Lattice QCD Spectroscopy

Maxwell T. Hansen

IWHSS 2020

November 16th, 2020

THE UNIVERSITY of EDINBURGH

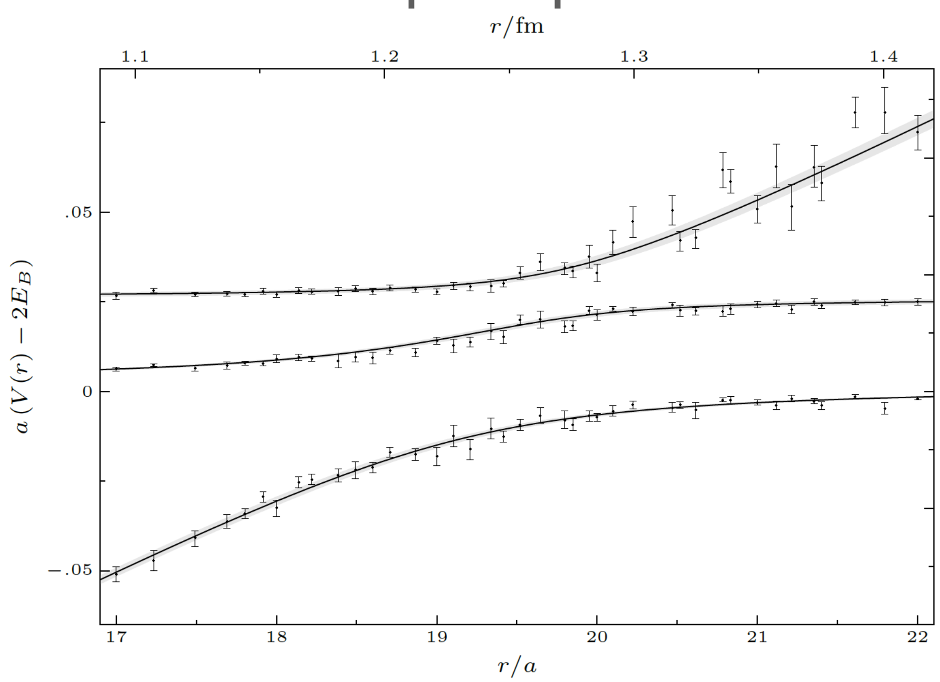
The strong force + the SM

- Collider phenomenology described by $SU(3) \times SU(2) \times U(1)$ + leptons, quarks, Higgs
  \+ neutrino masses

- Now know: QCD = theory of the ***strong*** strong force

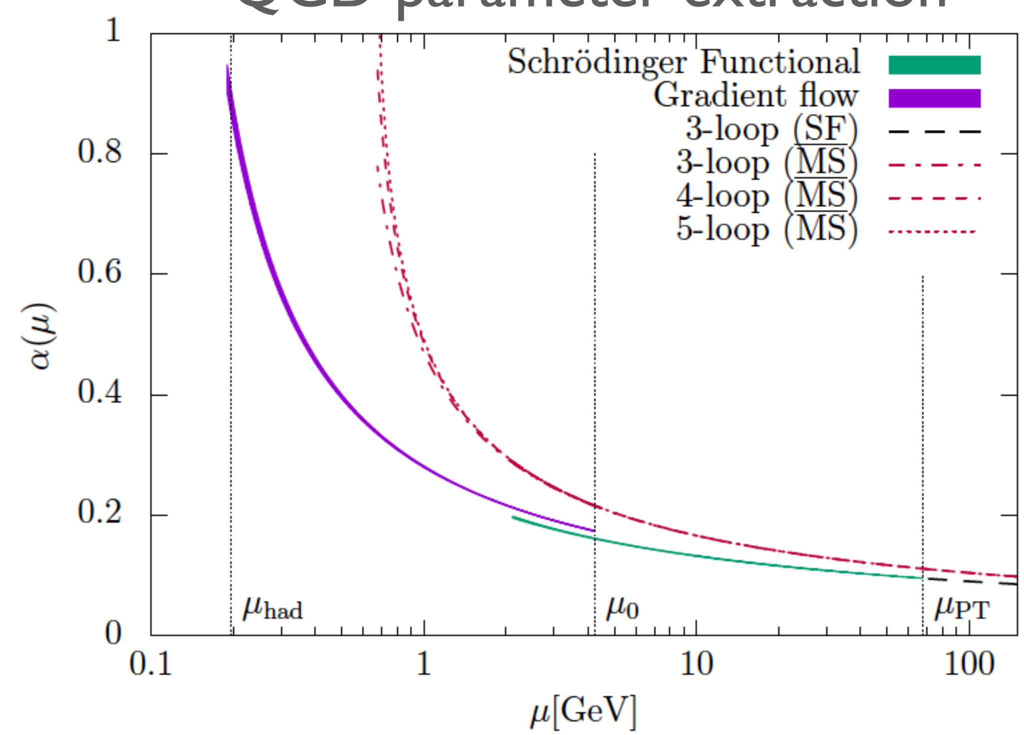
Non-numerical	Numerical (lattice QCD)
8-fold way	
Neutral pion decay $\pi^0 \to \gamma\gamma$	
ChPT predictions	
Large N_c expansion	

Inter-quark potential



• Bulava *et al.* (2019) •

QCD parameter extraction



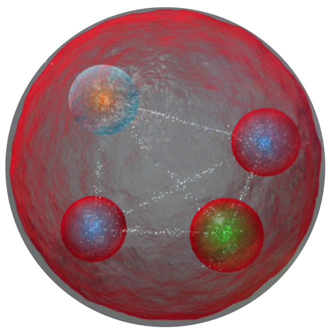
• Bruno *et al.* (2017) •

Precision hadron masses and matrix elements

FLAG 2019 Flavour Lattice Averaging Group

Multi-hadron observables

- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



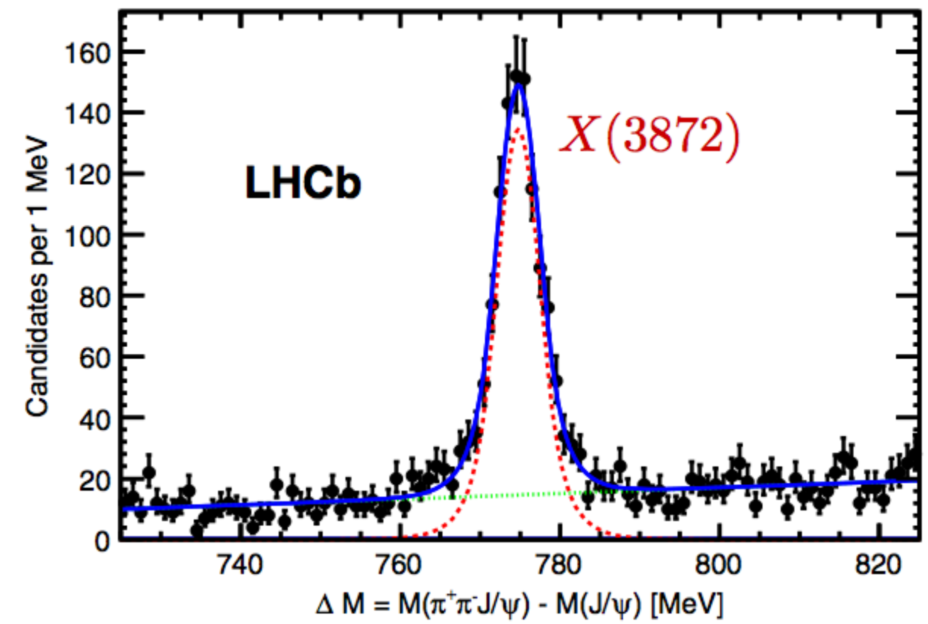
e.g. X(3872)

seems naturally *moleculer*

$$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$$

but… prompt production

• LHCb (PRD92, 2015) •



- Resonant enhancement in the electroweak

Resonant B decays $B \to \rho\,\ell\nu \to \pi\pi\,\ell\nu$ $B \to K^*\,\ell\ell \to K\pi\,\ell\ell$

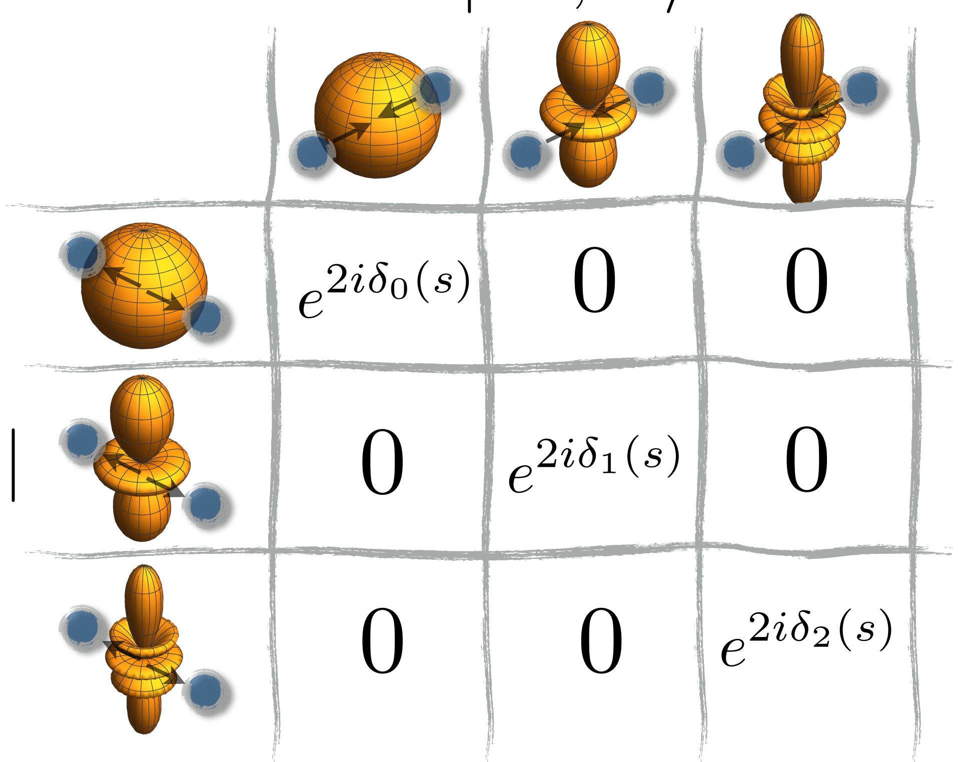
$$|X\rangle, |\rho\rangle\,, |K^*\rangle \quad \notin$$ **QCD Fock space**

QCD Fock space

- ❑ At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- ❑ Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \mathrm{in}\rangle$

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$



depends on $s = E_{\mathrm{cm}}^2$
and angular variables

diagonal in angular momentum

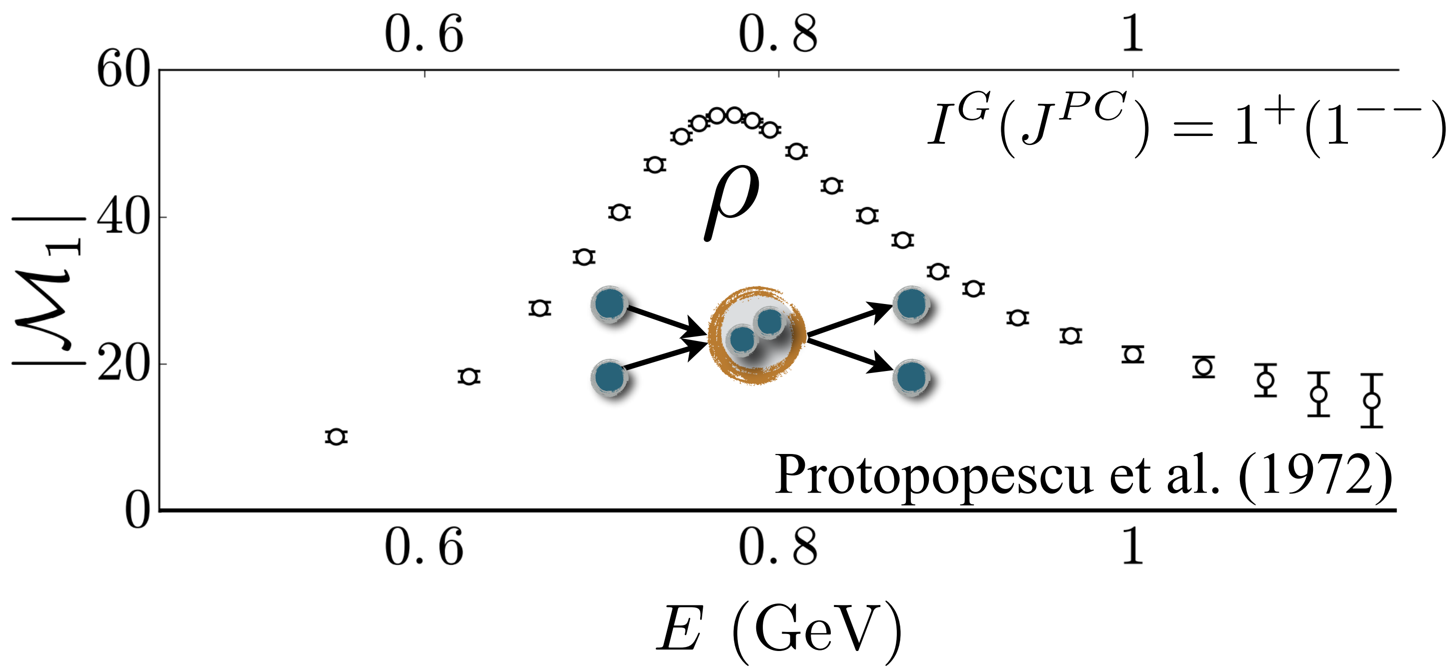
$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$

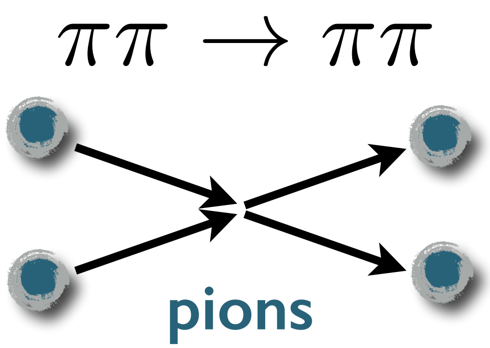
- ❑ ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

