The quantum effective action, external sources and a new exact RG equation

(a talk on the effective action with pictures)

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Based on work with Elizabeth Alexander, Björn Garbrecht, Jordan Nursey and Paul Saffin (not all at the same time)

Wednesday, 16th October, 2019

Università di Genova

Outline

- Introduction

- The **2PI effective action**
  [à la PM & P. M. Saffin '19, building on B. Garbrecht & PM '16]

  - Definition and properties

  - Single saddle point

  - Multiple saddle points and the Maxwell construction

  - Method of external sources
    [B. Garbrecht & PM '16]

- The **regulator-sourced 2PI effective action** and exact flow equations
  [E. Alexander, PM, J. Nursey & P. M. Saffin '19]

- Concluding remarks

What we want: A systematic way of dealing with non-perturbative phenomena in quantum field theory.

What we got: The quantum effective action (in various forms).
[R. Jackiw '74; J. M. Cornwall, R. Jackiw & E. Tomboulis '74; H. Verschelde & M. Coppens '92; M. E. Carrington '04; A. Pilaftsis & D. Teresi '13; J. Ellis, N. E. Mavromatos & D. P. Skliros '16]

What it's good for: Non-equilibrium phenomena
[J. S. Schwinger '61; G. Baym & L. P. Kadanoff '61; L. V. Keldysh '64; R. D. Jordan '86; E. Calzetta & B. L. Hu '88; J. P. Blaizot & E. Iancu '02; J. Berges '04; PM & A. Pilaftsis '13]
Symmetry breaking
[S. R. Coleman & E. J. Weinberg '73; J. Alexandre '12; J. Alexandre & A. Tsapalis '12]
Instantons/Solitons/Vacuum decay
[B. Garbrecht & PM '15; A. D. Plascencia & C. Tamarit '16]
Functional renormalisation group
[C. Wetterich '91 & '93; T. R. Morris '94; U. Ellwanger '94; M. Reuter '98; J. Berges, N. Tetradis & C. Wetterich '02; J. Pawlowski '07; H. Gies '12; O. J. Rosten '12]

The 2PI effective action

For illustration, let's work with a **zero-dimensional** Euclidean quantum field theory:
[PM & P. M. Saffin '19]

$$S(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$$

and write down the **partition function**

$$Z(J, K) = \mathcal{N}\int_{-\infty}^{+\infty} \mathrm{d}\Phi \, \exp\left[-\frac{1}{\hbar}\left(S(\Phi) - J\Phi - \frac{1}{2}K\Phi^2\right)\right]$$

in the presence of **external sources** J and K.

The **Schwinger function**

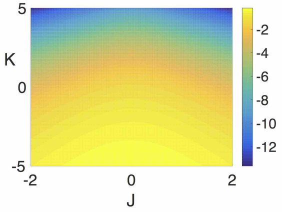
$$W(J, K) \equiv -\hbar \ln Z(J, K)$$

is **concave**.

Its gradients with respect to $-J$ and $-K/2$ are $\langle\Phi\rangle_{J,K}$ and $\langle\Phi^2\rangle_{J,K}$, respectively, i.e. the **one-** and **two-point functions**.

The 2PI effective action

$W(J, K) \equiv -\hbar \ln Z(J, K)$ for $m^2 = -1$ and $\lambda = 6$, i.e. non-convex classical potential:



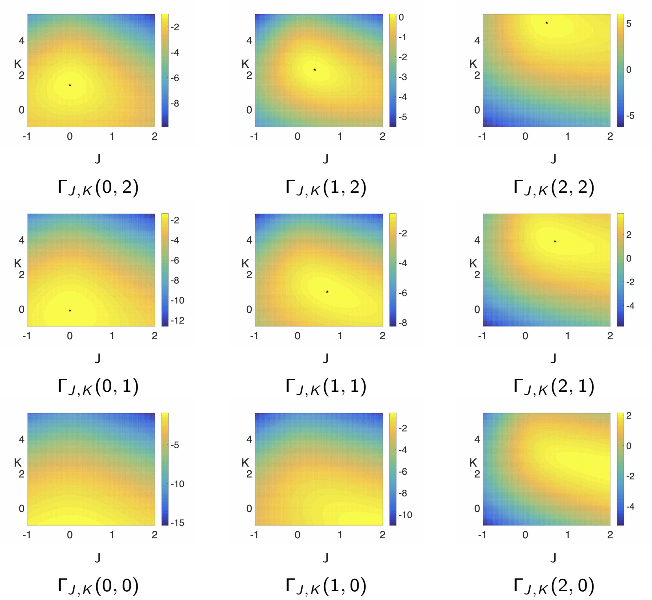
[PM & P. M. Saffin '19]

Introduce a function

$$\Gamma_{J,K}(\phi, \Delta) \equiv W(J, K) + J\phi + \frac{1}{2}K[\phi^2 + \hbar\Delta]$$

The variables ϕ and Δ determine the value of the maximum of this function and its position in the (J, K) plane ...

