Dynamic aperture limit caused by kinematic nonlinearity in extremely low beta B factories

K. Ohmi (KEK),
Super B workshop
16-19 Mar. 2010

Many thanks for collaboration with
E. Levichev and P. Piminov (BINP)

Collaboration with BINP

- E. Levichev and P. Piminov visited in KEK during Jan.-Feb. Mar. 2010.

- Crab waist solution in SuperKEKB
- Common understanding of the dynamic aperture issue for two very low beta B factories in the world. (and tau charm).

How code is required for dynamic aperture simulation in extremely low beta B factories?

- We would like to emphasize,
  - Kinematic nonlinearity at IR region
  - Nonlinear edige of IR quadrupoles.

Dynamic aperture evaluation using BINP and KEK codes

- E.L and P.P uses their own code
- SCTR and SAD in KEKB.
- Comparison is starting and will be continued.

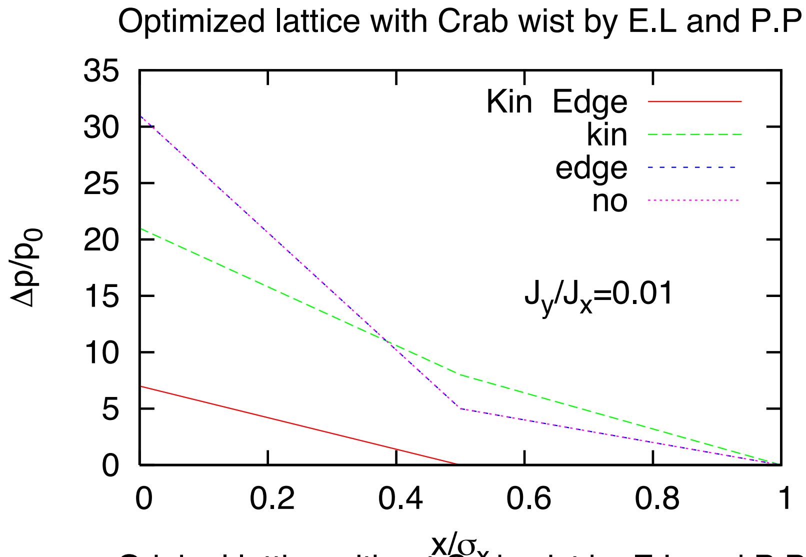
- In this presentation, I show preliminary results of SCTR (written by K. Ohmi).

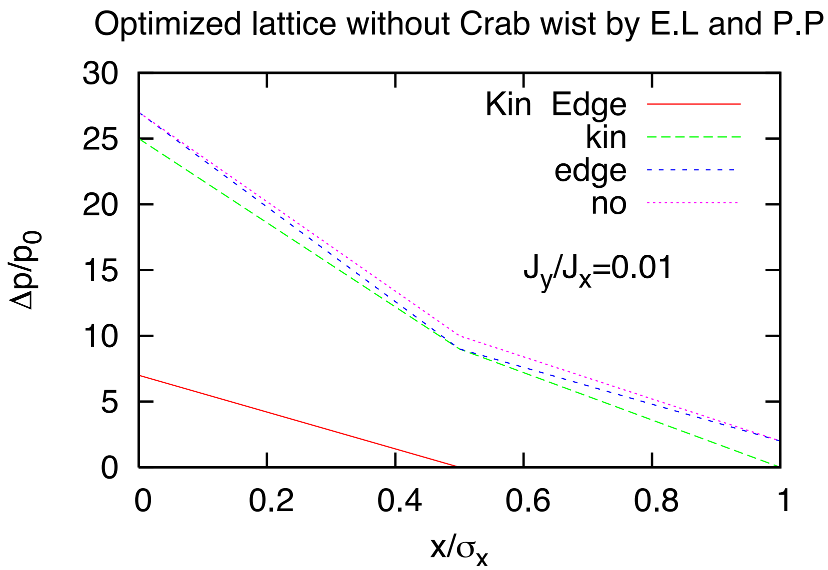
Dynamic aperture evaluation using SCTR codes