Resonance = complex pole

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate

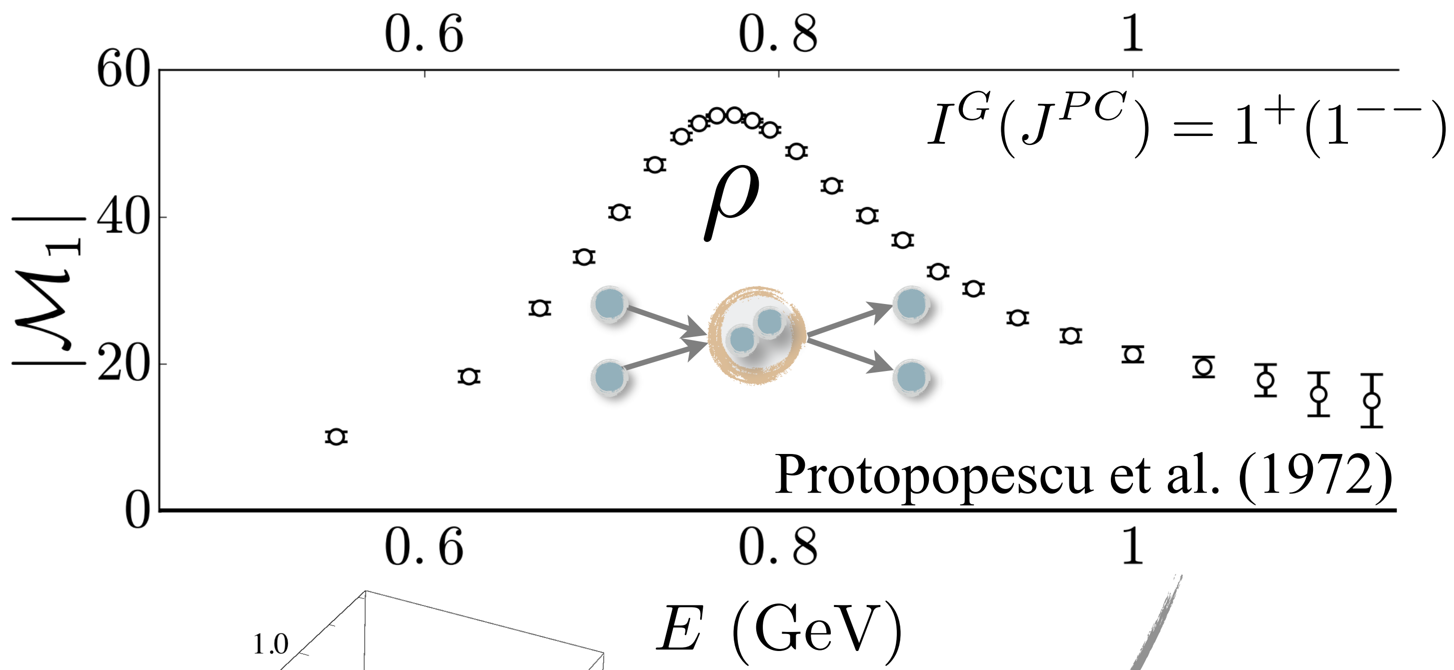


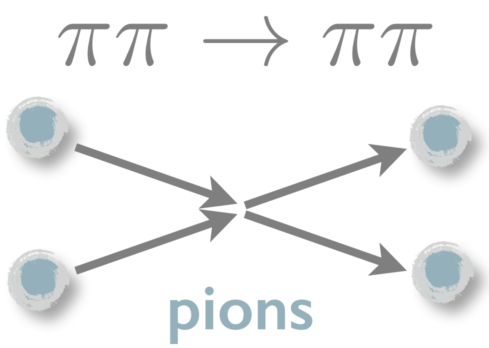


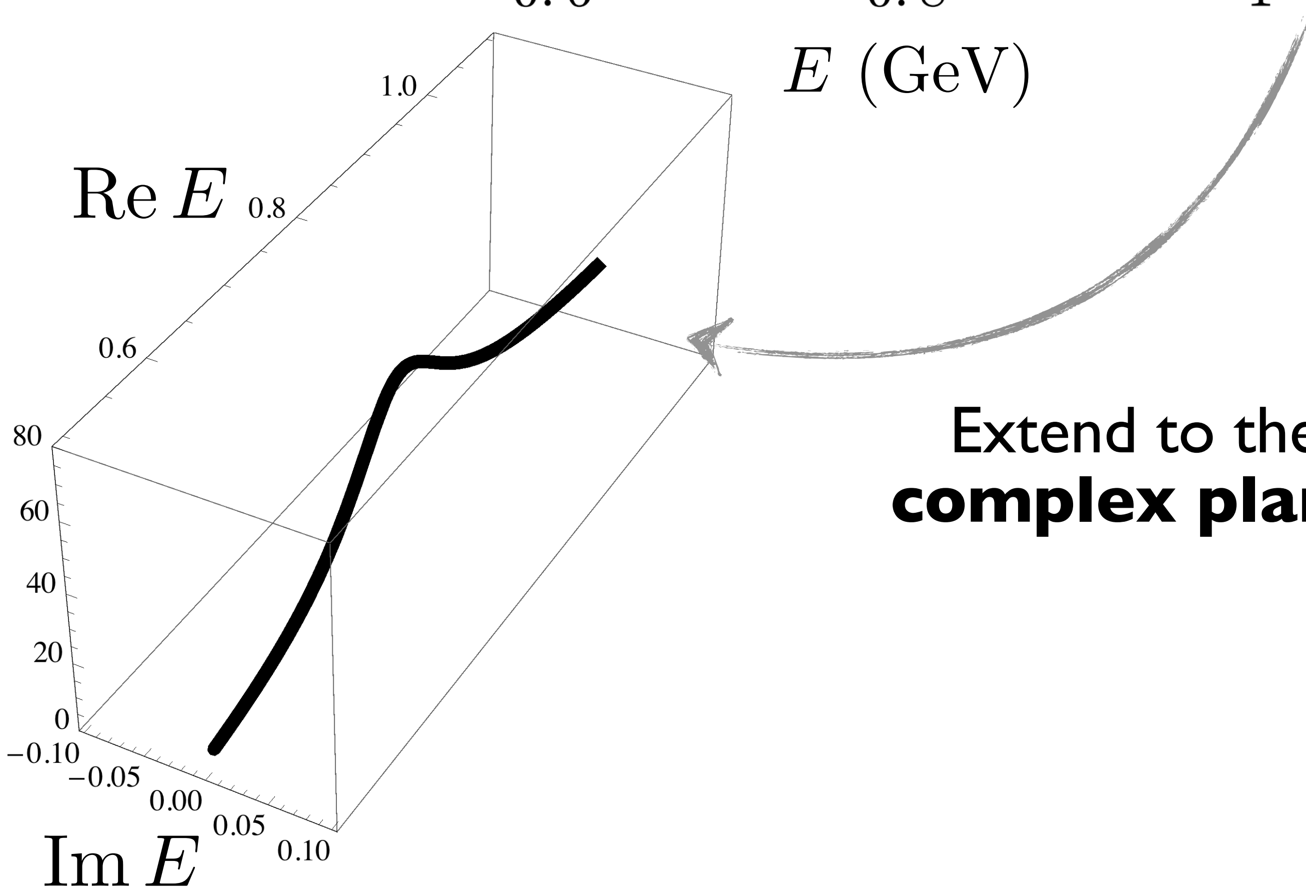
Resonance = complex pole

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





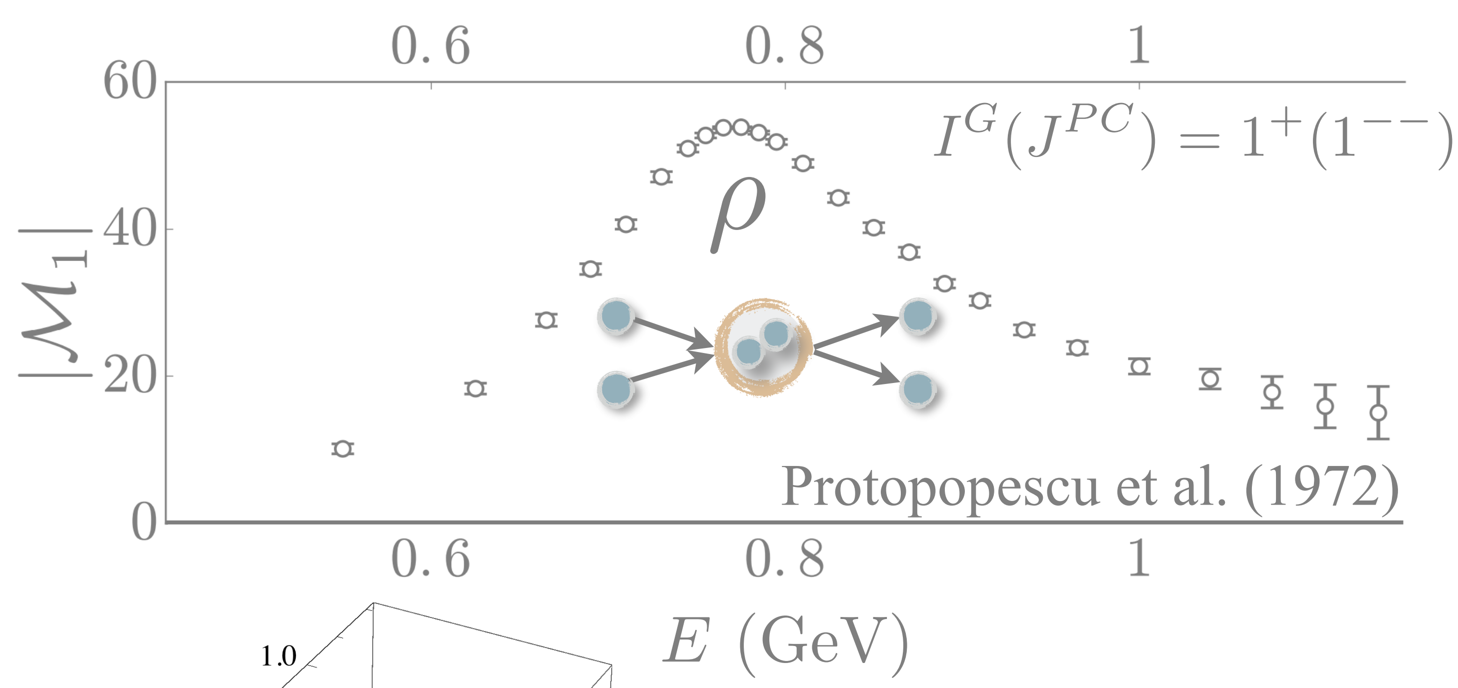


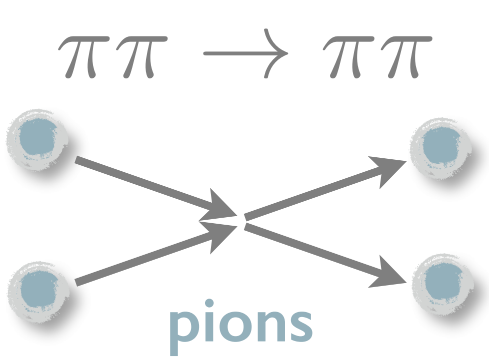
Extend to the **complex plane**

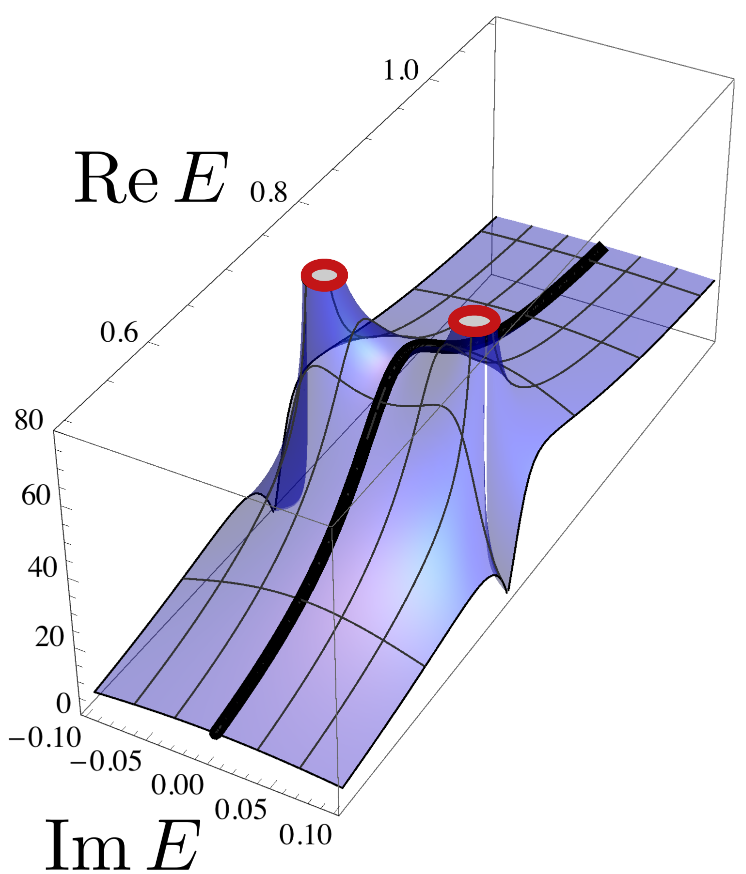
Resonance = complex pole

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

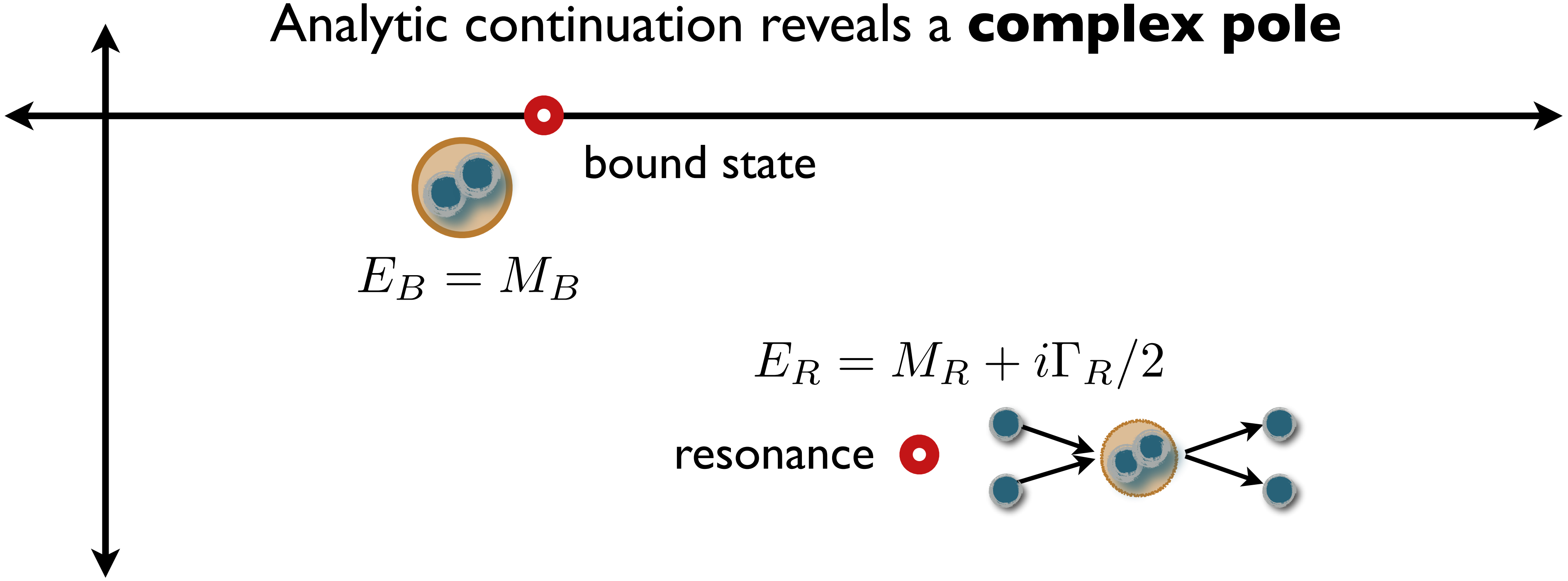
scattering rate







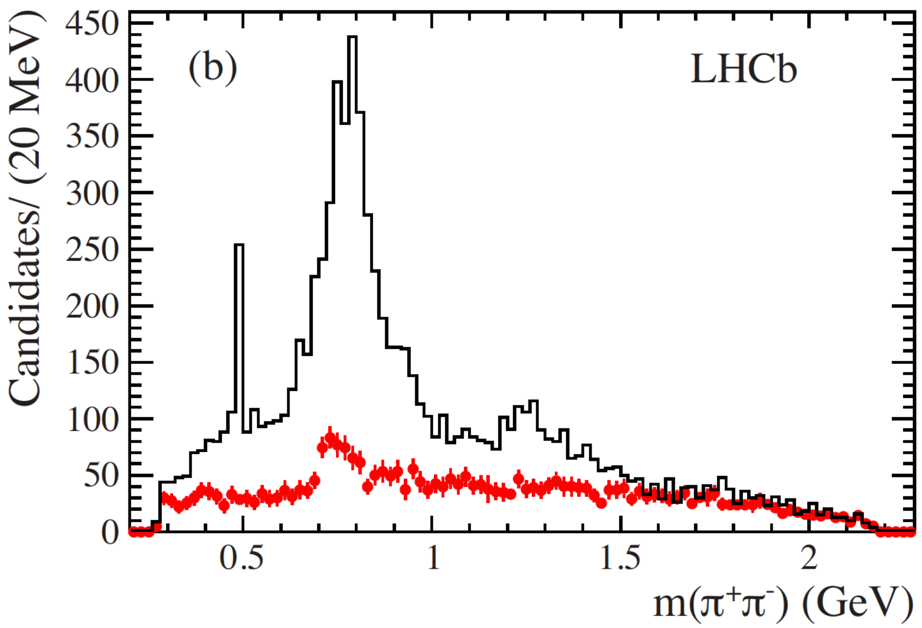
Analytic continuation reveals a **complex pole**



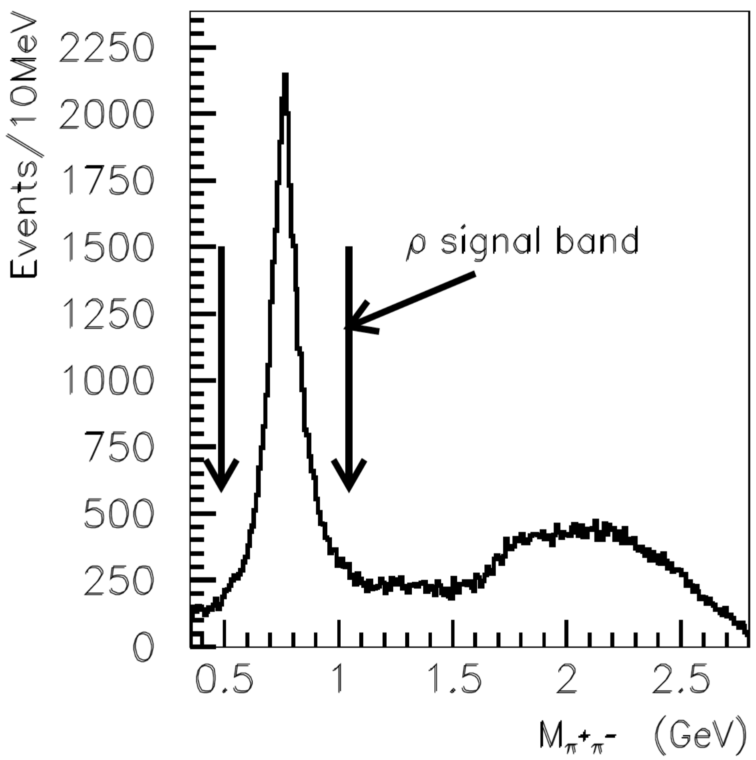
Pole is universal

- Resonances often seen in "production"

$$\overline{B}^0 \to J/\psi\,\pi^+\pi^-$$

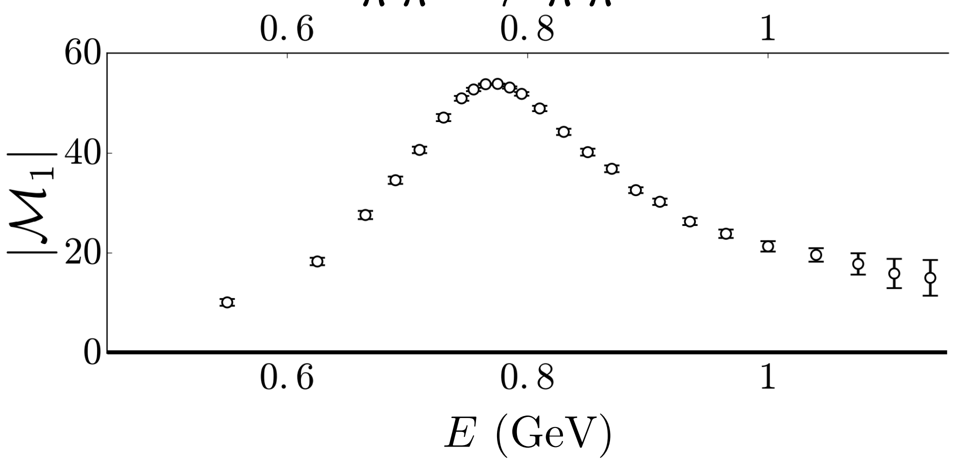


$$J/\psi \to \gamma\gamma\rho$$



(as opposed to scattering)

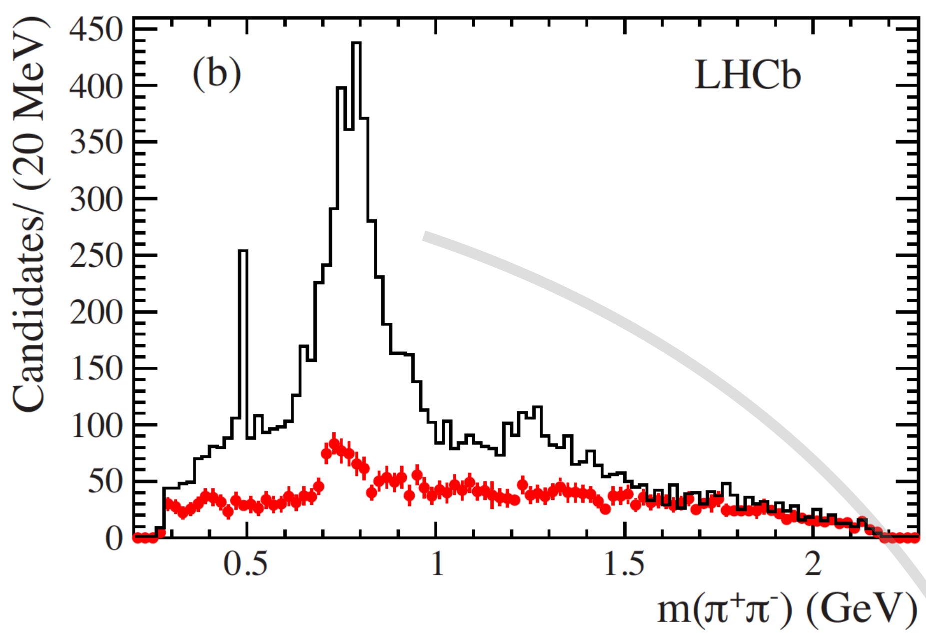
$$\pi\pi \to \pi\pi$$



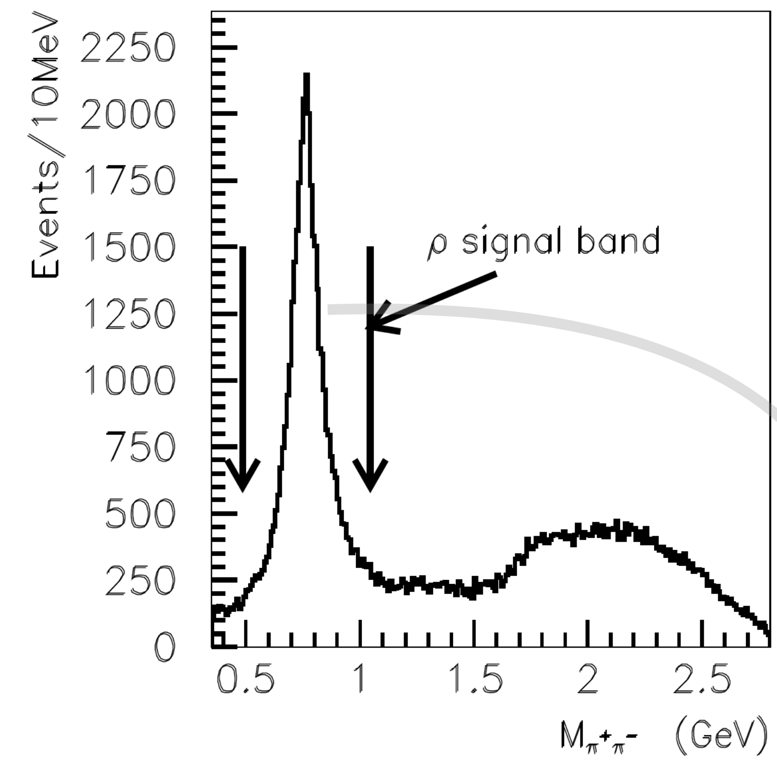
Pole is universal

- Resonances often seen in "production"