The 2PI effective action



[PM & P. M. Saffin '19]

The 2PI effective action

The **(double) Legendre transform**

$$\Gamma(\phi, \Delta) = \max_{J,K} \Gamma_{J,K}(\phi, \Delta)$$

corresponds to the value of these maxima as a function of ϕ and Δ.

The locations of the maxima correspond to **extremal sources** $\mathcal{J}$ and $\mathcal{K}$, defined by

$$\left.\frac{\partial \Gamma_{J,K}(\phi, \Delta)}{\partial J}\right|_{J=\mathcal{J}, K=\mathcal{K}} = 0 \qquad \left.\frac{\partial \Gamma_{J,K}(\phi, \Delta)}{\partial K}\right|_{J=\mathcal{J}, K=\mathcal{K}} = 0$$

The extremisation yields

$$\Gamma(\phi, \Delta) = W(\mathcal{J}, \mathcal{K}) + \mathcal{J}\phi + \frac{1}{2}\mathcal{K}[\phi^2 + \hbar\Delta]$$

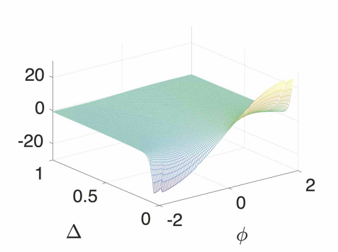
with

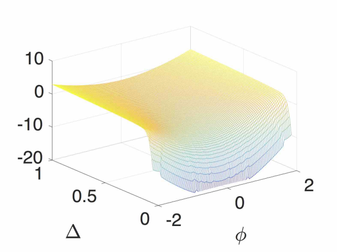
$$\phi = \hbar \frac{\partial}{\partial J} \ln Z(J, K)\Big|_{J=\mathcal{J}, K=\mathcal{K}} \qquad \hbar\Delta = 2\hbar \frac{\partial}{\partial K} \ln Z(J, K)\Big|_{J=\mathcal{J}, K=\mathcal{K}} - \phi^2$$

The 2PI effective action

Importantly, since the location of the maxima of $\Gamma_{J,K}(\phi, \Delta)$ depend on ϕ and Δ

$$\mathcal{J} \equiv \mathcal{J}(\phi, \Delta) \qquad\qquad \mathcal{K} \equiv \mathcal{K}(\phi, \Delta)$$





[PM & P. M. Saffin '19]

In corollary,

$$\phi \equiv \phi(\mathcal{J}, \mathcal{K}) \qquad\qquad \Delta \equiv \Delta(\mathcal{J}, \mathcal{K})$$

and they are related to the tangents to the Schwinger function.

The 2PI effective action

The extremal sources $\mathcal{J}$ and $\mathcal{K}$ are related to the tangents to $\Gamma(\phi, \Delta)$:

$$\frac{\partial \Gamma(\phi, \Delta)}{\partial \phi} = \mathcal{J}(\phi, \Delta) + \mathcal{K}(\phi, \Delta)\phi \qquad \frac{\partial \Gamma(\phi, \Delta)}{\partial \Delta} = \frac{\hbar}{2}\mathcal{K}(\phi, \Delta)$$

The right-hand sides are source terms, and the gradients of $\Gamma(\phi, \Delta)$ are the equations of motion for the one- and two-point functions ϕ and Δ.

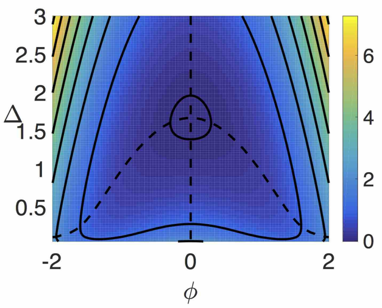
Since these are correct to all orders in $\hbar$, we are justified in calling $\Gamma(\phi, \Delta)$ a **quantum effective action**.

Why "2PI"?

The 2PI effective action: convexity

By definition of the Legendre transform, $\Gamma(\phi, \Delta)$ should be convex.