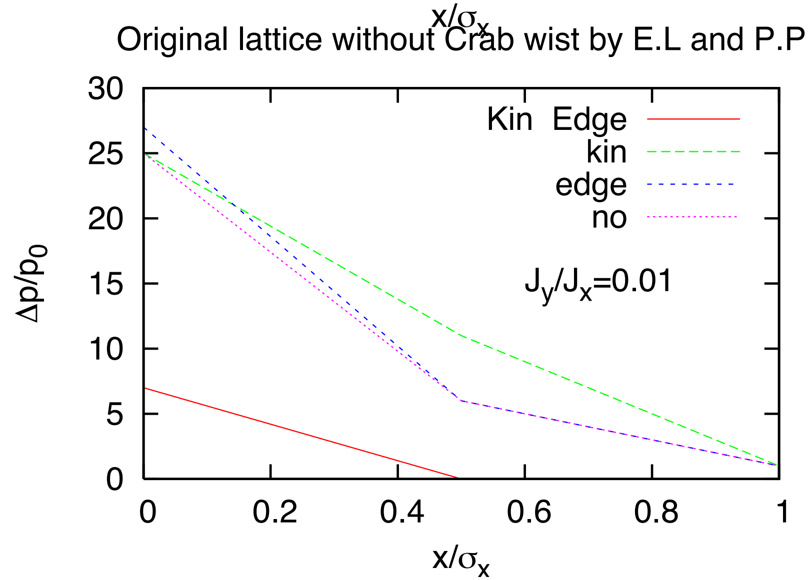
- SCTR is developed for space charge simulation of J-PARC.
- 6D symplectic code including PIC space charge solver (not used this work).
- The code equips tracking of kinematic nonlinearity and edge nonlinear field and is compatible with SAD.

Simulation for SuperB

Preliminary







- Dynamic aperture is very narrow.
- Edge and kinematic nonlinearity make worse drastically.

Study of kinematic and edge nonlinearity

- Model lattice with only IR drift space and IR quadrupoles.

- This study is under construction and preliminary.

Kinematic nonlinearity

- Drift space at the interaction region (IR).
- Hamiltonian contain nonlinear term for p_x, p_y, δ.

$$H = (1+\delta) - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

- This nonlinearity is not negligible for very low beta IR.
  - ★ Chromaticity and its higher order
  - ★ Octupole and higher order nonlinearity

Chromaticity and nonlinearity

$$H_n = (1+\delta) - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - \frac{p_x^2 + p_y^2}{2}$$
$$= -\frac{(p_x^2+p_y^2)\delta}{2(1+\delta)} + \frac{1}{8}\frac{(p_x^2+p_y^2)^2}{(1+\delta)^3} + \frac{1}{16}\frac{(p_x^2+p_y^2)^4}{(1+\delta)^5} + \cdots$$

- Subtract linear (drift) motion
- First term gives chromaticity
- Second and later give octupole and higher nonlinearity

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128}$$

Chromaticity

$$H_\xi = -\frac{(p_x^2 + p_y^2)\delta}{2(1+\delta)}$$

$$\bar{x} = x - p_x \frac{L_0\delta}{1+\delta} \equiv x + p_x L_\delta \qquad L_\delta = -\frac{L_0\delta}{1+\delta}$$

- The chromaticity is corrected by local chromaticity compensation.

$$\begin{pmatrix} 1 & L_\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} 1 & L_\delta \\ 0 & 1 \end{pmatrix}$$

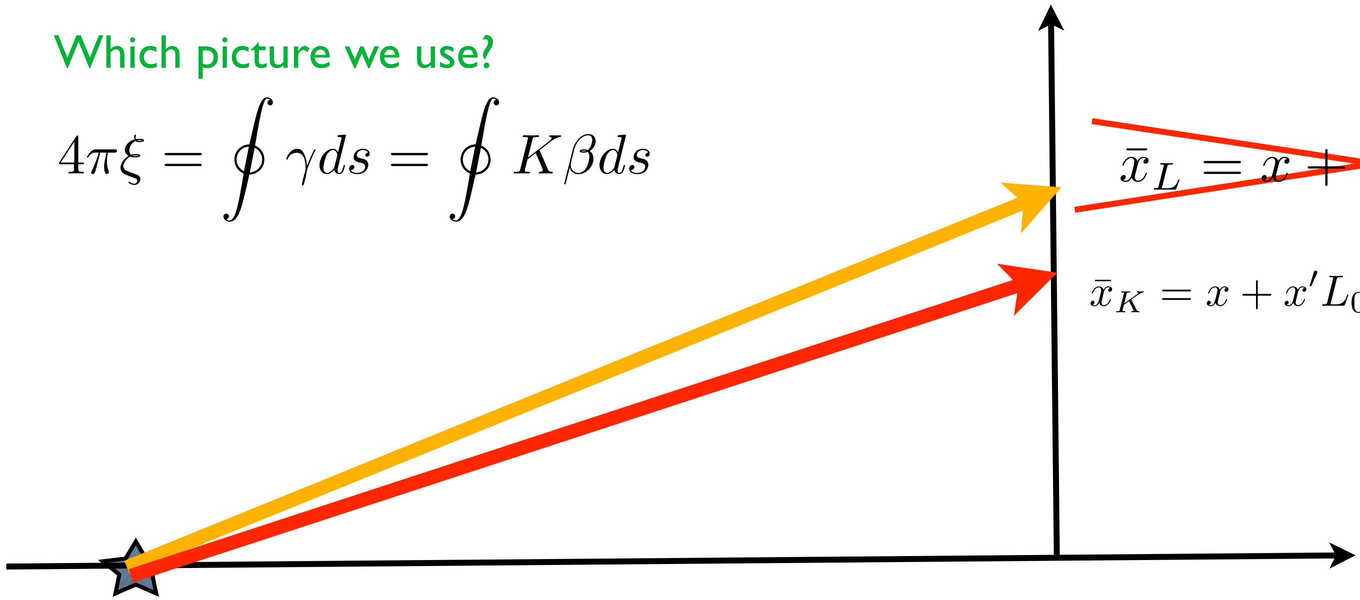
$$= \begin{pmatrix} \cos\mu - \frac{L_\delta \sin\mu}{\beta} & \left(\beta - \frac{L_\delta^2}{\beta}\right)\sin\mu + 2L_\delta\cos\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu - \frac{L_\delta\sin\mu}{\beta} \end{pmatrix}$$

