$

SUSY contributions to a_μ



Chargino $\tilde{\chi}^-$ contribution dominates over neutralino $\tilde{\chi}^0$

Large $\tan\beta$ limit (Czarnecki + Marciano '01):

$$
\begin{aligned}
\left|a_\mu^{\text{SUSY}}\right| &\approx \frac{\alpha(M_Z)}{8\pi\sin^2\theta_W}\,\frac{m_\mu^2}{M_{\text{SUSY}}^2}\,\tan\beta\left(1-\frac{4\alpha}{\pi}\ln\frac{M_{\text{SUSY}}}{m_\mu}\right)\\
&\approx 130\times10^{-11}\left(\frac{100\text{ GeV}}{M_{\text{SUSY}}}\right)^2\tan\beta
\end{aligned}
$$

Compare: $a_\mu^{\text{EW}} \approx 150\times10^{-11}$

To explain $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) \approx 290\times10^{-11}$

$$\Rightarrow M_{\text{SUSY}} \approx 135-425\text{ GeV} \qquad (4<\tan\beta<40)$$