But for the non-convex classical potential with $m^2 = -2$ and $\lambda = 6$, we find



[PM & P. M. Saffin '19]

This doesn't look convex; **what gives?**

The 2PI effective action: convexity

Convenient to work with the variables $\phi' \equiv \phi$ and $\Delta' \equiv \phi^2 + \hbar\Delta$ and the rescaled sources $\mathcal{J}' \equiv \mathcal{J}$ and $\mathcal{K}' \equiv \mathcal{K}/2$:
[PM & P. M. Saffin '19]

$$\Gamma(\phi, \Delta) = W(\mathcal{J}, \mathcal{K}) + \mathcal{J}'\phi' + \mathcal{K}'\Delta'$$

$$\frac{\partial \Gamma(\phi, \Delta)}{\partial \phi'} = \mathcal{J}' \qquad \frac{\partial \Gamma(\phi, \Delta)}{\partial \Delta'} = \mathcal{K}'$$

$$\phi' = -\frac{\partial W(\mathcal{J}, \mathcal{K})}{\partial \mathcal{J}'} \qquad \Delta' = -\frac{\partial W(\mathcal{J}, \mathcal{K})}{\partial \mathcal{K}'}$$

We consider the product
[cf. the 1PI case in J. Alexandre & A. Tsapalis '12]

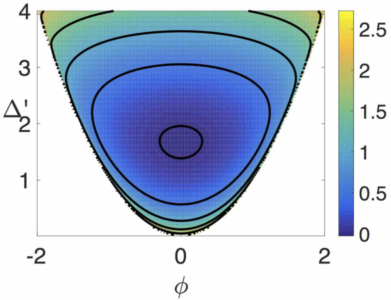
$$-\mathrm{Hess}(\Gamma)(\phi', \Delta') \cdot \mathrm{Hess}(W)(\mathcal{J}', \mathcal{K}') = \mathbb{I}$$

$-\mathrm{Hess}(W)(\mathcal{J}', \mathcal{K}')$ is a covariance matrix, i.e. positive definite.Thus, $\mathrm{Hess}(\Gamma)(\phi', \Delta')$

is positive definite, and $\Gamma(\phi, \Delta)$ is therefore convex, but with respect to ϕ and Δ'.

The 2PI effective action: convexity

Plotting $\Gamma(\phi, \Delta)$ as a function of ϕ and $\Delta' = \phi^2 + \hbar\Delta$, we see that it is convex:



[PM & P. M. Saffin '19]

Note that this is for a non-convex classical potential, with $m^2 = -2$ and $\lambda = 6$.

The 2PI effective action: single saddle point

Stationarity/saddle-point condition:

$$\left.\frac{\partial S(\Phi)}{\partial \Phi}\right|_{\Phi=\varphi} - \mathcal{J}(\phi,\Delta) - \mathcal{K}(\phi,\Delta)\varphi = 0$$

Define the two-point function

$$\mathcal{G} = \left[G^{-1}(\varphi) - \mathcal{K}(\phi,\Delta)\right]^{-1} \qquad G^{-1}(\varphi) = \left.\frac{\partial^2 S(\Phi)}{\partial \Phi^2}\right|_{\Phi=\varphi} = m^2 + \frac{\lambda}{2}\varphi^2$$

and expand $\Phi = \varphi + \sqrt{\hbar}\hat{\Phi}$ to obtain

$$\begin{aligned}\Gamma(\phi,\Delta) &= S(\varphi) + \hbar\Gamma_1(\varphi,\mathcal{G}) + \hbar^2\Gamma_2(\varphi,\mathcal{G}) + \hbar^2\Gamma_{1\text{PR}}(\varphi,\mathcal{G}) \\ &\quad + \mathcal{J}\left(\phi - \varphi\right) + \frac{1}{2}\mathcal{K}(\phi^2 - \varphi^2 + \hbar\Delta - \hbar\mathcal{G}) \\ \Gamma_1(\varphi,\mathcal{G}) &= \frac{1}{2}\left[\ln\left(\mathcal{G}^{-1}G(0)\right) + \mathcal{K}\mathcal{G}\right] = \frac{1}{2}\left[\ln\left(\mathcal{G}^{-1}G(0)\right) + G^{-1}\mathcal{G} - 1\right] \\ \Gamma_2(\varphi,\mathcal{G}) &= \frac{1}{8}\lambda\mathcal{G}^2 - \frac{1}{12}\lambda^2\varphi^2\mathcal{G}^3 \qquad \Gamma_{1\text{PR}}(\varphi,\mathcal{G}) = -\frac{1}{8}\lambda^2\varphi^2\mathcal{G}^3\end{aligned}$$

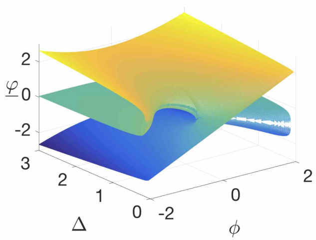
But $\varphi \equiv \varphi(\phi,\Delta)$, and we can expand the right-hand side around $\varphi - \phi = \mathcal{O}(\hbar)$:

$$\Gamma(\phi,\Delta) = S(\phi) + \hbar\Gamma_1(\phi,\Delta) + \hbar^2\Gamma_2(\phi,\Delta)$$

The 2PI effective action: multiple saddle points and the Maxwell construction

More generally, we have a set of saddle points $\{\varphi_i\} \equiv \{\varphi_i\}(\phi, \Delta)$, where both the type and number depend on (ϕ, Δ).