What is kinematical nonlinearity

- Why nonlinearity appears although particles simply move straight.

Which picture we use?

$$4\pi\xi = \oint \gamma ds = \oint K\beta ds$$



~~$\bar{x}_L = x + x'L_0 = x + p_x L_0$~~

$$\bar{x}_K = x + x'L_0 = x + \frac{p_x}{\sqrt{(1+\delta)^2 + p_x^2 + p_y^2}} L_0$$

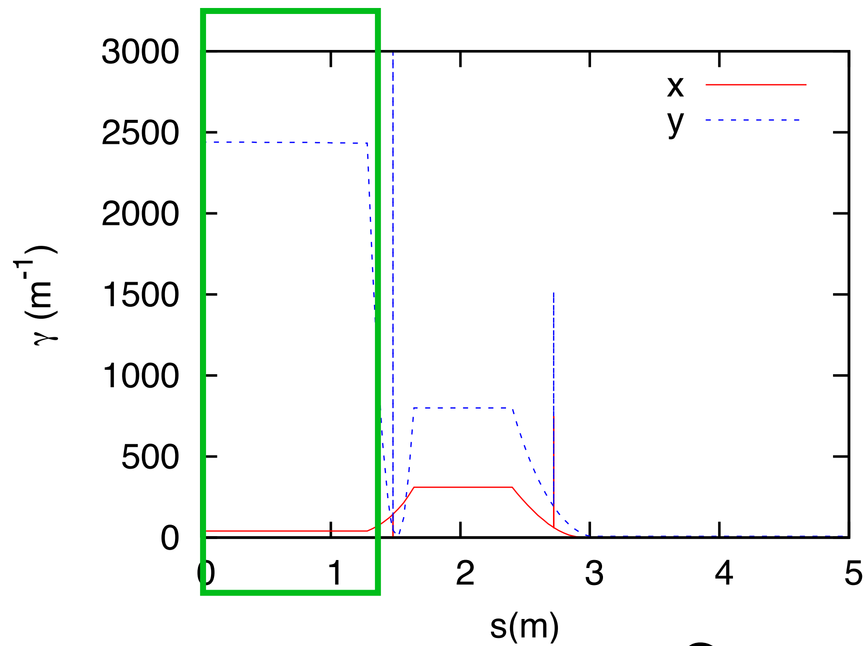
$$\bar{x}_K - \bar{x}_L = -\frac{p_x\delta}{1+\delta} - \frac{p_x(p_x^2+p_y^2)}{2(1+\delta)^3} + \ldots$$

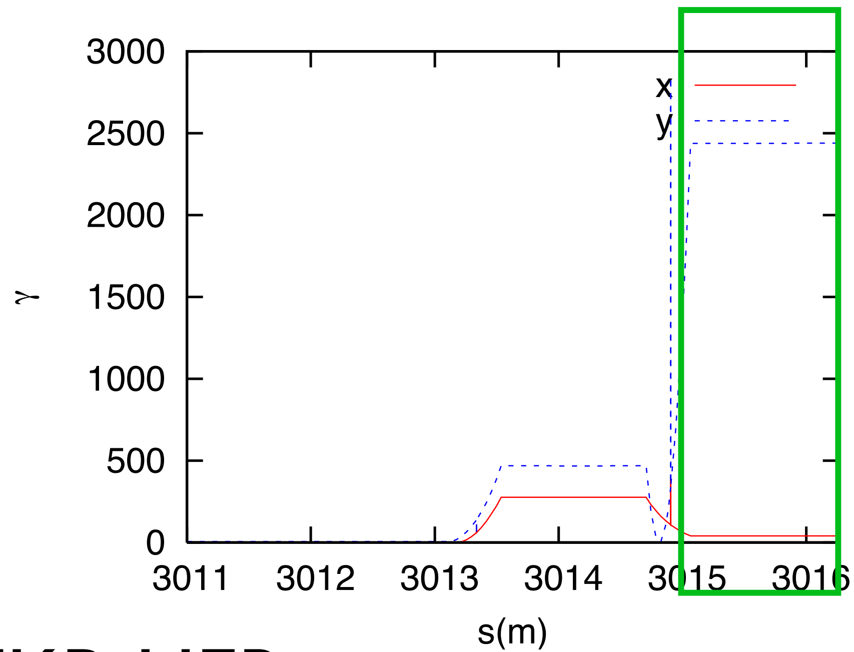
Chromaticity   Kinematic nonlinearity

$p^2 \sim \gamma J$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

- These nonlinearities are strong at high γ region near IP.
- We consider only L_0 area in this presentation. It means that the effects are underestimated, especially for horizontal.