$\overline{B}^0 \to J/\psi\, \pi^+\pi^-$

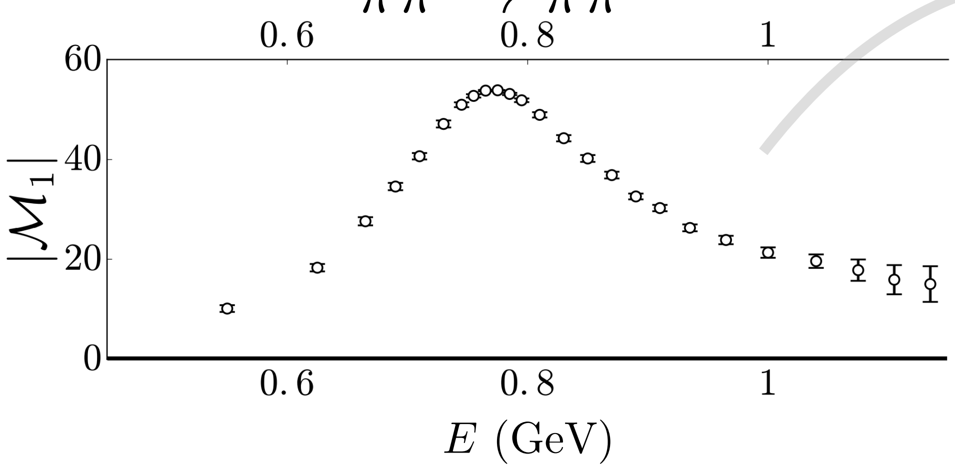


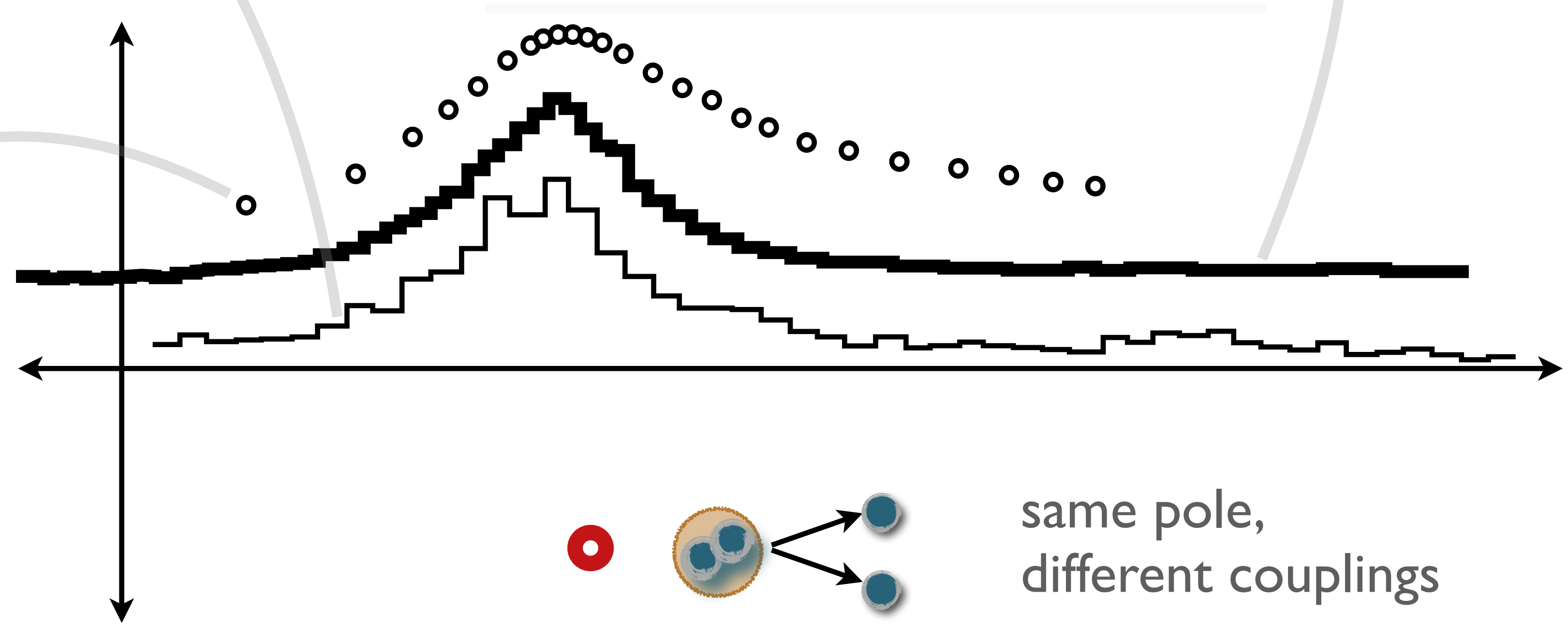
$J/\psi \to \gamma\gamma\rho$



(as opposed to scattering)

$\pi\pi \to \pi\pi$



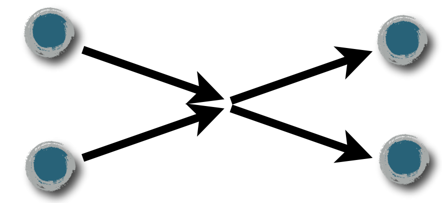


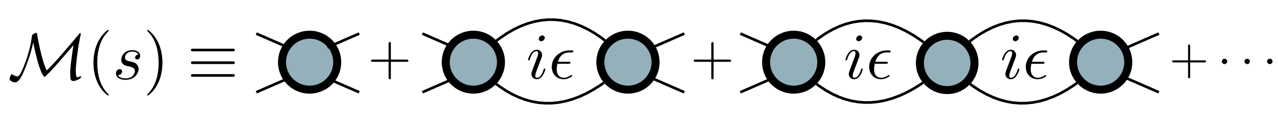
same pole,
different couplings

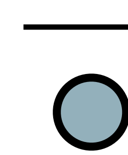
Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself

  For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

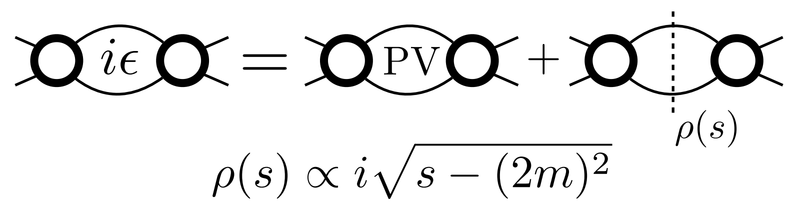






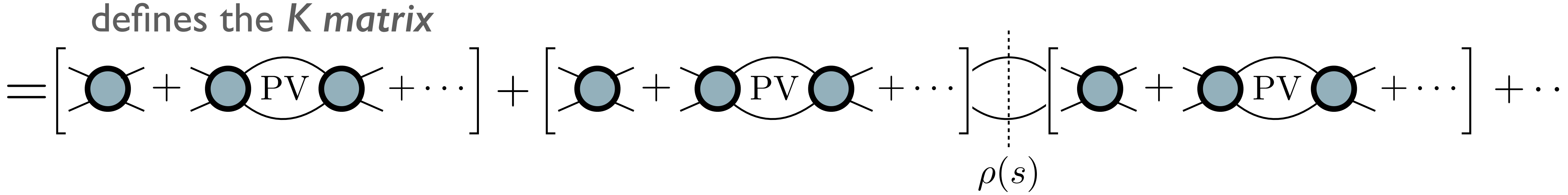
propagating pion

interaction kernel



cutting rule

defines the *K* ***matrix***



$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

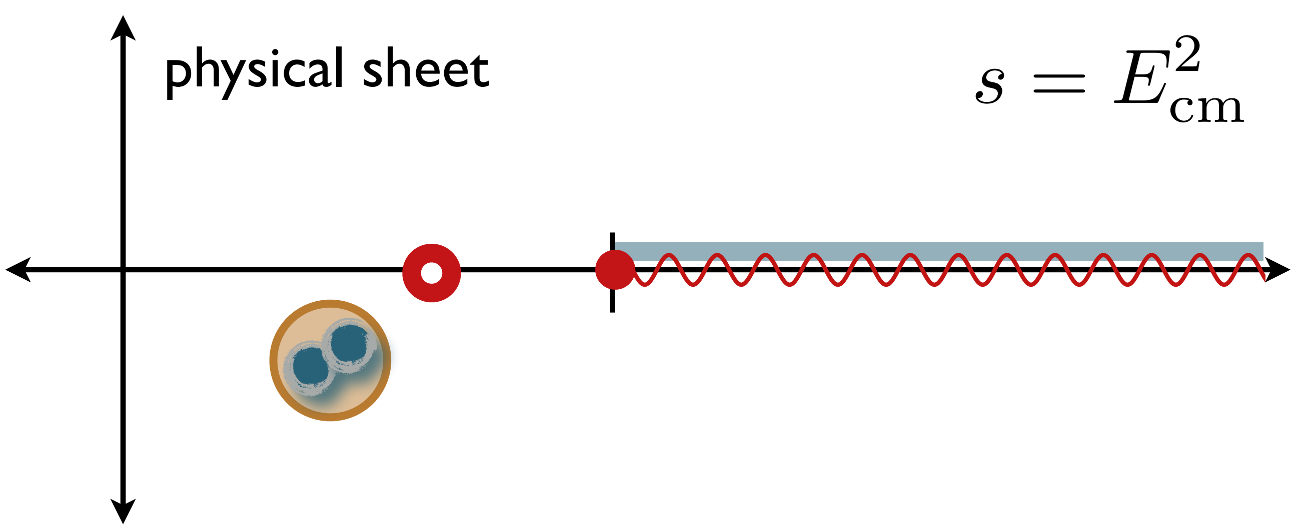
branch-cut singularity

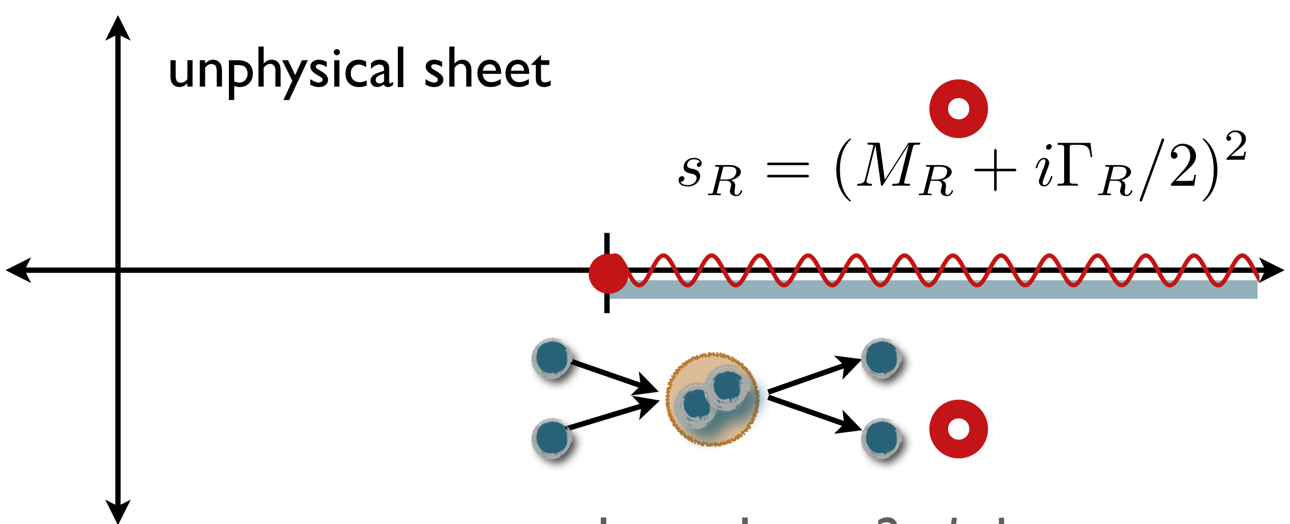
$\sqrt{s-(2m)^2}$

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - \rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto i\sqrt{s-(2m)^2}$$

- Each channel generates a *square-root cut* → doubles the number of sheets





resonance poles only on *2nd* sheet

- Important lessons:

Details of analyticity = important for quantitative understanding

Cutting analysis separates…

(i) long-distance kinematic singularities

(ii) short-distance/microscopic physics (depending on interaction details)

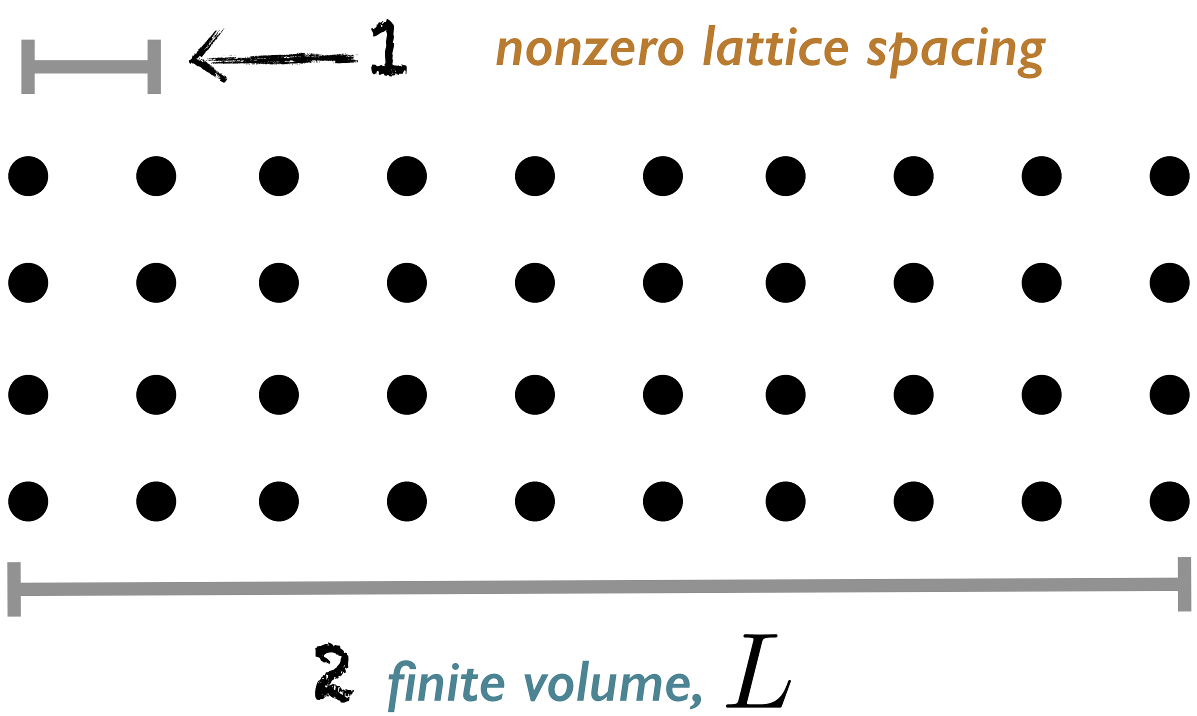
Microscopic physics *via Lattice QCD*

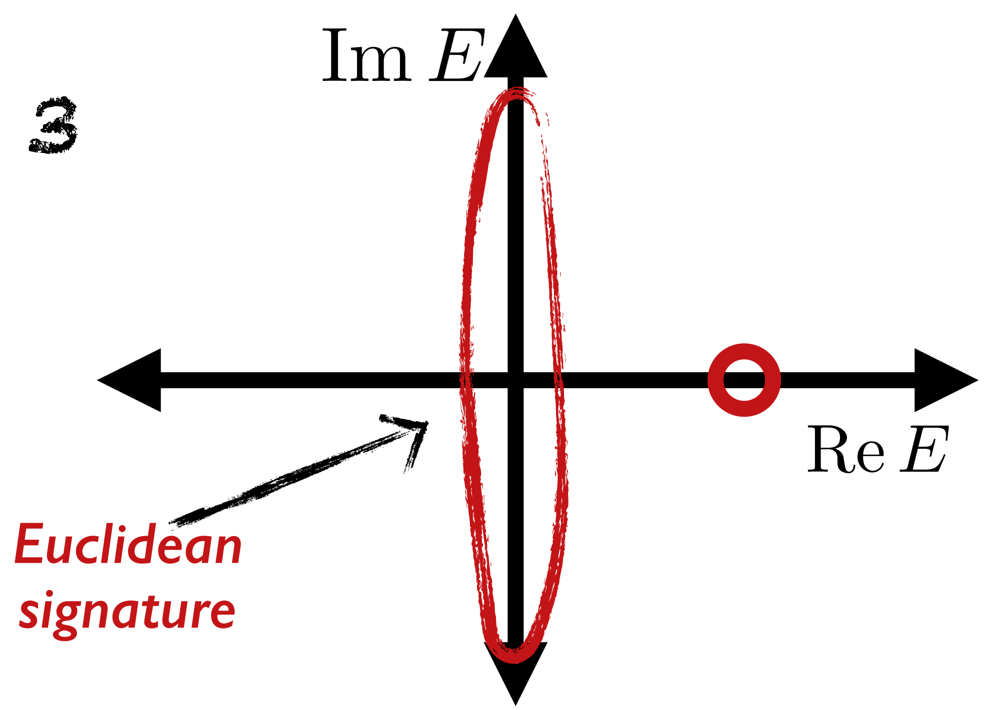
$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