For $m^2 = -1$ and $\lambda = 6$, we have 1 to 3 saddles, depending on (ϕ, Δ):



[PM & P. M. Saffin '19]

Don't mix up your ϕ's and φ's!

If the saddle points are "reasonably well separated"

$$Z(\mathcal{J}, \mathcal{K}) \approx \sum_i Z_i(\mathcal{J}, \mathcal{K})$$

The 2PI effective action: multiple saddle points and the Maxwell construction

Suppose there are two contributing saddle points, $\varphi_\pm(\phi,\Delta) = \tilde{\varphi}_\pm + \hbar\delta\varphi_\pm(\phi,\Delta)$:
[PM & P. M. Saffin '19]

$$\Gamma(\phi,\Delta) = \frac{(\tilde{\varphi}_+ - \phi)\tilde{\Gamma}_- + (\phi - \tilde{\varphi}_-)\tilde{\Gamma}_+}{\tilde{\varphi}_+ - \tilde{\varphi}_-} - \frac{1}{2}\mathcal{K}(\tilde{\varphi}_+ - \phi)(\phi - \tilde{\varphi}_-)$$
$$- \hbar \ln\left[\left(\frac{\phi - \tilde{\varphi}_-}{\tilde{\varphi}_+ - \phi}\right)^{\frac{\tilde{\varphi}_+ - \phi}{\tilde{\varphi}_+ - \tilde{\varphi}_-}} + \left(\frac{\tilde{\varphi}_+ - \phi}{\phi - \tilde{\varphi}_-}\right)^{\frac{\phi - \tilde{\varphi}_-}{\tilde{\varphi}_+ - \tilde{\varphi}_-}}\right] + \frac{\hbar}{2}\mathcal{K}\Delta$$

In the limit $\mathcal{K} \to 0$, we recover the 1PI result:
[J. Alexandre & A. Tsapalis '12]

$$\Gamma(\phi) = \frac{(\tilde{\varphi}_+ - \phi)\tilde{\Gamma}_- + (\phi - \tilde{\varphi}_-)\tilde{\Gamma}_+}{\tilde{\varphi}_+ - \tilde{\varphi}_-} - \hbar \ln\left[\left(\frac{\phi - \tilde{\varphi}_-}{\tilde{\varphi}_+ - \phi}\right)^{\frac{\tilde{\varphi}_+ - \phi}{\tilde{\varphi}_+ - \tilde{\varphi}_-}} + \left(\frac{\tilde{\varphi}_+ - \phi}{\phi - \tilde{\varphi}_-}\right)^{\frac{\phi - \tilde{\varphi}_-}{\tilde{\varphi}_+ - \tilde{\varphi}_-}}\right]$$

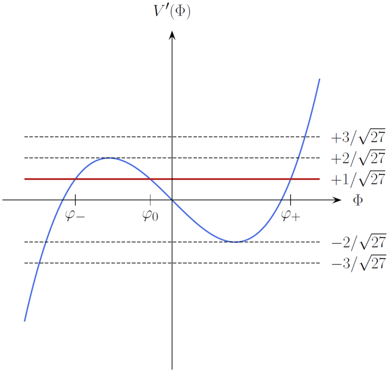
giving the **Maxwell construction** in the limit $\hbar \to 0$:

$$\Gamma(\phi) = \frac{(\tilde{\varphi}_+ - \phi)\tilde{V}_- + (\phi - \tilde{\varphi}_-)\tilde{V}_+}{\tilde{\varphi}_+ - \tilde{\varphi}_-}$$

For how this works in higher dimensions, see [R. J. Rivers '84; PM & P. M. Saffin '19].

The 2PI effective action: multiple saddle points and the Maxwell construction

- $\Gamma(\phi)$ is monotonic only for $\tilde{\varphi}_- < \phi < \tilde{\varphi}_+$.
- We hit branch points at $\phi = \tilde{\varphi}_\pm$ when we no longer have multiple saddles.
- For $\phi > \tilde{\varphi}_+$ or $\phi < \tilde{\varphi}_-$, $\Gamma(\phi) \to V(\phi)$.



[PM & P. M. Saffin '19]

The values on the right-hand side are $\mathcal{J} \equiv \mathcal{J}[\phi]$, with $\mathcal{K} = 0$.