L_0=1.4m

SuperKEKB-HER

C=3016.2m

Kinematic nonlinearity

$$e^{-:H_K:L_0} M_0 e^{-:H_K:L_0}$$

M_0: Linear (matrix) transformation for ring

- L : distance between IP and QD0.

$$\begin{aligned} H_K &= (1+\delta) - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - \frac{p_x^2 + p_y^2}{2}(1-\delta) \\ &= \frac{\delta^2 (p_x^2 + p_y^2)}{2(1+\delta)} + \frac{1}{8}\frac{(p_x^2+p_y^2)^2}{(1+\delta)^3} + \frac{1}{16}\frac{(p_x^2+p_y^2)^4}{(1+\delta)^5} + \ldots \end{aligned}$$

- Here only linear chromaticity is corrected by the last term of H_K.

- If higher order chromaticity is corrected, first term is eliminated.

Amplitude dependent Tune shift due to the kinematic nonlinearity

$$2L_0 H_K \approx \frac{p_y^4}{4} L_0 = \frac{J_y^2}{\beta_{y,0}^2} L_0$$

$$e^{-:H_K:2L_0}$$

$$\Delta\nu_y = \frac{1}{2\pi} L_0 \frac{J_y}{\beta_{y,0}^2} = \frac{J_y}{2\times 10^6}$$

Aperture may be $J_y = 10^{-8}$ m

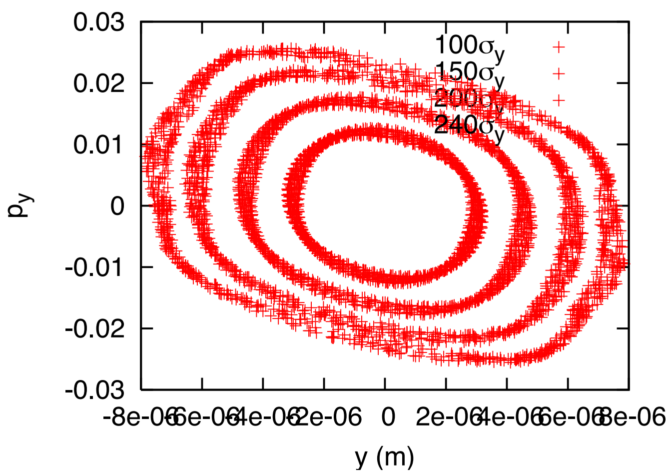
- For horizontal tune shift, defocusing of QD0 and distance to QF1 should be taken into account.

Dynamic aperture limit due to the kinematic nonlinearity

- Simple aperture simulation using H_K.

$$e^{-:H_K:L_0} M_0 e^{-:H_K:L_0}$$

- L0=0.4m, ε_x=2nm, ε_y=5pm, β_x=2cm, β_y=0.2mm
- $A_y=(240\sigma_y)^2/\beta_y=2.9\times10^{-7}$ m (x=0)
- $A_x=A_y/0.01=(88\sigma_x)^2/\beta_x=1.5\times10^{-5}$ m (J_y/J_x=0.01)



Crab waist

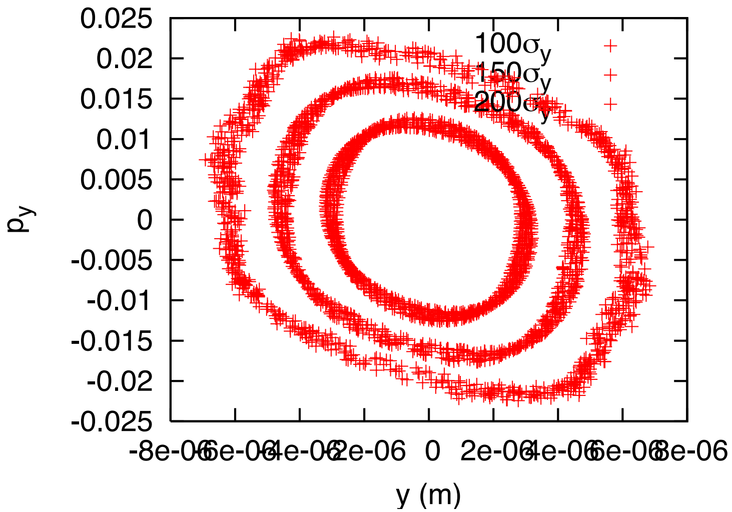
$$e^{-:H_K:L_0} e^{-:xp_y^2:/\theta} M_0 e^{:xp_y^2:/\theta} e^{-:H_K:L_0}$$

$$M_0 \boxed{e^{:xp_y^2:/\theta} e^{-:H_K:L_0} e^{-:H_K:L0} e^{-:xp_y^2:/\theta}}$$

- If H_K is negligible, crab waist nonlinearity is cancelled outside of IR.
- Kinematic nonlinearity breaks to cancel between the crab waist sextupoles.
- x^3 terms of the sextupole may affect something but are neglected now.
- The fact that the kinematic nonlinearity affect the aperture for crab waist scheme has been investigated by H. Koiso using SAD since several years ago.