Microscopic physics *via Lattice QCD*

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*

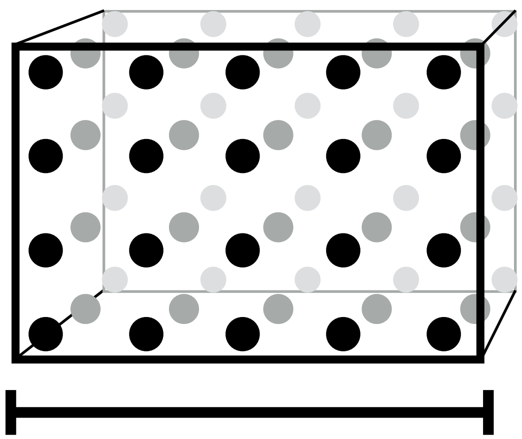




Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

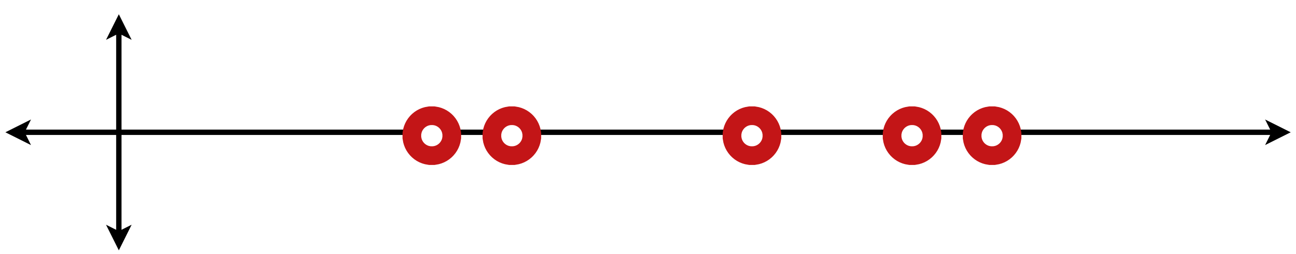
(but physical masses → increasingly common)

Difficulties for multi-hadron observables

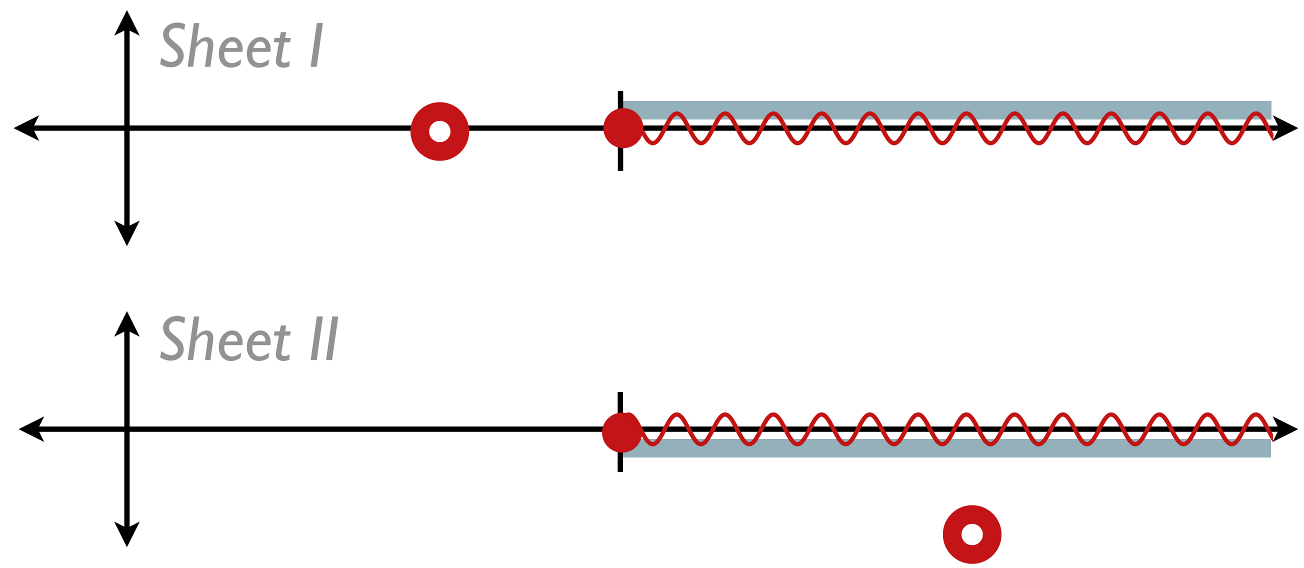


- The *finite volume*…
  - *Discretizes* the spectrum
  - *Eliminates* the branch cuts and extra sheets
  - *Hides* the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



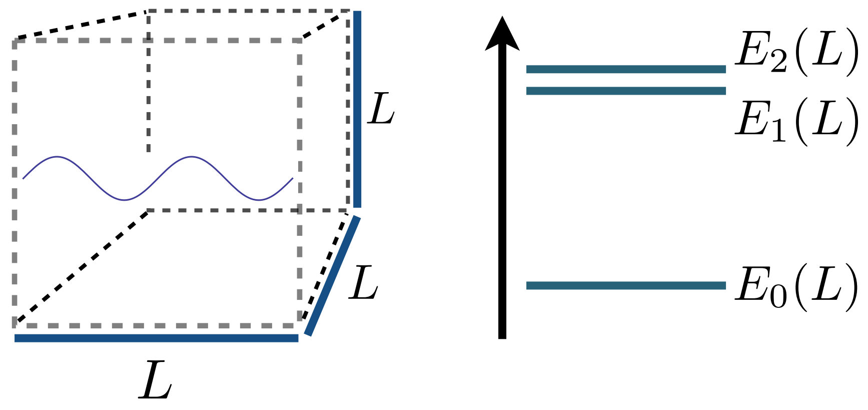
- LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

The finite-volume as a tool

- Finite-volume set-up

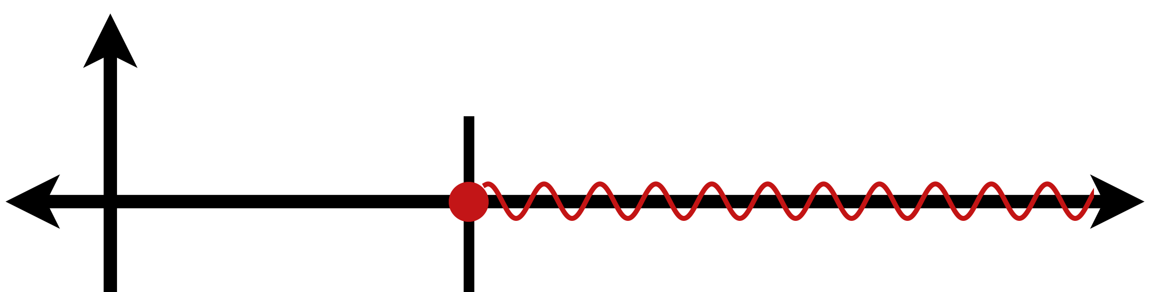


- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

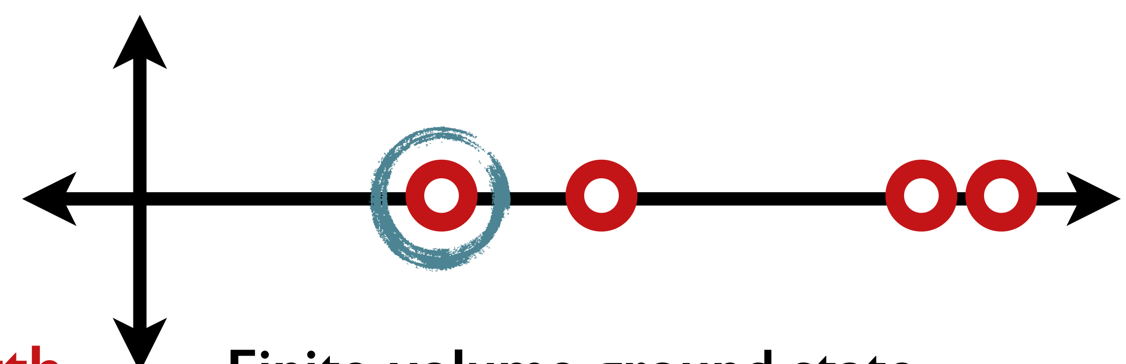
- *L* is large enough to neglect $e^{-M_\pi L}$
- *T* and lattice also negligible

- Scattering leaves an ***imprint*** on finite-volume quantities



$$E_0 = 2M_\pi$$

Infinite-volume threshold



Finite-volume ground state

scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

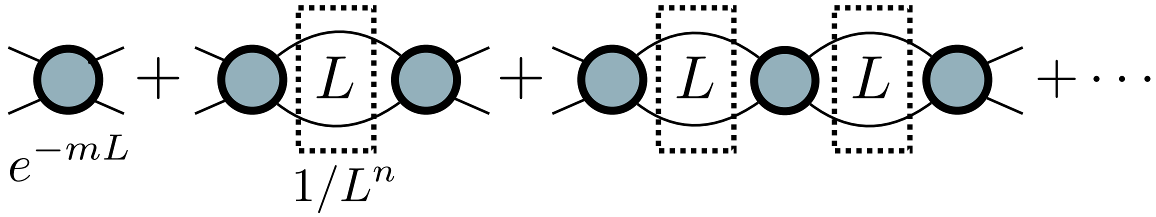
$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

Derivation

- Consider the finite-volume correlator:

$\mathcal{M}_L(P) =$



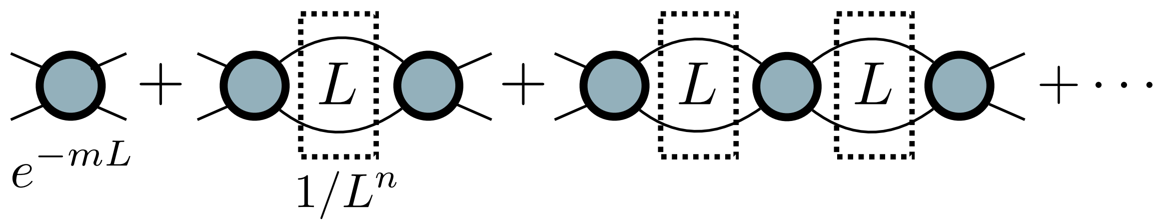
$\square = \sum_{\mathbf{k}}$

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?

- Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

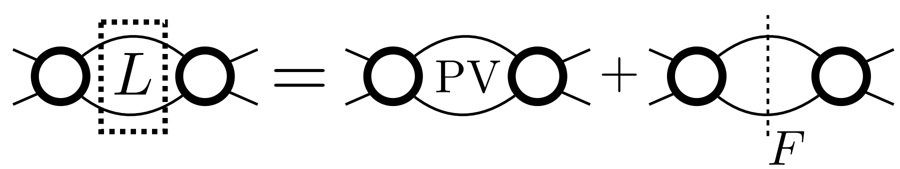
Derivation

- Consider the finite-volume correlator:

$\mathcal{M}_L(P) =$ 

$\square = \sum_{\mathbf{k}}$

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?

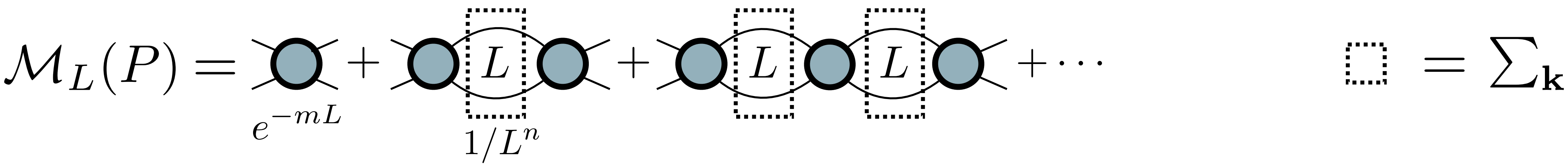


F = matrix of known geometric functions

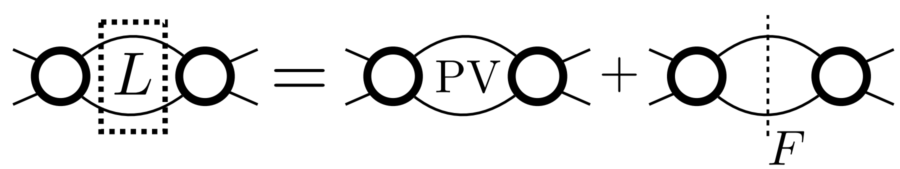
• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

Derivation

- Consider the finite-volume correlator:

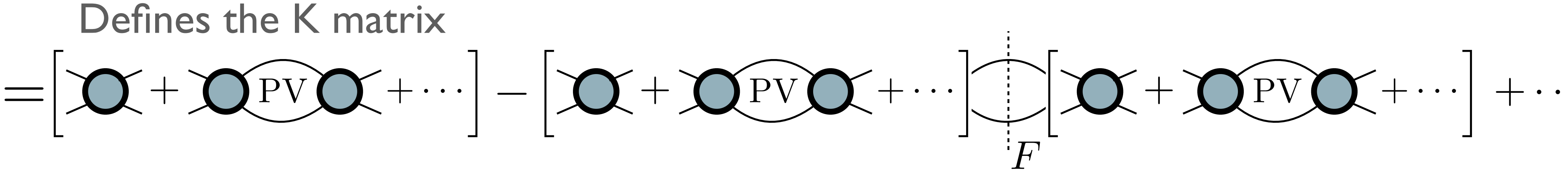


For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

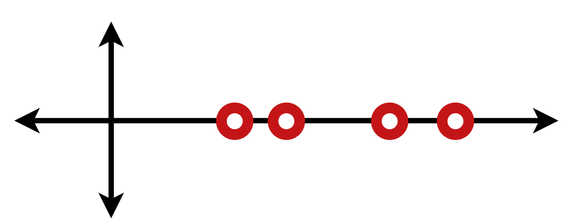


F = matrix of known geometric functions

Defines the K matrix



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

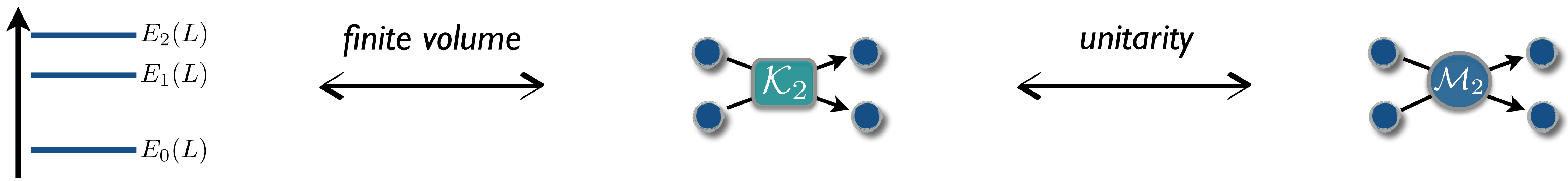


• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

Result

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to ***non-degenerate masses, multiple channels, spinning particles***

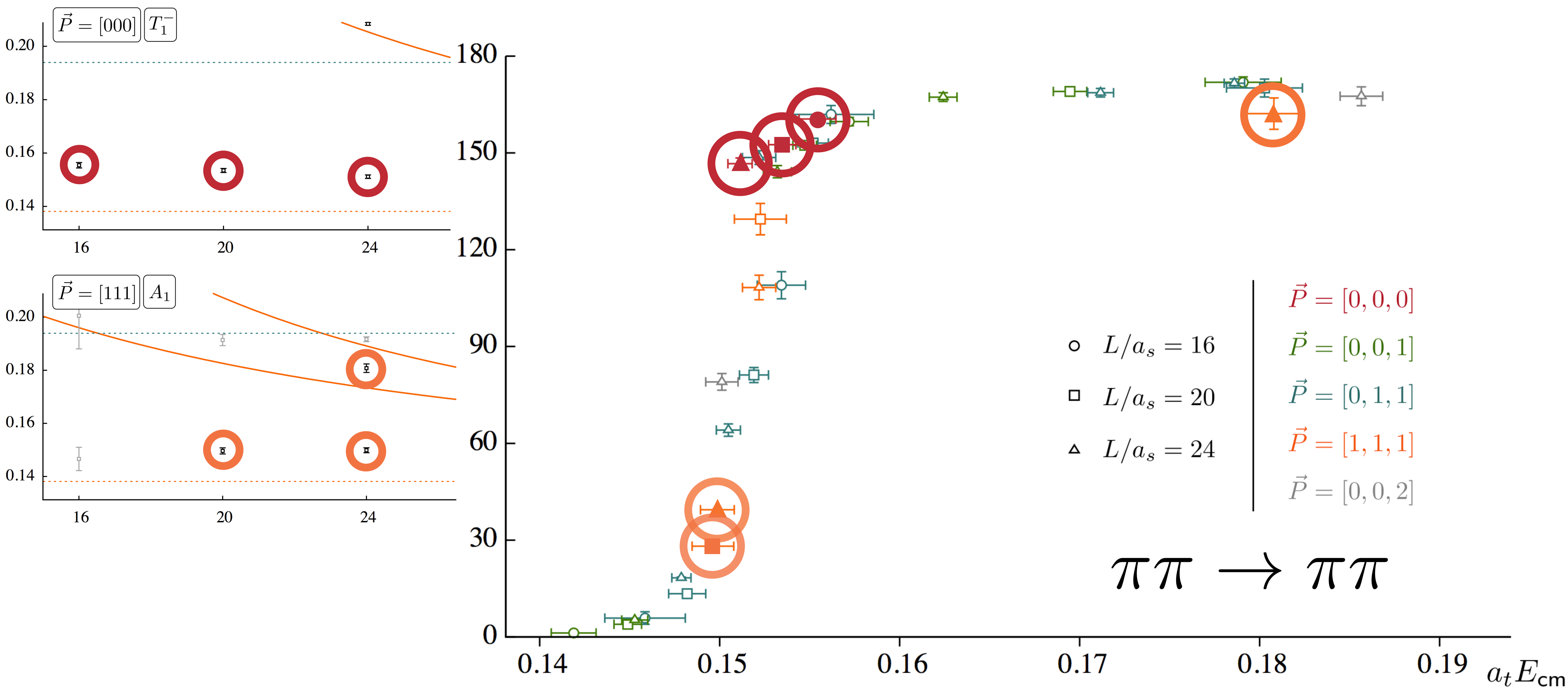
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