The 2PI effective action: method of external sources [B. Garbrecht & PM '16]

Folklore: The physical limit corresponds to vanishing external sources.

Reality: Setting $\mathcal{J}(\phi, \Delta)$ and $\mathcal{K}(\phi, \Delta)$ to zero constrains $\phi \equiv \phi(\mathcal{J}, \mathcal{K})$ and $\Delta \equiv \Delta(\mathcal{J}, \mathcal{K})$, yielding the CJT effective action with an important difference:
[J. M. Cornwall, R. Jackiw & E. Tomboulis '74]

We can choose the sources $\mathcal{J}(\phi, \Delta)$ and $\mathcal{K}(\phi, \Delta)$, such that the saddle point of the partition function coincides with the quantum trajectory by demanding

$$\frac{\delta S[\Phi]}{\delta \Phi}\bigg|_{\Phi=\varphi} - \mathcal{J}(\phi, \Delta) - \mathcal{K}(\phi, \Delta)\varphi = \frac{\delta \Gamma[\phi, \Delta]}{\delta \phi}\bigg|_{\phi=\varphi, \Delta=\mathcal{G}} = 0$$

This requires

$$\mathcal{J}(\varphi, \mathcal{G}) + \mathcal{K}(\varphi, \mathcal{G})\varphi = 0$$

and it can be proven that this is the case.
[B. Garbrecht & PM '16; PM & P. M. Saffin '19]

This is important when the quantum trajectory is non-perturbatively far away from the classical trajectory, e.g., as in tunnelling problems in radiatively generated potentials.
[E. J. Weinberg '93; B. Garbrecht & PM '15 & '16]

The 2PI effective action: method of external sources

But we can do more:
[B. Garbrecht & PM '16]

- Setting $\mathcal{J}$ to zero and choosing $\mathcal{K}$ to be local yields the **2PPI effective action** of Verschelde and Coppens.
  [H. Verschelde & M. Coppens '92]

- Constraining the sources by, e.g., the Ward identities, yields results in the spirit of the **symmetry-improved effective action** of Pilaftsis and Teresi.
  [A. Pilaftsis & D. Teresi '13]

- Choosing $\mathcal{K}$ to be the regulator of the **renormalisation group evolution** yields ...
  [E. Alexander, PM, J. Nursey & P. M. Saffin '19]

Interlude: the 1PI average effective action

The average 1PI effective action is defined as
[C. Wetterich '91]

$$\Gamma^{\mathrm{1PI}}_{\mathrm{av}}[\phi,\mathcal{R}_k] = W[\mathcal{J},\mathcal{R}_k] + \mathcal{J}_x\phi_x + \frac{1}{2}\phi_x\mathcal{R}_{k,xy}\phi_y \qquad \phi_x = -\frac{\delta W[\mathcal{J},\mathcal{R}_k]}{\delta\mathcal{J}_x}$$

where $\mathcal{R}_{k,xy}$ is the inverse FT of the **regulator** (kills fluctuations with $q^2 > k^2$).

Requiring

$$\partial_k\phi_x = -\partial_k\frac{\delta W[\mathcal{J},\mathcal{R}_k]}{\delta\mathcal{J}_x} \overset{!}{=} 0$$

implies $\mathcal{J}[\phi] \equiv \mathcal{J}_k[\phi]$ and

$$\partial_k W[\mathcal{J}_k,\mathcal{R}_k] = -\phi_x\partial_k\mathcal{J}_{k,x} - \frac{1}{2}\left(\hbar\Delta_{k,xy} + \phi_x\phi_y\right)\partial_k\mathcal{R}_{k,xy}$$

$$\Delta_{k,xy} = -\frac{\delta^2 W[\mathcal{J}_k,\mathcal{R}_k]}{\delta\mathcal{J}_{k,x}\delta\mathcal{J}_{k,y}}$$

The **Wetterich-Morris-Ellwanger equation**:
[C. Wetterich '93; T. R. Morris '94; U. Ellwanger '94]

$$\partial_k\Gamma^{\mathrm{1PI}}_{\mathrm{av}}[\phi,\mathcal{R}_k] = -\frac{\hbar}{2}\,\mathrm{Tr}\left(\Delta_k * \partial_k\mathcal{R}_k\right)$$