Dynamic aperture limit due to the crab waist

- $A_y=(210\sigma_y)^2/\beta_y=2.2\times10^{-7}$ (x=0)
- $A_x=A_y/0.01=(37\sigma_x)^2/\beta_x=2.7\times10^{-6}$ m ($J_y/J_x=0.01$)
- Horizontal kinematic term is underestimated.



Quadrupole edge nonlinearity

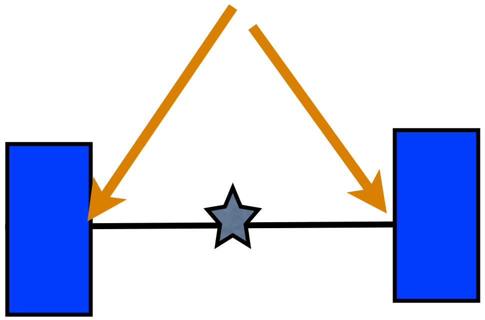
- Quadrupole nonlinearity at its face of IR

$$H_E = -\frac{k}{1+\delta}\frac{yp_y(3x^2+y^2)-xp_x(3y^2+x^2)}{12}$$

$$k = \frac{eB'}{p_0 B}$$

k~6m^{-2}

$$e^{:H_0:}e^{-:H_E:}e^{-:H_0:} = e^{-:H_{E,0}}$$



$$H_{E,0} \approx -\frac{kL_0^3}{1+\delta}\frac{p_y^4 - p_x^4}{12}$$

$$H_{E,0} \approx -\frac{kL_0^3}{1+\delta}\frac{p_y^4 - p_x^4}{12} \approx -\frac{kL_0^3}{1+\delta}\left(\frac{J_y^2}{12\beta_{y,0}^2} - \frac{J_x^2}{12\beta_{x,0}^2}\right)$$

$$\Delta\nu_x \approx \frac{kL_0^3}{12\pi(1+\delta)}\frac{J_x}{\beta_{x,0}^2} \approx J_y/40$$

$\Delta\nu_x$ is too optimistic since QF1 is not included.

$$\Delta\nu_y \approx -\frac{kL_0^3}{12\pi(1+\delta)}\frac{J_y}{\beta_{y,0}^2} \approx \frac{J_y}{4\times 10^6}$$

Dynamic aperture limit due to the Quadrupole edge

- Simple aperture simulation using $H_{E,0}$.

$$e^{-:H_{E,0}}e^{-:H_K:L_0}M_0e^{-:H_K:L_0}e^{-:H_{E,0}}$$

$$e^{-:H_{E,0}:}e^{-:H_K:L_0}e^{-:xp_y^2:/\theta}M_0e^{:xp_y^2:/\theta}e^{-:H_K:L_0}e^{-:H_{E,0}:}$$

Summary

- Kinematic nonlinearity and quadrupole edge nonlinearity is indispensable to the dynamic aperture calculation for the very low beta factories.
- Kinematic and chromatic nonlinearity between two crab waist sextupoles has to be cancelled.
- Quadrupole edge nonlinearity mainly located in IR has to be cancelled.

Memo: Inside SCTR code

- Canonical variable, Pi's are momentum in ordinary definition in the classical mechanics.

$$x, p_x = P_x/P_0, y, p_y = P_y/P_0, z = v(t_0 - t), \delta = (P - P_0)/P_0$$

- Hamiltonian

$$H = \frac{E(\delta)}{P_0 v_0} - \left(1 + \frac{x}{\rho}\right)\left[(1+\delta^2) - \left(p_x - \hat{A}_x\right)^2 - \left(p_y - \hat{A}_y\right)^2\right]^{1/2} - \left(1 + \frac{x}{\rho}\right)\hat{A}_s$$

Kinematic slippage

$$H = H_0 + H_1$$

$$H_0 = \frac{E}{P_0 v_0} - (1+\delta)$$

$$H_1 = (1+\delta) - \left(1 + \frac{x}{\rho}\right)\sqrt{(1+\delta)^2 - (p_x - \hat{A}_x)^2 - (p_y - \hat{A}_y)^2} - \left(1 + \frac{x}{\rho}\right)\hat{A}_s$$