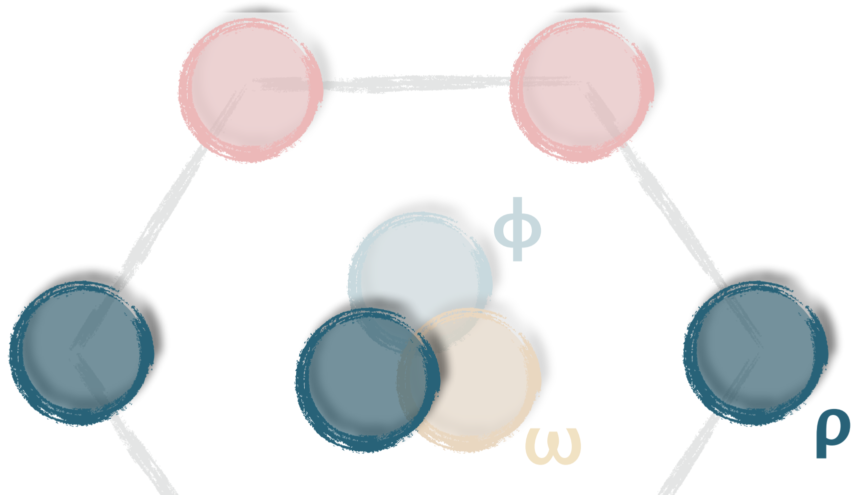
$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$

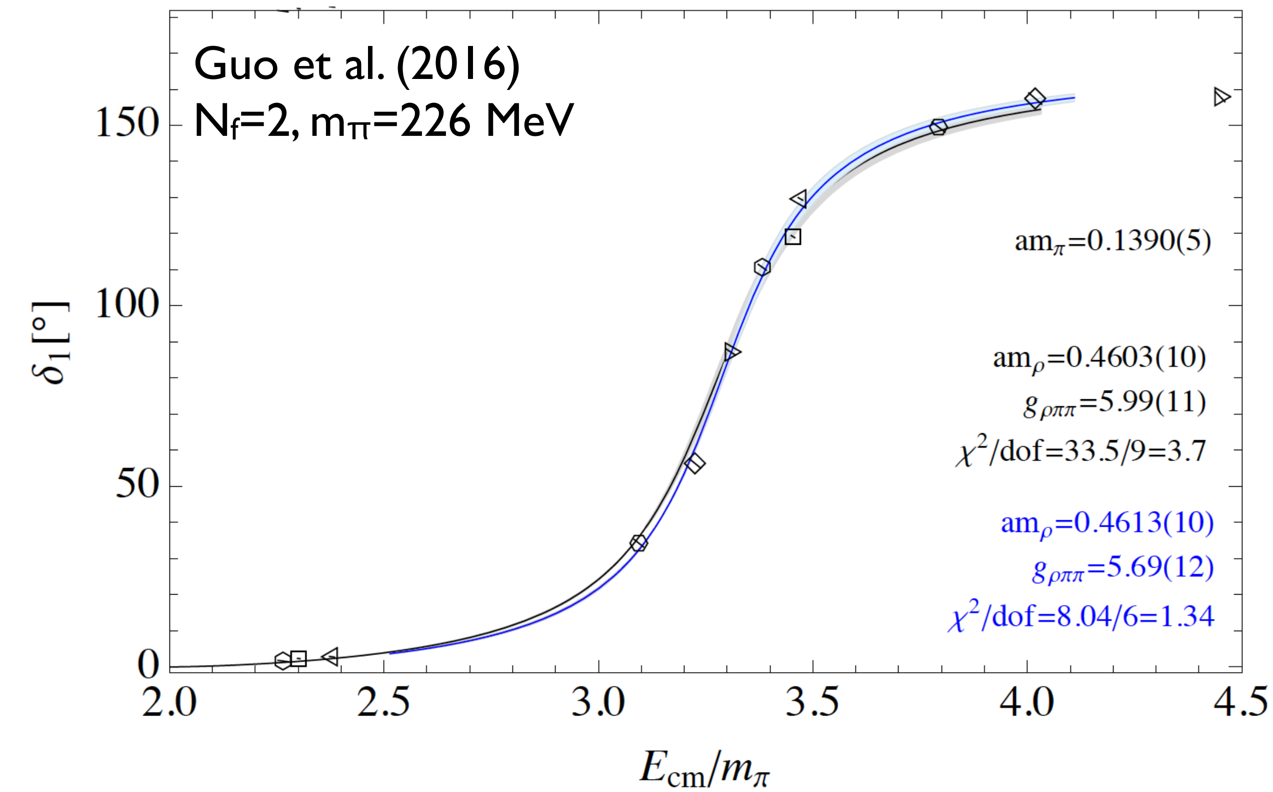


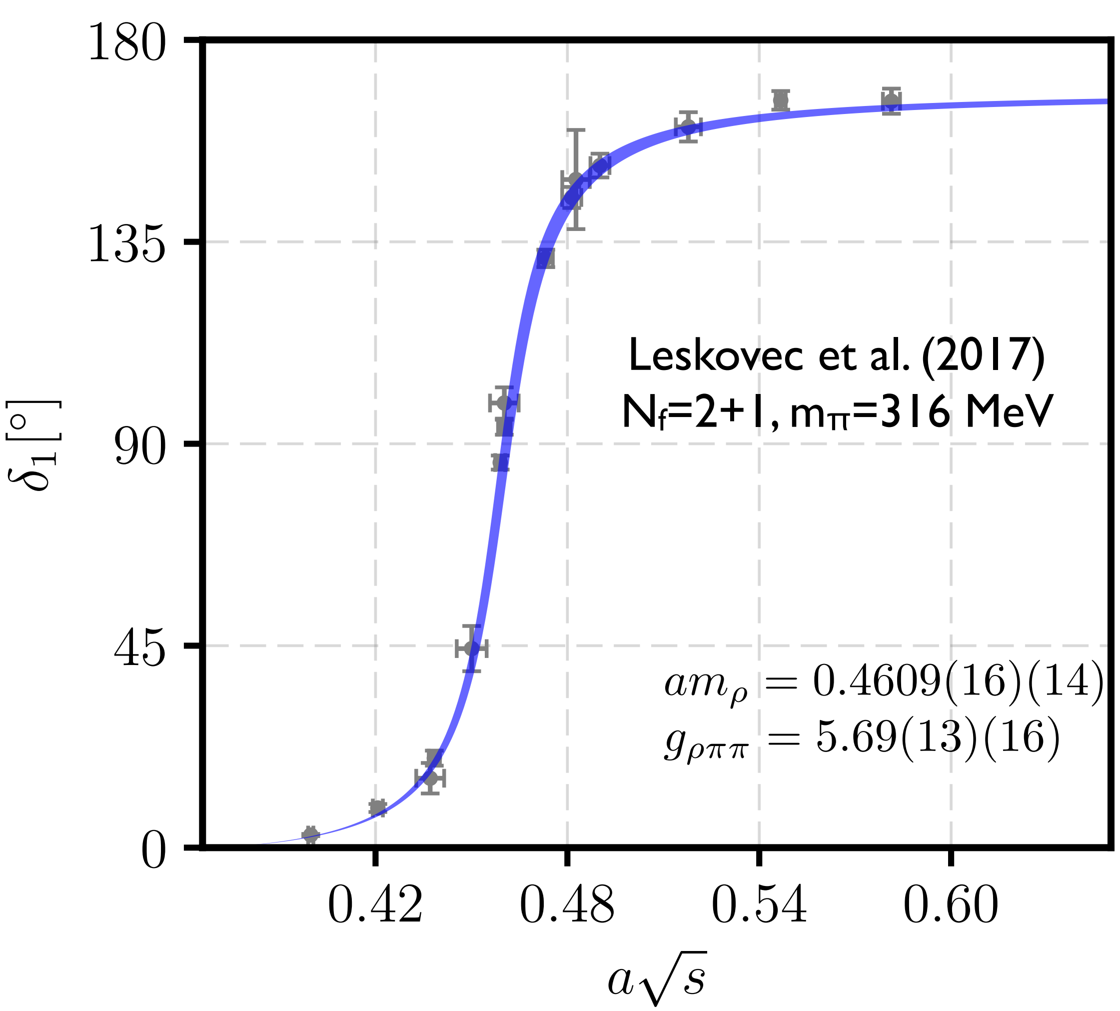
• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

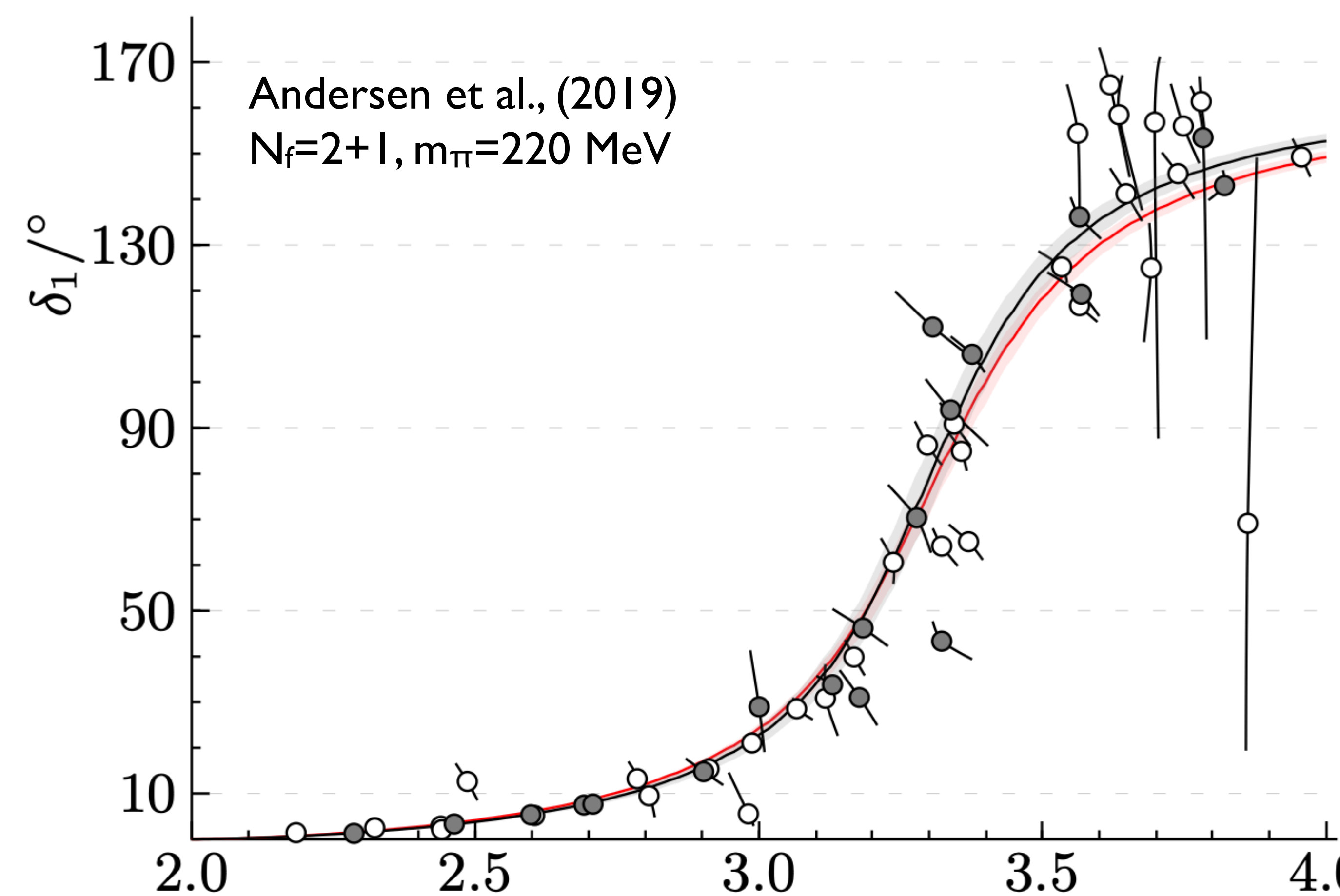
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$



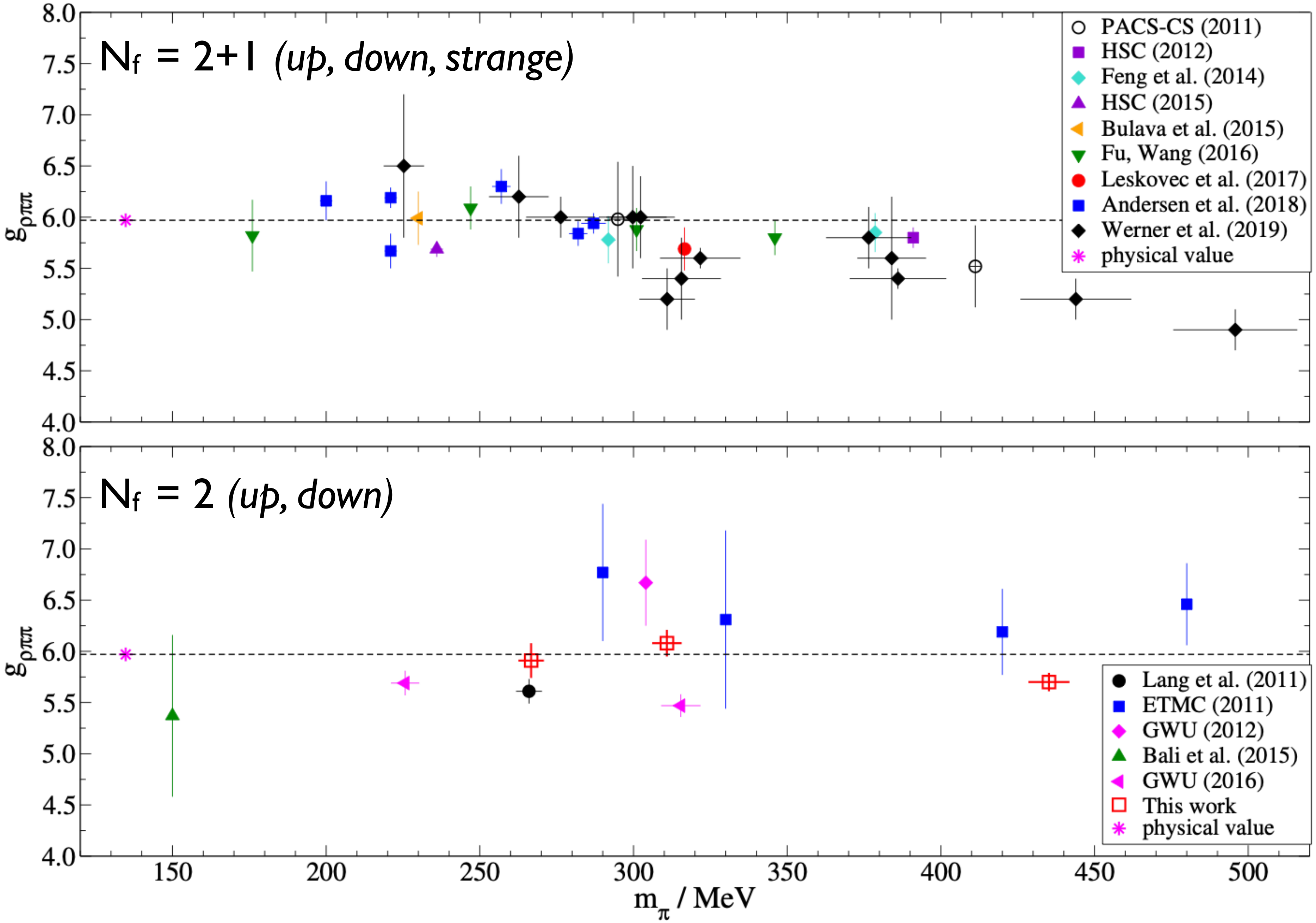






$\rho \to \pi\pi$

$$I^G(J^{PC}) = 1^+(1^{--})$$



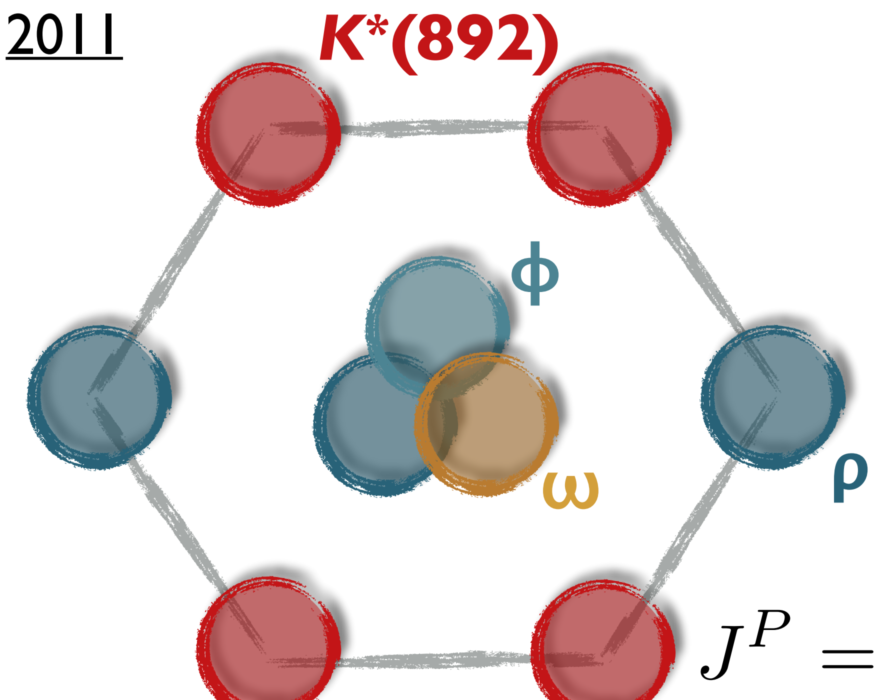
• Erben et al., Phys.Rev. **D** 101 (2020) 5, 054504 •

$\rho \rightarrow \pi\pi$

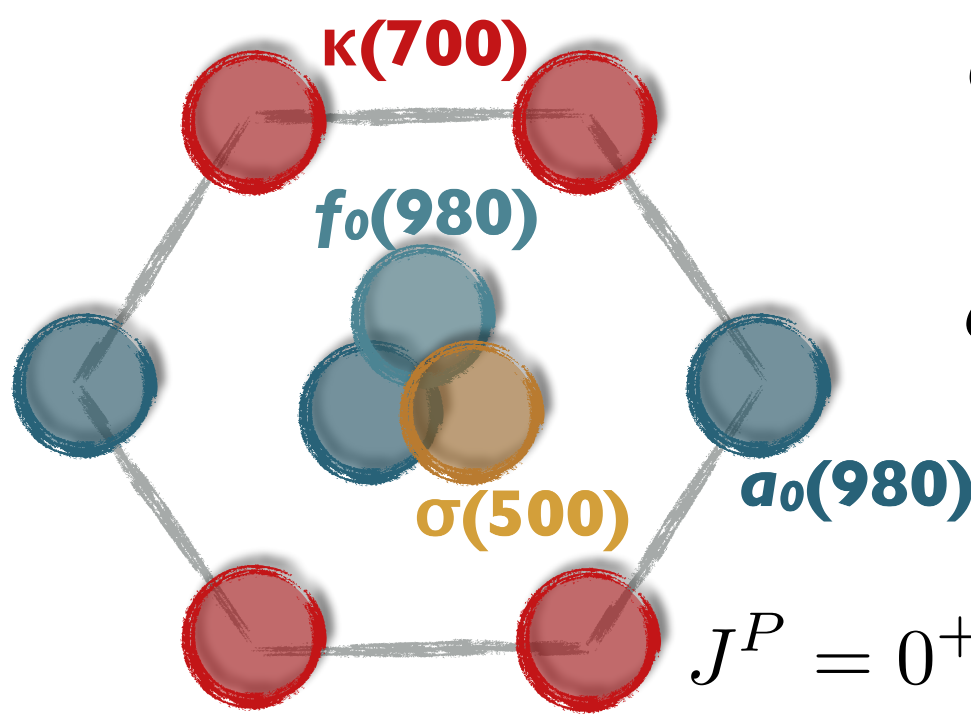
- [ ] CP-PACS/PACS-CS 2007, 2011
- [ ] ETMC 2010
- [ ] Lang et al. 2011
- [ ] HadSpec 2012, **2016**
- [ ] Pellisier 2012
- [ ] RQCD 2015
- [ ] Guo et al. 2016
- [ ] Fu et al. 2016
- [ ] Bulava et al. 2016
- [ ] Alexandrou et al. 2017
- [ ] Andersen et al. 2018
- [ ] Fischer et al. 2020
- [ ] Erben et al. 2020