The regulator-sourced 2PI effective action and exact flow equations

Instead, starting from the 2PI effective action,
[E. Alexander, PM, J. Nursey & P. M. Saffin '19]

$$\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta] = \frac{\delta \Gamma^{2\mathrm{PI}}[\phi,\Delta]}{\delta \phi_x}\,\partial_k \phi_x + \frac{\delta \Gamma^{2\mathrm{PI}}[\phi,\Delta]}{\delta \Delta_{xy}}\,\partial_k \Delta_{xy}$$

$$\partial_k \phi_x = -\partial_k \frac{\delta W[\mathcal{J},\mathcal{K}]}{\delta \mathcal{J}_x} = 0$$

$$\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta] = \frac{\hbar}{2}\,\mathcal{K}_{xy}[\phi,\Delta]\partial_k \Delta_{xy}$$

Now choose $\mathcal{K}_{xy}[\phi,\Delta] = \mathcal{R}_{k,xy}$ to be the inverse FT of the **regulator**:

$$\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta] = \frac{\hbar}{2}\,\mathrm{Tr}\left(\mathcal{R}_k * \partial_k \Delta\right)$$

$$\boxed{\begin{aligned} \partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta_k] &= +\frac{\hbar}{2}\mathrm{STr}\left(\mathcal{R}_k \partial_k \Delta_k\right) \\ \partial_k \Gamma^{1\mathrm{PI}}_{\mathrm{av}}[\phi,\mathcal{R}_k] &= -\frac{\hbar}{2}\mathrm{STr}\left(\Delta_k \partial_k \mathcal{R}_k\right) \end{aligned}}$$

The regulator-sourced 2PI effective action and exact flow equations

Boundary conditions:

- As $k \to 0$, $\mathcal{R}_k \to 0$, and both the regulator-sourced 2PI and average 1PI effective actions, coincide with the 1PI effective action $\Gamma^{1\mathrm{PI}}[\phi] = W[\mathcal{J}] + \mathcal{J}_x \phi_x$.
- As $k \to \infty$, all fluctuations are killed, and both the regulator-sourced 2PI and average 1PI effective actions, coincide with the bare action S.

Closure: It follows from the convexity of the 2PI effective action that

$$-\frac{\delta^2 \Gamma^{2\mathrm{PI}}[\phi, \Delta_k]}{\delta\phi_x \delta\phi_y} \frac{\delta^2 W[\mathcal{J}_k, \mathcal{K}_k]}{\delta \mathcal{J}_{k,x} \delta \mathcal{J}_{k,y}} - \frac{\delta^2 \Gamma^{2\mathrm{PI}}[\phi, \Delta_k]}{\delta\phi_x \delta\Delta'_{k,yz}} \frac{\delta^2 W[\mathcal{J}_k, \mathcal{K}_k]}{\delta \mathcal{J}_{k,x} \delta \mathcal{K}'_{k,yz}} = 1$$

$$\frac{\delta^2 \Gamma^{2\mathrm{PI}}[\phi, \Delta_k]}{\delta\phi_x \delta\phi_y} \Delta_{k,xy} + \frac{\delta^2 \Gamma^{2\mathrm{PI}}[\phi, \Delta_k]}{\delta\phi_x \delta\Delta'_{k,yz}} \frac{\delta\phi_x}{\delta \mathcal{K}'_{k,yz}} = 1$$

But $\delta\phi_x \delta/\mathcal{K}'_{k,yz} = 0$ and therefore

$$\Delta^{-1}_{k,xy} = \frac{\delta^2 \Gamma^{2\mathrm{PI}}[\phi, \Delta_k]}{\delta\phi_x \delta\phi_y} = \frac{\delta^2 S[\phi]}{\delta\phi_x \delta\phi_y} - \mathcal{R}_{k,xy} + \mathcal{O}(\hbar)$$

So we have two closed systems with the same boundary conditions but different evolution equations, and therefore different RG flows!

The regulator-sourced 2PI effective action and exact flow equations

Employing the **derivative expansion**, we make the Ansatz ($\rho \equiv \phi^2/2$)

$$\Gamma^{2\mathrm{PI}}[\phi, \Delta_k] = \int \mathrm{d}^d x \left[U_k(\rho) + \frac{1}{2} Z_k(\rho, (\partial\phi)^2) \partial_\mu \phi \partial_\mu \phi + \mathcal{O}(\partial^4) \right]$$

$$U_k(\rho) = \frac{1}{2} g_k \left(\rho - \bar{\rho}_k\right)^2 + \Lambda_k$$

and introduce dimensionless variables

$$\kappa_k = \bar{Z}_k k^{2-d} \bar{\rho}_k \qquad \lambda_k = \bar{Z}_k^{-2} k^{d-4} g_k$$

with $\bar{Z}_k \equiv Z_k(\bar{\rho}_k, k^2)$, giving

$$U_k(\rho) = \frac{1}{2} k^d \lambda_k (\bar{Z}_k k^{2-d} \rho - \kappa)^2 + \Lambda_k$$