- H_1 characterizes particle motion in ultra-relativistic limit.
- Kinematic slippage, effect of velocity <c.
- Not important for electron machines

$$\frac{dz}{ds} = \frac{\partial H_0}{\partial \delta} = \frac{1}{P_0 v_0}\frac{\partial E}{\partial \delta} - 1 = \frac{v(\delta) - v_0}{v_0} \approx \frac{\delta}{\gamma_0^2}$$

Drift space

- Expand by 4-th order

$$H_1 = (1+\delta) - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

$$\begin{aligned}
x_2 &= x_1 + \left[\frac{p_x}{1+\delta} + \frac{(p_x^2+p_y^2)p_x}{2(1+\delta)^3}\right]\Delta s \\
y_2 &= y_1 + \left[\frac{p_y}{1+\delta} + \frac{(p_x^2+p_y^2)p_y}{2(1+\delta)^3}\right]\Delta s \\
z_2 &= z_1 - \left[\frac{p_x^2+p_y^2}{2(1+\delta)^2} + \frac{3(p_x^2+p_y^2)^2}{8(1+\delta)^4}\right]\Delta s
\end{aligned}$$

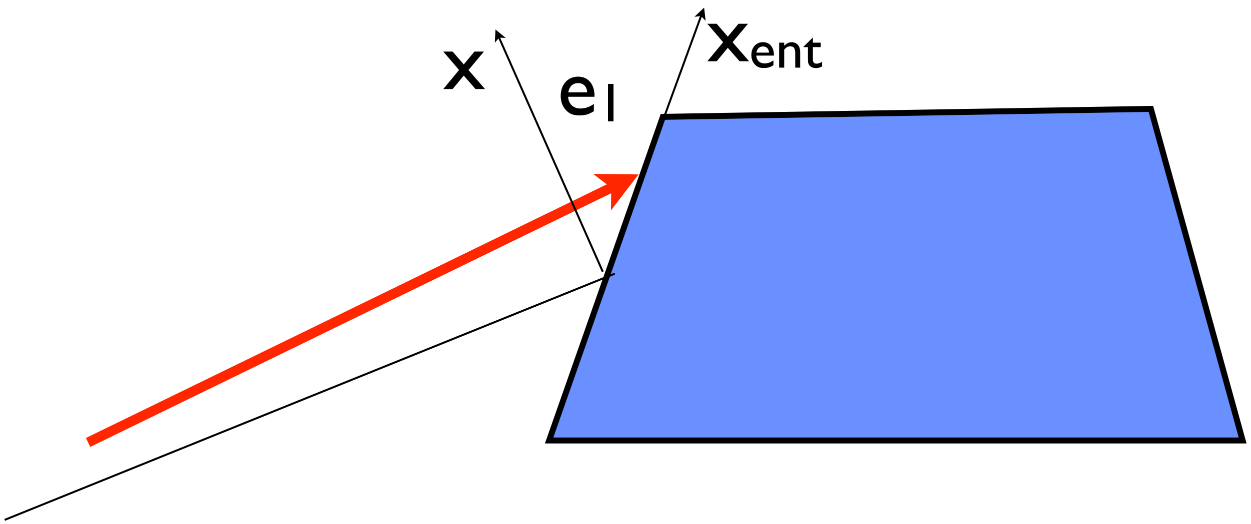
Bend

- Place particles at the entrance face
- Edge nonlinearity
- Back to original coordinates
- Edge focusing
- Body integration
- Edge focusing
- Place particles at the exit face
- Edge nonlinearity
- Back to original coordinates

Transfer to entrance face

- Leading order

$$
\begin{aligned}
x_{ent} &= \frac{x}{\cos e_1} \\
p_{x,ent} &= p_x \cos(e_1) + (1+\delta)\sin(e_1) \\
z_{ent} &= z - \tan(e_1)x
\end{aligned}
$$



Bend edge transformation in the entrance face coordinate

- Linear fringe

$$H = \frac{F_1^2}{24\rho} p_x \delta - \frac{F_1}{6\rho^2} p_y^2$$

$$x = x + \frac{F_1^2}{24\rho}\delta \qquad p_y = p_y + \frac{F_1}{6\rho^2} y \qquad z = z + \frac{F_1^2}{24\rho} p_x$$

- Nonlinear edge

$$H = \frac{1}{2\rho}\frac{y^2 p_x}{(1+\delta)\cos e_1}$$

$$\begin{aligned} x &= x + \frac{y^2}{2\rho\cos e_1(1+\delta)} \\ p_y &= p_y - \frac{p_x y}{\rho\cos e_1(1+\delta)} \\ z &= z - \frac{p_x y^2}{2\rho\cos e_1(1+\delta)^2} \end{aligned}$$