$\sigma \rightarrow \pi\pi$

- [ ] Prelovsek et al. 2010
- [ ] Fu 2013
- [ ] Wakayama 2015
- [ ] Howarth and Giedt 2017
- [ ] Briceño et al. 2017
- [ ] Guo et al. 2018



$J^P = 1^-$



$J^P = 0^+$

$\kappa \rightarrow K\pi$

$K^* \rightarrow K\pi$

- [ ] Lang et al. 2012
- [ ] Prelovsek et al. 2013
- [ ] **Wilson et al. 2015**
- [ ] RQCD 2015
- [ ] **Brett et al. 2018**
- [ ] Wilson et al. 2019
- [ ] Rendon et al. 2020

$b_1 \rightarrow \pi\omega, \pi\phi$

- [ ] **Woss et al. 2019**

$a_0(980) \rightarrow \pi\eta, K\overline{K}$

- [ ] **Dudek et al. 2016**

$\sigma, f_0, f_2 \rightarrow \pi\pi, K\overline{K}, \eta\eta$

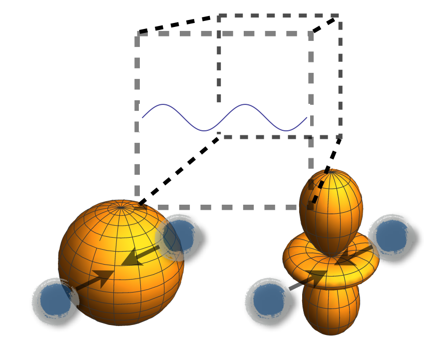
- [ ] **Briceño et al. 2017**

See the recent review by Briceño, Dudek and Young

Coupled channels

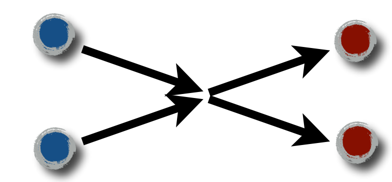
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp} \\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0 \\ 0 & F_b\end{pmatrix}\right] = 0$

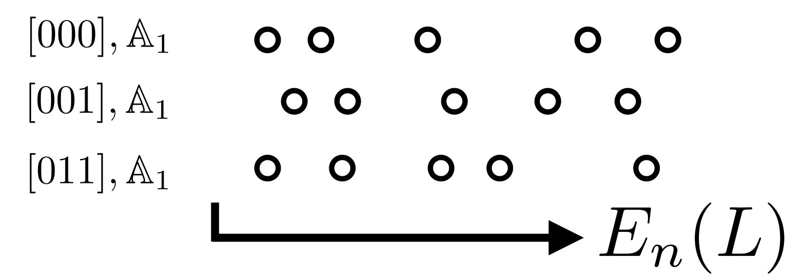


- Workflow…

Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→ Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→ Vary L and P to recover a dense set of energies

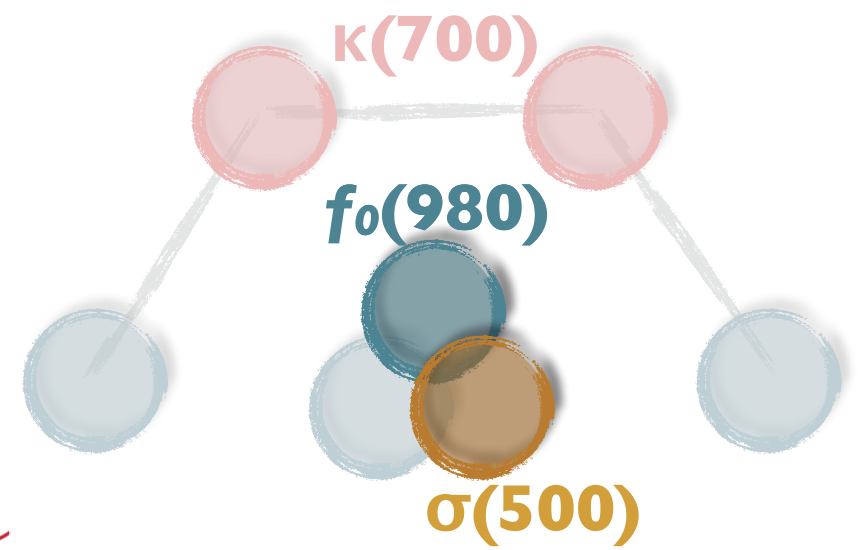


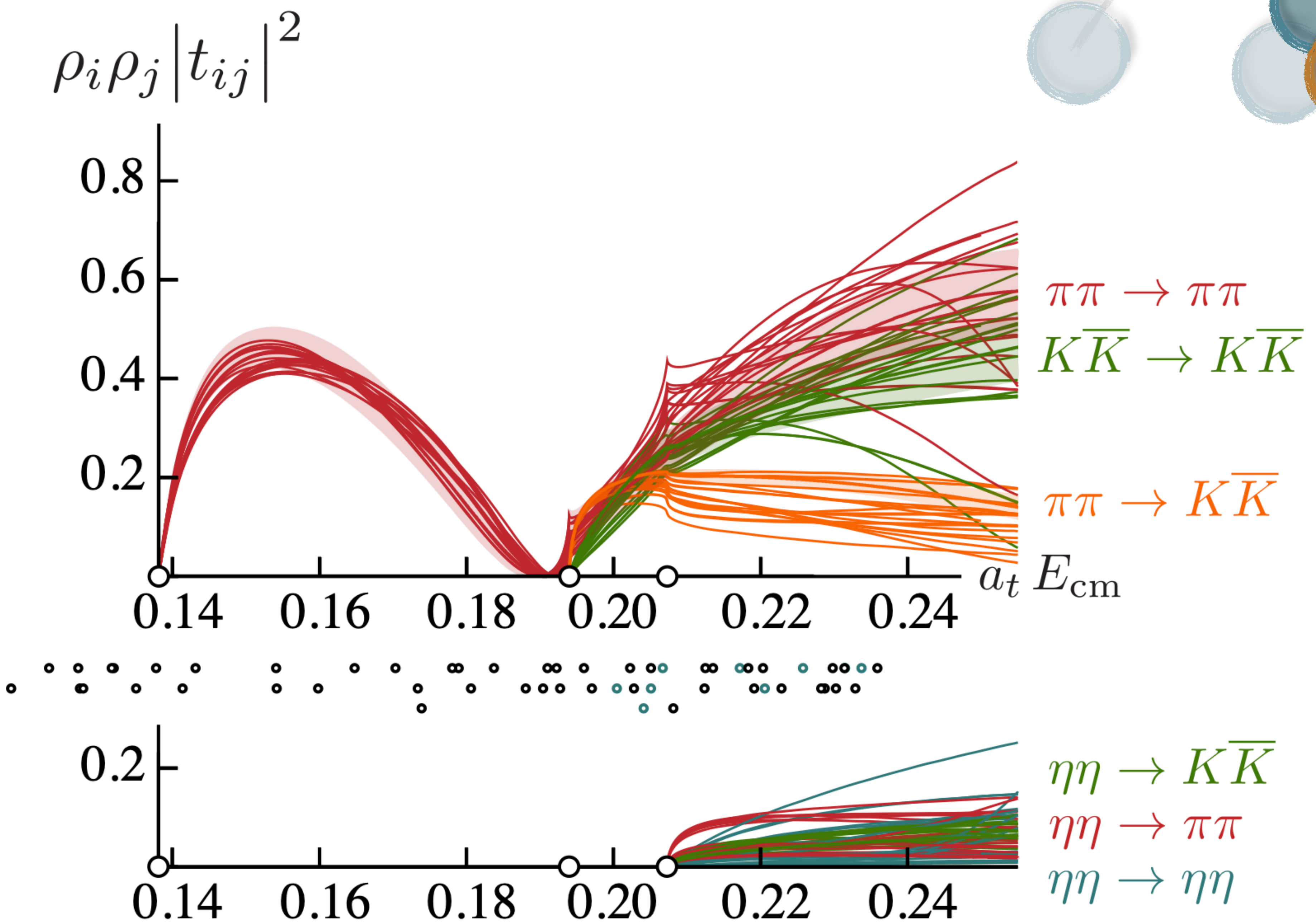
Identify a broad list of K-matrix parametrizations
polynomials and poles | EFT based | dispersion theory based

→ **Perform global fits to the finite-volume spectrum**

$\sigma, f_0 \to \pi\pi, K\overline{K}, \eta\eta$

$I^G(J^{PC}) = 0^+(0^{++})$

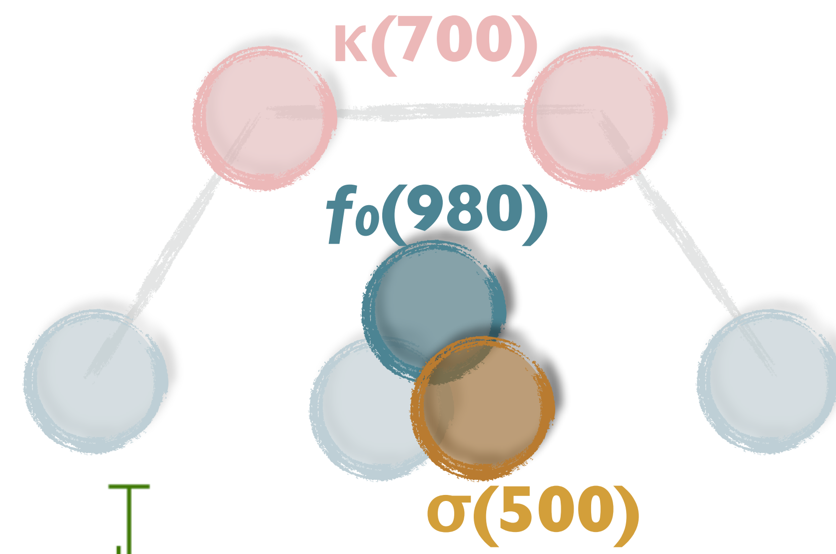


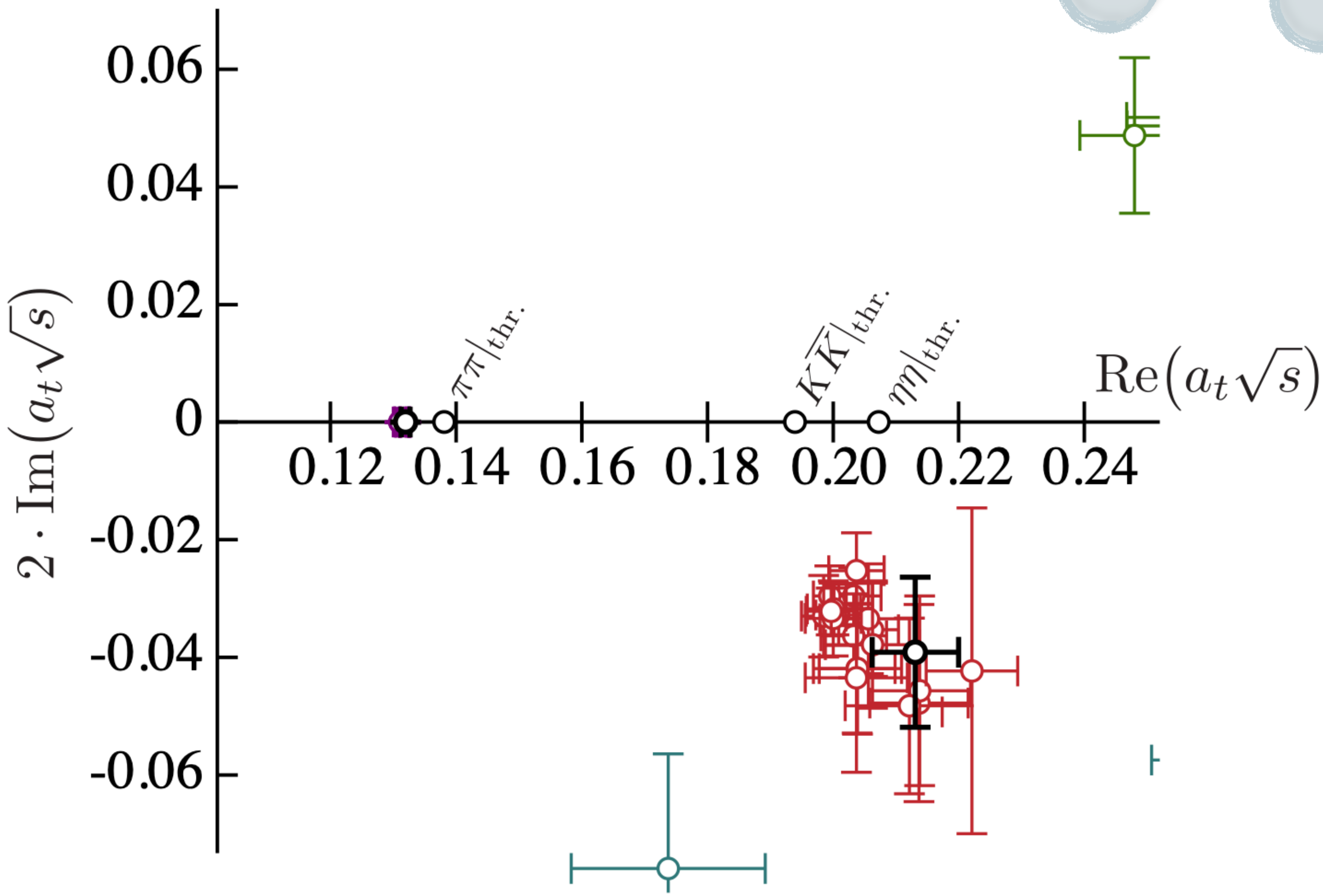


- Briceno et al., Phys.Rev. **D** 97 (2018) 5, 054513 •

$\sigma, f_0 \to \pi\pi, K\overline{K}, \eta\eta$

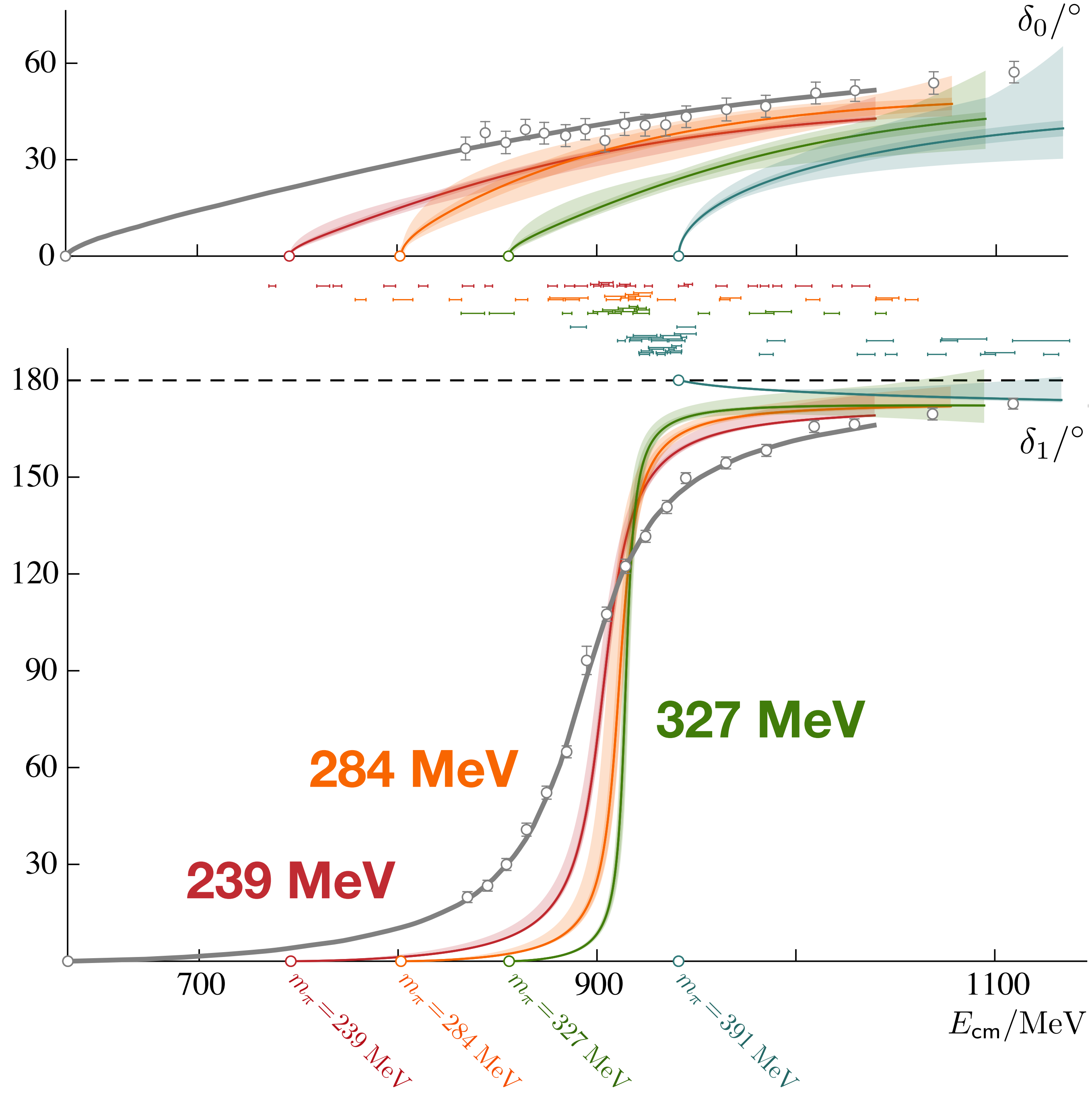
$$I^G(J^{PC}) = 0^+(0^{++})$$

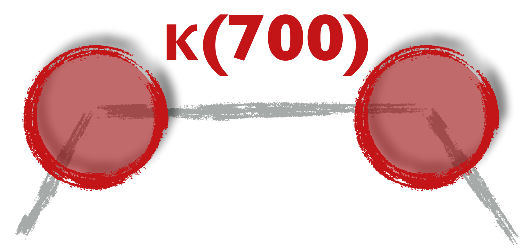




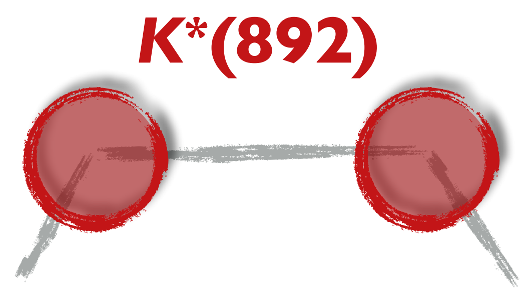
- Briceno et al., Phys.Rev. **D** 97 (2018) 5, 054513