The Ansatz for the two-point function is

$$\Delta_k(\rho, q^2) = \frac{1}{Z_k(\rho, q^2) q^2 - \mathcal{R}_k(q^2) + U_k'(\rho) + 2\rho U_k''(\rho)}$$

and we take the **Litim regulator**
[D. F. Litim '02]

$$\mathcal{R}_k(q^2) = \bar{Z}_k \left(q^2 - k^2\right) \Theta\left(k^2 - q^2\right)$$

The regulator-sourced 2PI effective action and exact flow equations

Regulator-sourced 2PI:
[E. Alexander, PM, J. Nursey & P. M. Saffin '19]

$$\partial_t U_k(\rho) = +\frac{1}{2}\int_q \mathcal{R}_k(q^2)\partial_t\Delta_k(\rho,q^2) \qquad \partial_t\Lambda_k = \frac{8v_d k^d}{d(d+2)}\,\frac{1}{(1+2\kappa_k\lambda_k)^2}$$

$$\partial_t\kappa_k = (2-d)\kappa_k + \frac{48v_d}{d(d+2)}\,\frac{1}{(1+2\kappa_k\lambda_k)^3}$$

$$\partial_t\lambda_k = (d-4)\lambda_k + \frac{432v_d}{d(d+2)}\,\frac{\lambda_k^2}{(1+2\kappa_k\lambda_k)^4}$$

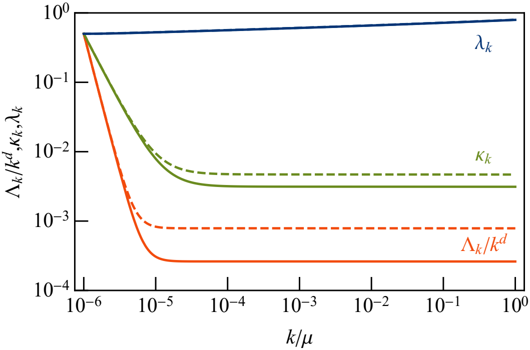
Wetterich-Morris-Ellwanger:
[C. Wetterich '93; T. R. Morris '94; U. Ellwanger '94]

$$\partial_t U_k(\rho) = -\frac{1}{2}\int_q \Delta_k(\rho,q^2)\partial_t\mathcal{R}_k(q^2) \qquad \partial_t\Lambda_k = \frac{4v_d k^d}{d}\,\frac{1}{1+2\kappa_k\lambda_k}$$

$$\partial_t\kappa_k = (2-d)\kappa_k + \frac{12v_d}{d}\,\frac{1}{(1+2\kappa_k\lambda_k)^2}$$

$$\partial_t\lambda_k = (d-4)\lambda_k + \frac{72v_d}{d}\,\frac{\lambda_k^2}{(1+2\kappa_k\lambda_k)^3}$$

The regulator-sourced 2PI effective action and exact flow equations



[E. Alexander, PM, J. Nursey & P. M. Saffin '19]

- Solid: regulator-sourced 2PI. Dashed: Wetterich-Morris-Ellwanger.
- The flow of κ_k is faster (in $d = 4$); the flow of λ_k is slower (in $d = 4$).
- This is in a perturbative regime . . .

Concluding remarks

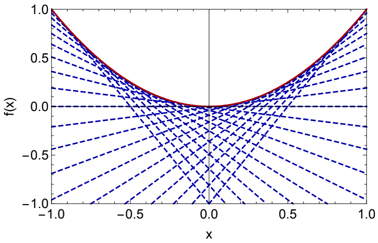
- *It pays to be pedantic when it comes to the quantum effective action.*

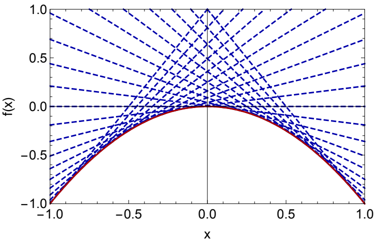
- We can exploit the sources to:
  - Improve our perturbation theory.
  - Improve symmetry properties.
  - Map between different realisations of the effective action.
  - Study the exact RG flow.

- We **do not** recover the Wetterich-Morris-Ellwanger equation; which is correct?
  - Significant differences in non-perturbative regimes?
  - Implications for the asymptotic safety programme?

- Lots to do ...