Bend body

$$H_1 = (1+\delta) - \left(1+\frac{x}{\rho}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \left(x + \frac{x^2}{2\rho}\right)B_y + \frac{K_1}{2}(x^2+y^2) + \Delta B_y x + \Delta B_x y + K_{1s}xy$$

$$H_1 = -\frac{x}{\rho} + x\hat{B}_y + H_2 + H_3 = H_2 + H_3$$

- Linear term in H is eliminated by $\hat{B} = \frac{eB}{p_0} = \frac{1}{\rho}$

$$H_2 = -\frac{1}{\rho}x\delta + \frac{p_x^2 + p_y^2}{2} + \frac{1}{\rho^2}x^2 + \frac{K_1}{2}(x^2 - y^2)$$

Linear and nonlinear transformations

$$e^{-:H_2:s} = \begin{pmatrix} \cos\phi & \rho\sin\phi & 0 & 0 & 0 & \rho(1-\cos\phi) \\ -\frac{1}{\rho}\sin\phi & \cos\phi & 0 & 0 & 0 & \sin\phi \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\phi & -\rho(1-\cos\phi) & 0 & 0 & 1 & \rho\sin\phi - s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}$$

- The transformations are carried out with several steps.

$$(e^{-H_2:L/2N} e^{-:H_3:L/N} e^{-H_2:L/2N})^N$$

$$H_3 = \frac{(p_x^2+p_y^2)\delta}{2(1+\delta)} + \frac{(p_x^2+p_y^2)^2}{8(1+\delta)^3} + \frac{x}{\rho}\frac{p_x^2+p_y^2}{2(1+\delta)^2} + x\Delta\hat{B}_y + y\Delta\hat{B}_x + K_{1s}xy$$

Nonlinear transformation for H_3

$$\begin{aligned} x &= x + \left[-\frac{p_x \delta}{1+\delta} + \frac{p_x (p_x^2 + p_y^2)}{2(1+\delta)^3} \right] s \\ y &= y + \left[-\frac{p_y \delta}{1+\delta} + \frac{p_y (p_x^2 + p_y^2)}{2(1+\delta)^3} \right] s \\ z &= z - \left[\frac{p_x^2 + p_y^2}{2(1+\delta)^2} + \frac{3(p_x^2 + p_y^2)^2}{8(1+\delta)^4} \right] s \end{aligned}$$

- Additional transformation of 3rd term

$$\begin{aligned} x &= \frac{x}{1 - \frac{p_x s}{\rho(1+\delta)}} \\ y &= y + \frac{x p_y s}{\rho} \\ p_x &= p_x - \frac{p_x^2 + p_y^2}{2\rho(1+\delta)} \\ z &= z \end{aligned}$$

Quadrupole

- Edge nonlinearity
- Linear Fringe
- Body
- Linear Fringe
- Edge nonlinearity

Quad nonlinear edge

$$H = -\frac{k}{1+\delta}\frac{yp_y(3x^2+y^2) - xp_x(3y^2+x^2)}{12}$$

- 2nd order integrator

$$\begin{aligned}
\bar{x} &= x + \frac{k}{1+\delta}(x^3+3xy^2) + \frac{k^2}{96(1+\delta)^2}x(x^2-y^2)^2 \\
\bar{y} &= y - \frac{k}{1+\delta}(3x^2y+y^3) + \frac{k^2}{96(1+\delta)^2}y(x^2-y^2)^2
\end{aligned}$$

- Implicit relation for $p_{x,y}$, but solved easy.

$$\begin{aligned}
p_x &= \bar{p}_x\left[1 + \frac{k}{4(1+\delta)}(x^2+y^2) + \frac{k^2}{96(1+\delta)^2}(5x^2-y^2)\right] \\
&+ \bar{p}_y\left[-\frac{k}{2(1+\delta)}xy + \frac{k^2}{24(1+\delta)^2}(x^2-y^2)xy\right] \\
p_y &= \bar{p}_y\left[1 - \frac{k}{4(1+\delta)}(x^2+y^2) + \frac{k^2}{96(1+\delta)^2}(x^2-5y^2)\right] \\
&+ \bar{p}_x\left[\frac{k}{2(1+\delta)}xy - \frac{k^2}{24(1+\delta)^2}(x^2-y^2)xy\right] \\
\bar{z} &= z + \frac{k}{12}\frac{y\bar{p}_y(3x^2+y^2) - x\bar{p}_x(3y^2+x^2)}{(1+\delta)^2}
\end{aligned}$$

Quad linear fringe

- Linear transformation and chromatic correction

$$H = -a_0(xp_x - yp_y)\delta$$

$$\begin{aligned}
\bar{x} &= x - a_0 x\delta \qquad \bar{p}_x = \frac{p_x}{1 - a_0\delta} \\
\bar{y} &= y + a_0 y\delta \qquad \bar{p}_y = \frac{p_x}{1 + a_0\delta} \\
\bar{z} &= z - a_0(x\bar{p}_x - y\bar{p}_y)\delta
\end{aligned}$$

$$\begin{aligned}
\bar{x} &= e^{a_0}x + f_2 p_x \qquad p_x = e^{-a_0}p_x \\
\bar{y} &= e^{-a_0}y - f_2 p_y \qquad p_y = e^{a_0}p_y
\end{aligned}$$