$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$

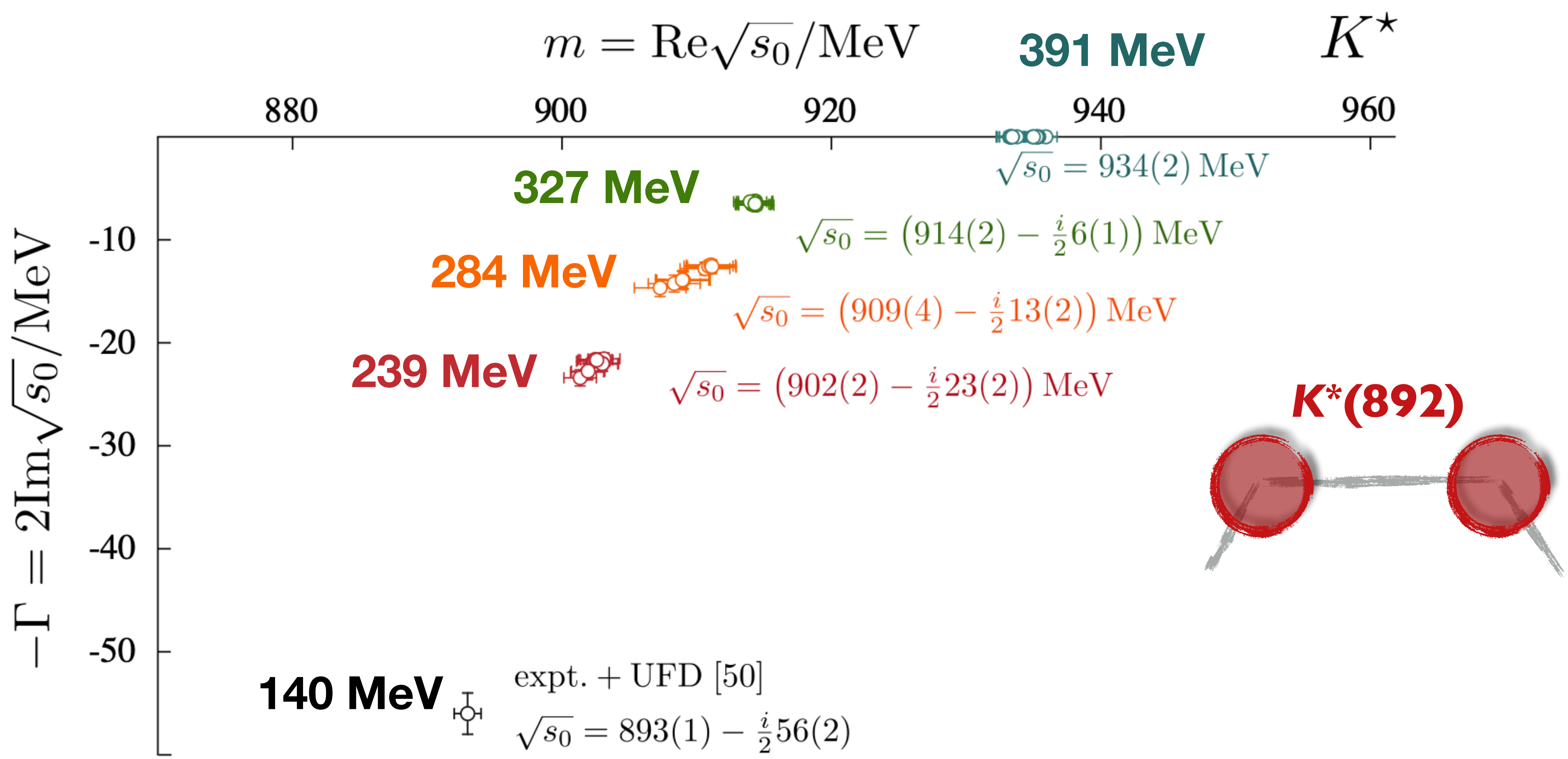


$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

$\kappa, K^* \to K\pi$

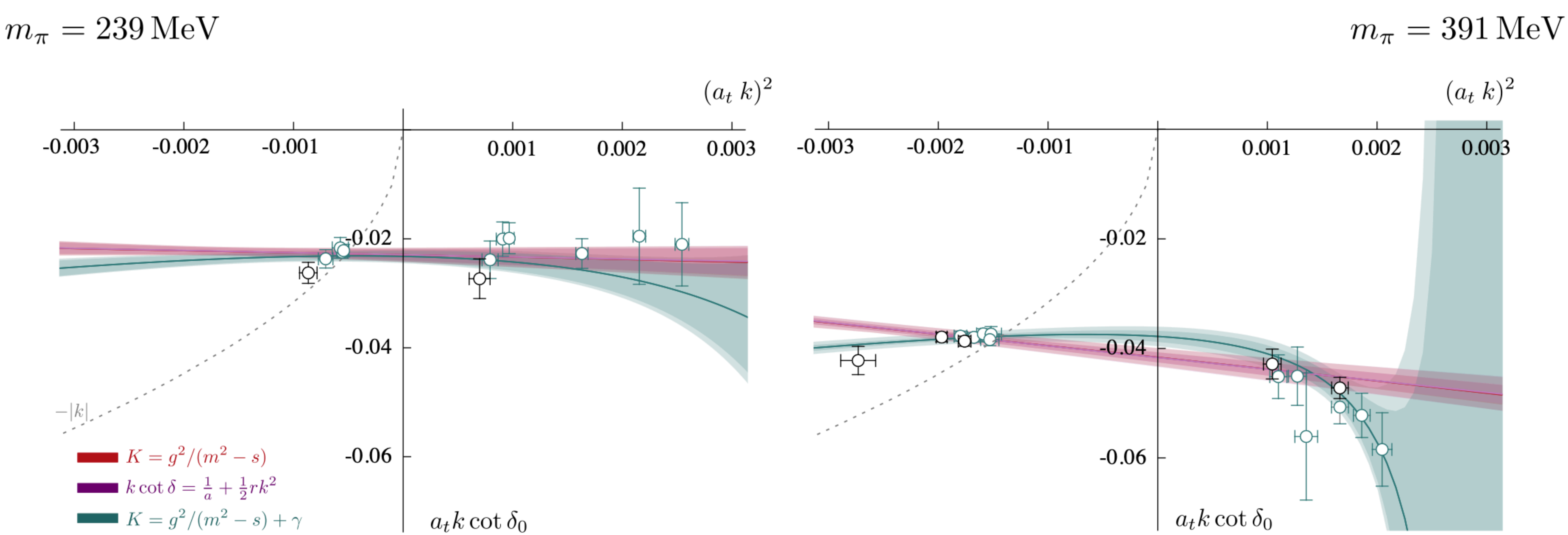
$I(J^P) = 1/2(1^-)$



• Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

$D^*_{s_0}(2317) \to DK$

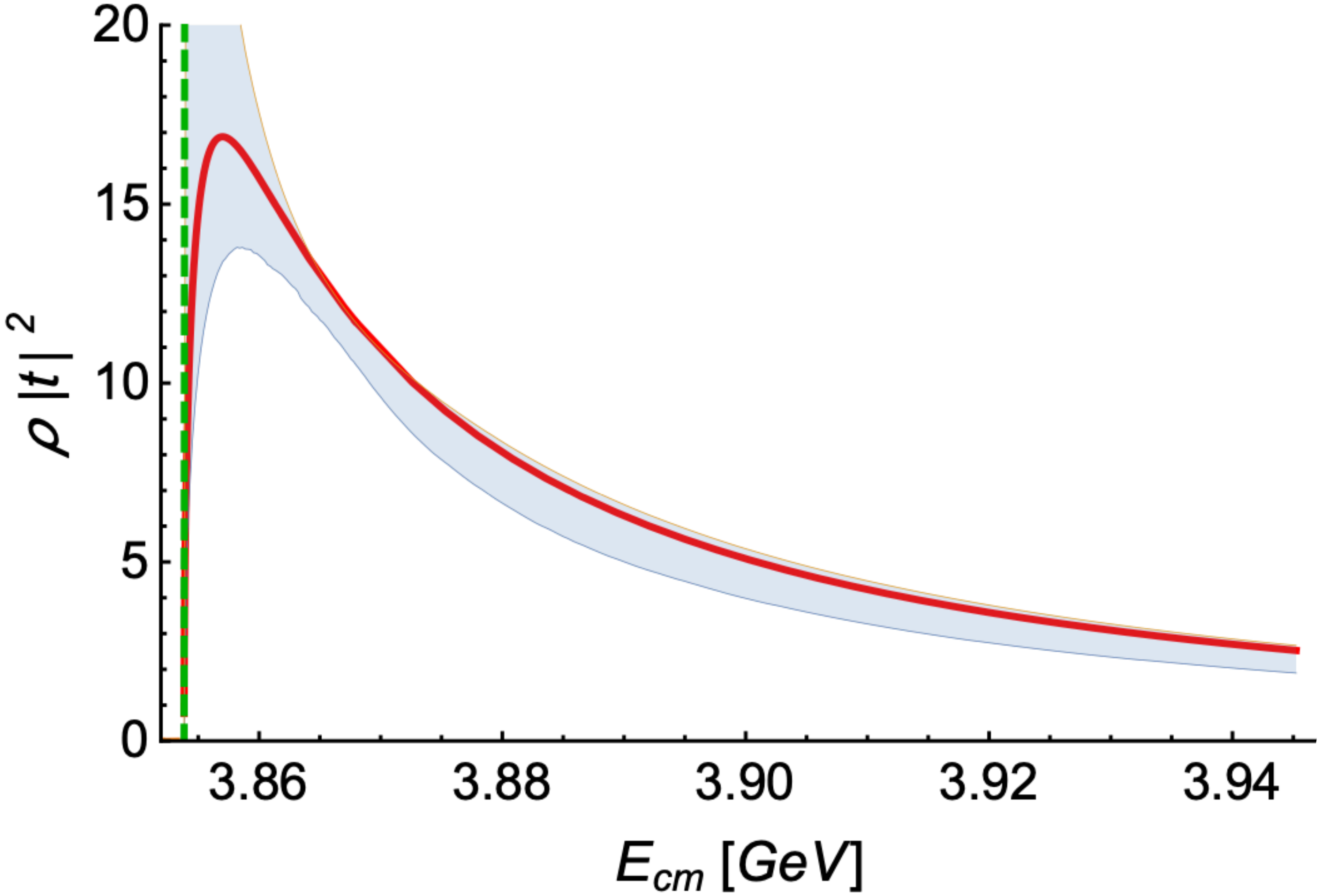
$I(J^P) = 0(0^+)$

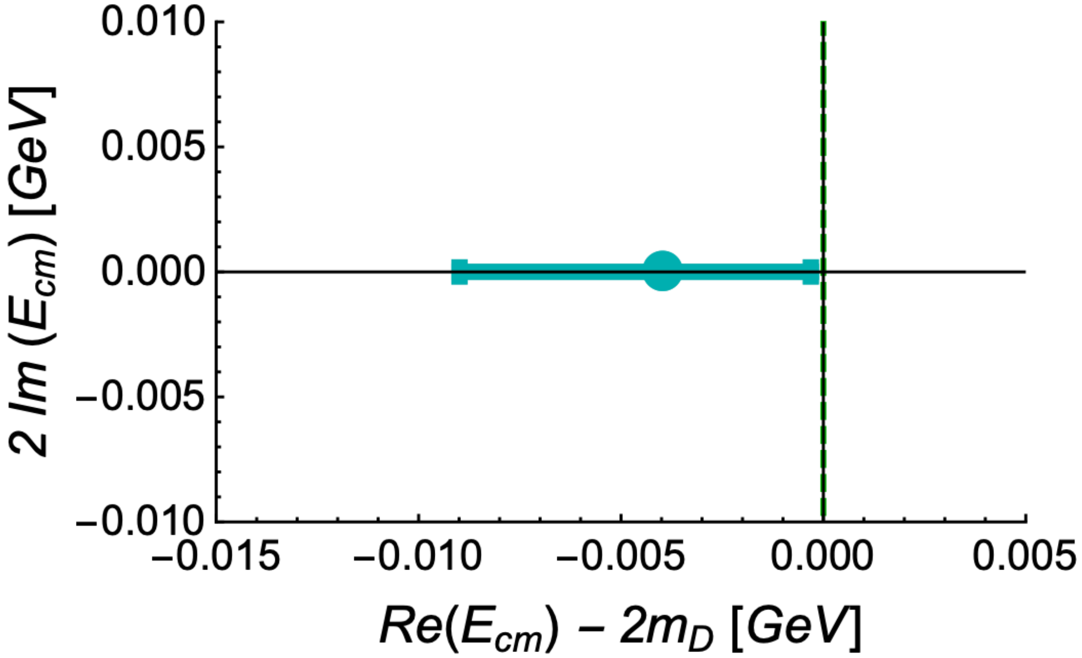


- Cheung et al. (Aug 14, 2020), *2008.06432 [hep-lat]*

$D\bar{D}, D_s\bar{D}_s$

$J^{PC} = 0^{++}, 2^{++}$

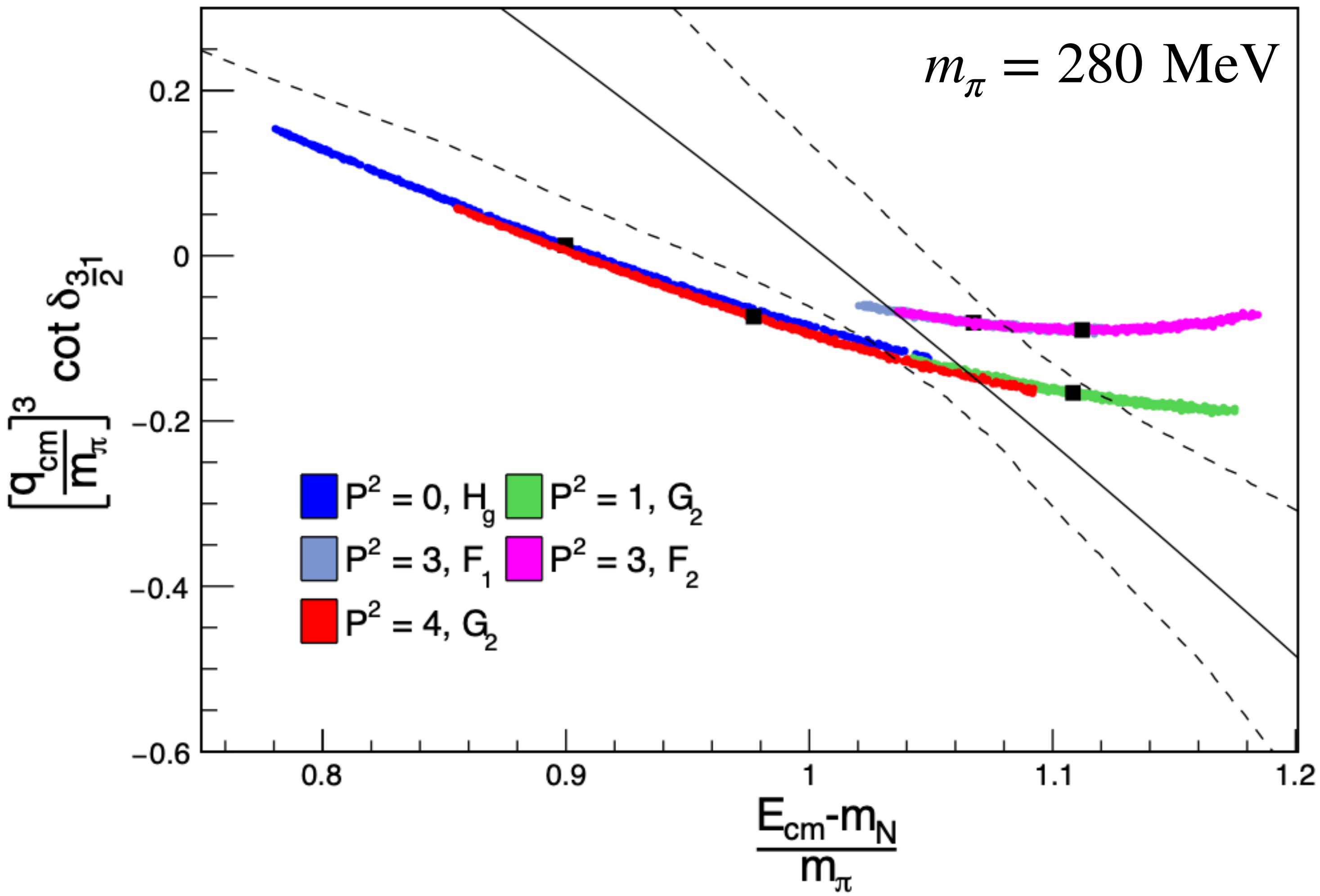




• Prelovsek et al. (November 4, 2020), *2011.02542 [hep-lat]* •

$\Delta(1232) \to N\pi$

$$I(J^P) = 3/2(3/2^+)$$



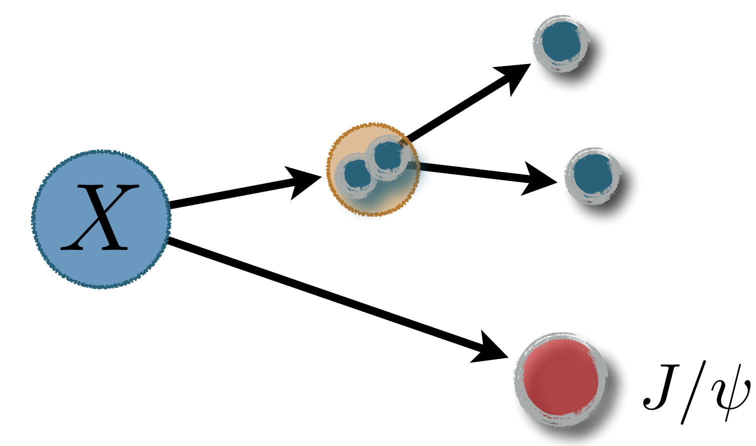
$$\frac{m_\Delta}{m_\pi} = 4.734(56), \qquad g^{\rm BW}_{\Delta N\pi} = 19.0(7.4),$$
$$(m_\pi a_{\frac{3}{2}2})^{-5} = 0.00(10), \quad (m_\pi a_{\frac{5}{2}2})^{-5} = 0.00(12),$$
$$\chi^2/\text{d.o.f.} = 4.17.$$