The Legendre transform

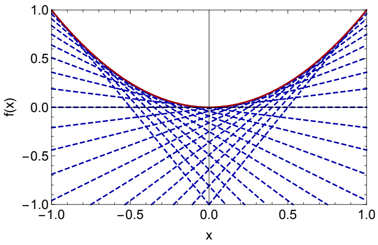


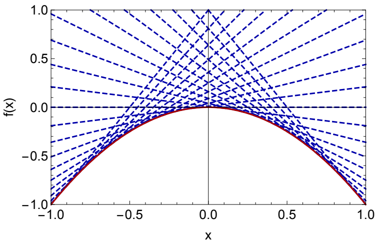


[M. Deserno '12 (unpublished); PM '16 (unpublished)]

- A function f that is strictly convex or concave on an interval $I \in \mathbb{R}$ has a second-derivative of definite sign.
- Its first derivative $f'(x)$ is monotonic, single-valued and invertible on I.
- We can express f as the set of ordered pairs $\{(x, f(x)) | x \in I, f(x) \in \mathbb{R}\}$ **or** the envelope of the tangents to f.
- The **Legendre transform** maps $\{(x, f(x))\}$ to $\{(x^*, f^*(x^*) = -^*f(x^*))\}$, specifying the gradients and intercepts of the tangents. ($* \equiv$ **convex conjugate**.)

The Legendre transform





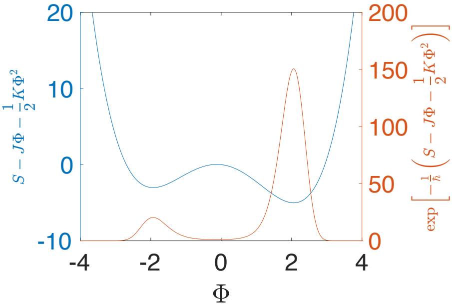
[M. Deserno '12 (unpublished); PM '16 (unpublished)]

- Define $w(x) \equiv x^* x$.
- If $f(x)$ is convex (concave), $w(x) - f(x)$ will have a maximum (minimum):

$$f^*(x^*) \equiv \begin{cases} \min_{x \in I}\{f(x) - x^* x\} \,, & f(x) \text{ convex} \\ \max_{x \in I}\{f(x) - x^* x\} \,, & f(x) \text{ concave} \end{cases}$$

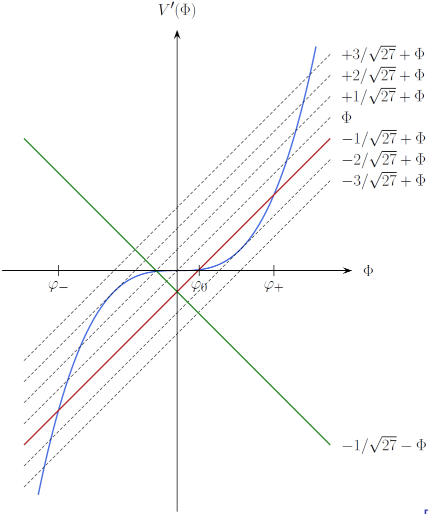
$$^*f(x^*) \equiv \begin{cases} \max_{x \in I}\{x^* x - f(x)\} \,, & f(x) \text{ convex} \\ \min_{x \in I}\{x^* x - f(x)\} \,, & f(x) \text{ concave} \end{cases}$$

Backup: middle saddle



[PM & P. M. Saffin '19]

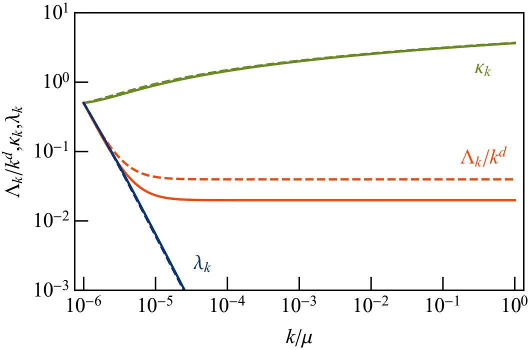
Backup: saddles with $\mathcal{K} \neq 0$



[PM & P. M. Saffin '19]

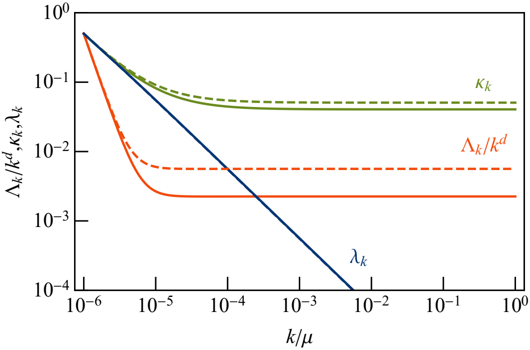
The values on the right-hand side are $\mathcal{J} + \mathcal{K}\Phi$, with $|\mathcal{K}| = 1$.

Backup: $d = 2$



[E. Alexander, PM, J. Nursey & P. M. Saffin '19]

Backup: $d = 3$



[E. Alexander, PM, J. Nursey & P. M. Saffin '19]