- Andersen et al., Phys.Rev. **D** 97 (2018) 1, 014506 •

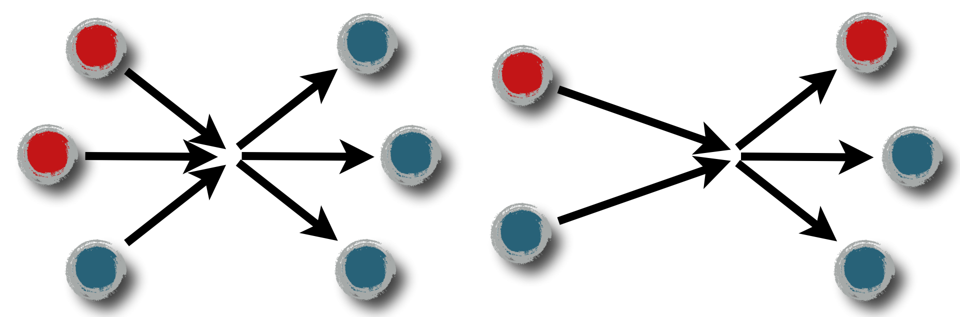
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays



Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

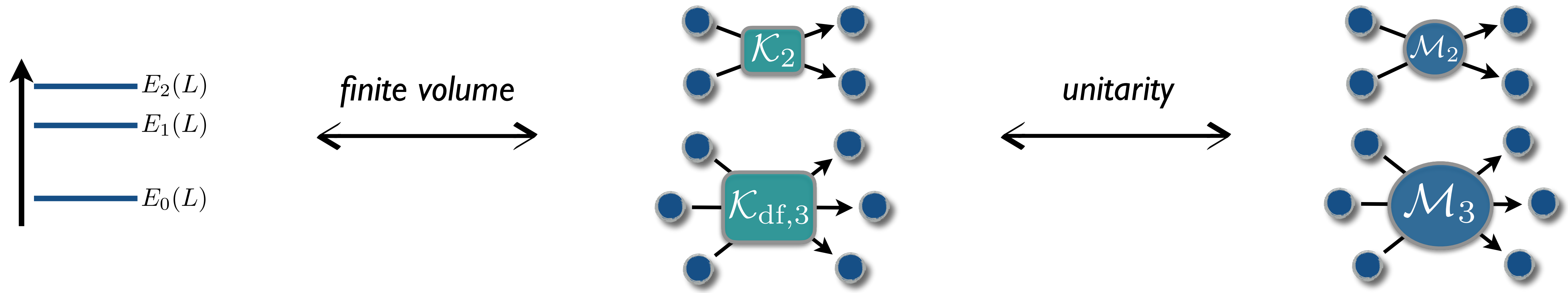
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\ a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Status



Holds only for three-particle energies

Neglects e^{-mL}

Generalized to ***two-to-three, non-identical, non-degenerate masses***

Encodes angular momentum mixing

• MTH, Sharpe (2014,2015) • Briceño, MTH, Sharpe (2017) •
Blanton, Romero-López, Sharpe (2020) • Hansen, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

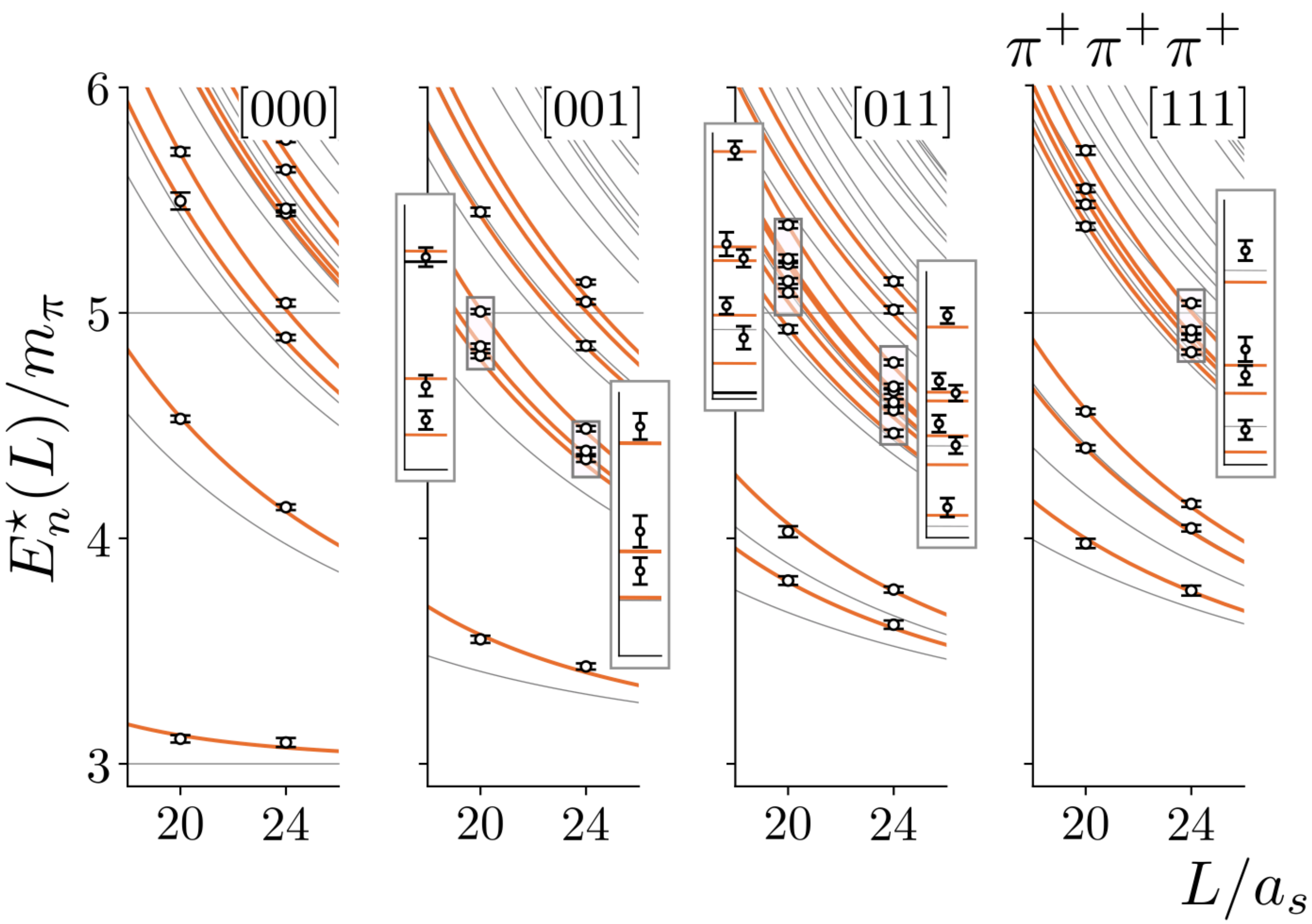
Recent Review: ***Lattice QCD and Three-particle Decays of Resonances***
MTH and Sharpe, *1901.00483*

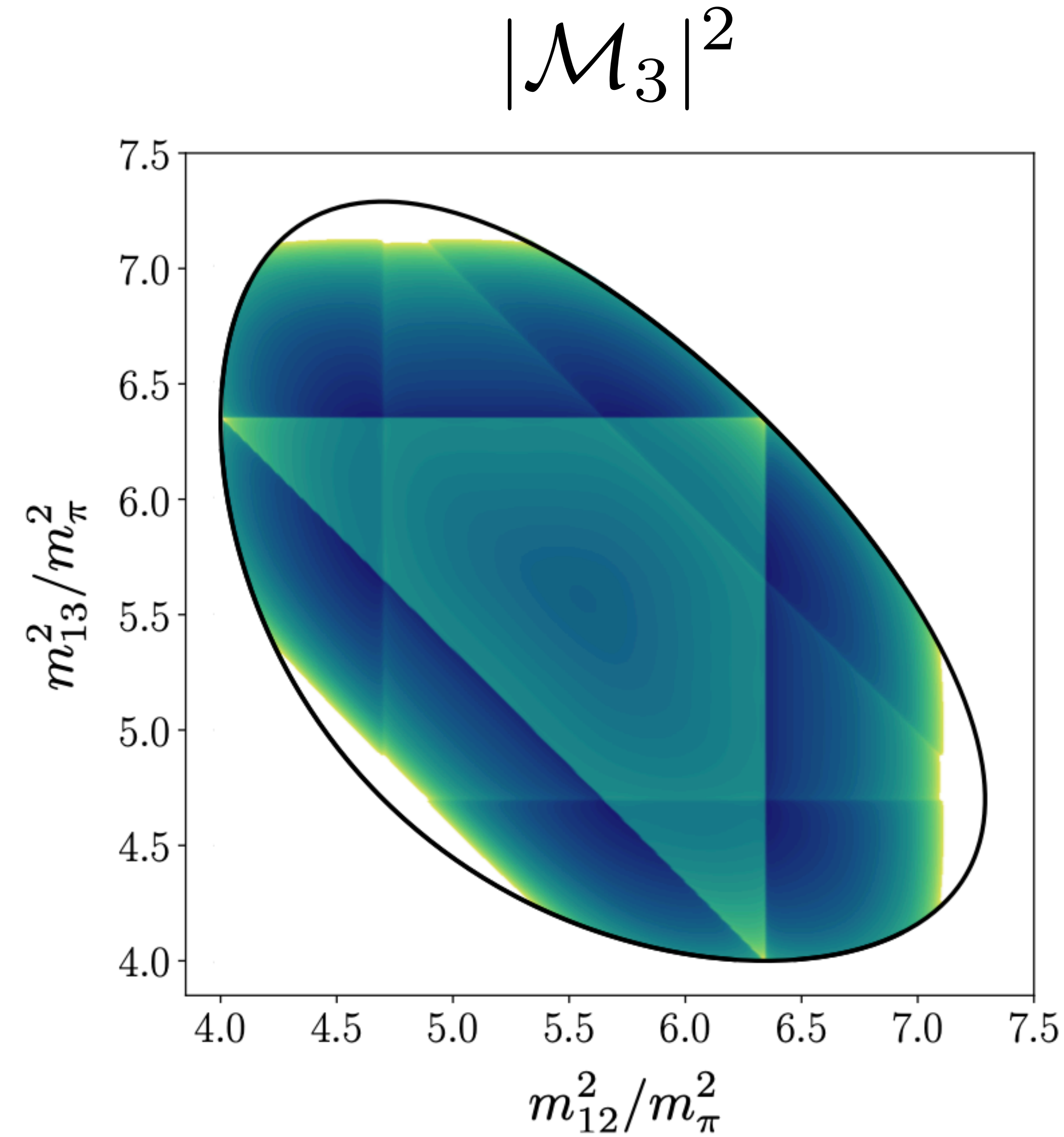
See also…

• Hammer, Pang, Rusetsky (2017) • Döring, Mai (2017) •

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

- Entire work-flow now implemented

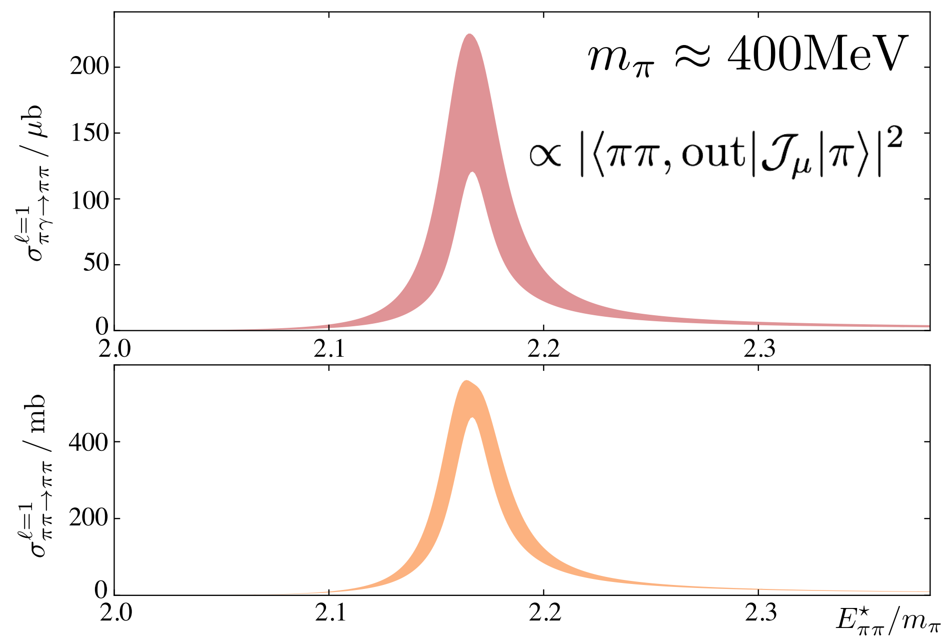




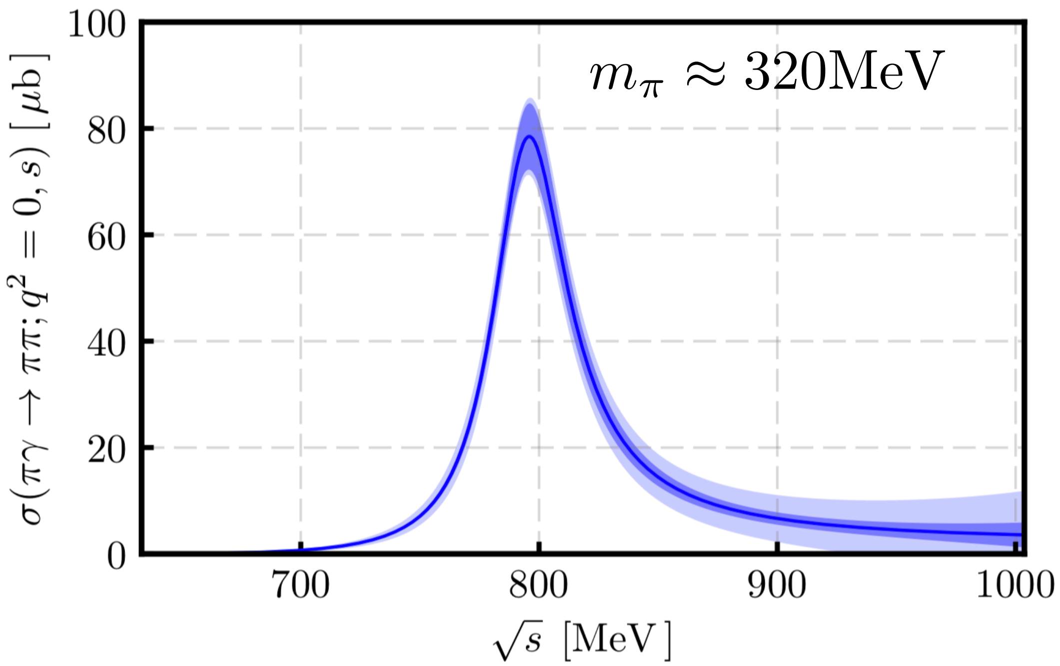
- MTH et al. (Sep. 10, 2020), *2009.0493 [hep-lat], to appear in PRL* •

Not discussed here

- One-to-two transition amplitudes: $\pi\gamma^\star \to \rho \to \pi\pi$



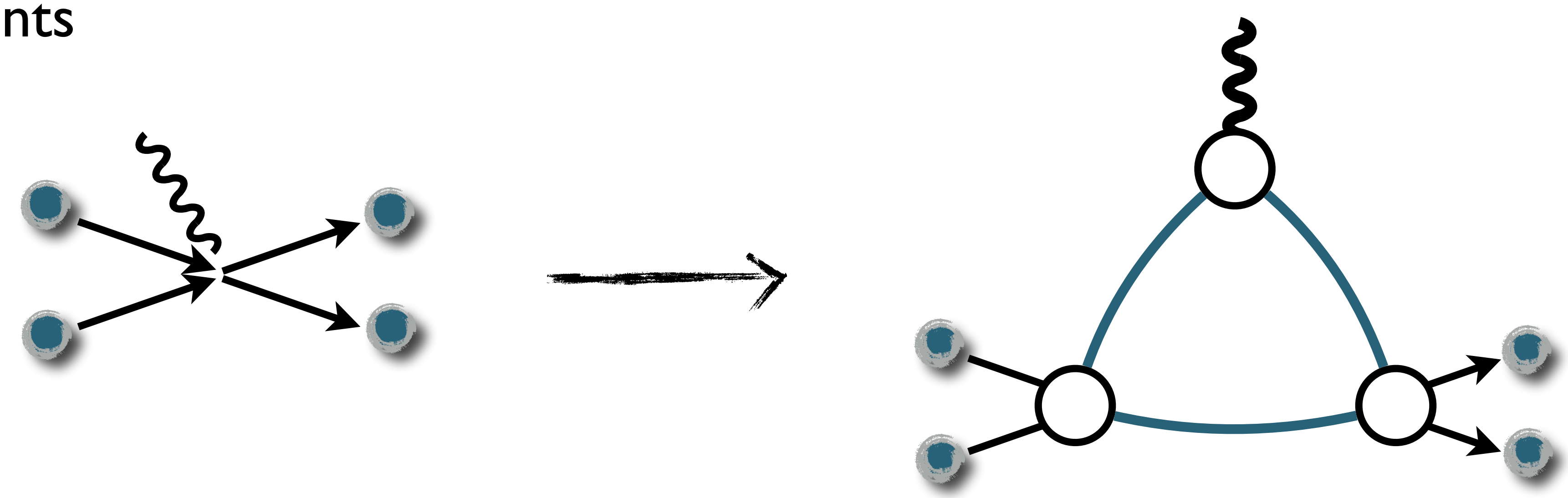
Briceño et. al., Phys. Rev. D93, 114508 (2016)



Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

- Two-to-two matrix elements

$$\langle \pi\pi, \text{out}|\mathcal{J}_\mu|\pi\pi, \text{in}\rangle \equiv$$



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018)

Conclusions

- LQCD is in the era of 'rigorous resonance spectroscopy'
- The finite-volume = ***a useful tool***
- Challenges and progress

  formal analysis was technical → *ground work is now set*

  scattering demands high precision excited states → *advanced algorithms make this possible*

  many calculations at unphysical quark masses → *physical-mass scattering now appearing*

  → *varying masses probes resonance structure*

  3-body amplitude is highly singular → *intermediate K matrix is not*

- Next steps…

  complete 3-particle formalism → *extend to N-particle formalism*

  extend studies involving an external current

  push more channels into the precision regime

Big Picture

A thriving field, with much more to come…

Thanks for listening!