Quad Body

$$H_1 = (1+\delta) - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{K_1}{2}(x^2 + y^2)$$

$$H_1 = H_2 + H_3$$

$$H_2 = \frac{p_x^2 + p_y^2}{2} + \frac{K_1}{2}(x^2 - y^2)$$

$$H_3 = -\frac{(p_x^2 + p_y^2)\delta}{2(1+\delta)} + \frac{(p_x^2 + p_y^2)^2}{8(1+\delta)^3}$$

$$(e^{-H_2:L/2N} e^{-:H_3:L/N} e^{-H_2:L/2N})^N$$

Linear motion

- QF

$$e^{-:H_2:s} = \begin{pmatrix} \cos\sqrt{K_1}s & \frac{1}{\sqrt{K_1}}\sin\sqrt{K_1}s & 0 & 0 & 0 & 0 \\ -\sqrt{K_1}\sin\sqrt{K_1}s & \cos\sqrt{K_1}s & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\sqrt{K_1}s & \frac{1}{\sqrt{K_1}}\sinh\sqrt{K_1}s & 0 & 0 \\ 0 & 0 & \sqrt{K_1}\sinh\sqrt{K_1}s & \cosh\sqrt{K_1}s & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Nonlinear transformation for H_3

$$
\begin{aligned}
x &= x + \left[-\frac{p_x \delta}{1+\delta} + \frac{p_x (p_x^2 + p_y^2)}{2(1+\delta)^3} \right] s \\
y &= y + \left[-\frac{p_y \delta}{1+\delta} + \frac{p_y (p_x^2 + p_y^2)}{2(1+\delta)^3} \right] s \\
z &= z - \left[\frac{p_x^2 + p_y^2}{2(1+\delta)^2} + \frac{3(p_x^2 + p_y^2)^2}{8(1+\delta)^4} \right] s
\end{aligned}
$$

Solenoid

$$H_1 = (1+\delta) - \sqrt{(1+\delta)^2 - (p_x - by/2)^2 - (p_y + bx/2)^2} = H_2 + H_3$$

$$b = \frac{eB_z}{p_0}$$

- Linear transformation

$$H_2 = \frac{(p_x - by/2)^2}{2} + \frac{(p_y + bx/2)^2}{2}$$

- Nonlinear transformation

$$H_3 = -\frac{\delta}{2(1+\delta)}\left[(p_x - by/2)^2 + (p_y + bx/2)^2\right] + \frac{1}{8(1+\delta)^3}\left[(p_x - by/2)^2 + (p_y + bx/2)^2\right]^2$$

Linear transformation

- Equation of motion

$$x'' + iy'' = ib(x' + iy').$$

- Solution $x' + iy' = e^{ibs}(x'_0 + iy'_0)$

$$x + iy = \frac{e^{ibs} - 1}{ib}\left[x'_0 + iy'_0\right] + x_0 + iy_0$$

- Transformation for (x,p_x,y,p_y)

$$M_{sol} = \begin{pmatrix} \frac{1+\cos bs}{2} & \frac{\sin bs}{b} & -\frac{\sin bs}{2} & -\frac{1-\cos bs}{b} \\ -\frac{b\sin bs}{4} & \frac{1+\cos bs}{2} & \frac{b(1-\cos bs)}{4} & -\frac{\sin bs}{2} \\ \frac{\sin bs}{2} & \frac{1-\cos bs}{b} & \frac{1+\cos bs}{2} & \frac{\sin bs}{b} \\ -\frac{b(1-\cos bs)}{4} & \frac{\sin bs}{2} & -\frac{b\sin bs}{4} & \frac{1+\cos bs}{2} \end{pmatrix}$$

Nonlinear transformation

- The first term in H_3. Second term is not written with an explicit form

$$H_{31} = -\frac{\delta}{2(1+\delta)}\left[(p_x - by/2)^2 + (p_y + bx/2)^2\right]$$

$$\begin{aligned}
\bar{x} &= x - \frac{\delta}{1+\delta}(\bar{p}_x - by/2) \\
\bar{y} &= y - \frac{\delta}{1+\delta}(\bar{p}_y + bx/2) \\
\bar{z} &= z - \frac{1}{2(1+\delta)^2}\left[(\bar{p}_x - by/2)^2 + (\bar{p}_y + bx/2)^2\right] \\
p_x &= \bar{p}_x - \frac{b\delta}{2(1+\delta)}(\bar{p}_y + bx/2) \\
p_y &= \bar{p}_y + \frac{b\delta}{2(1+\delta)}(\bar{p}_x - by/2)
\end{aligned}$$

$$\begin{aligned}
\bar{p}_x &= \frac{p_x + b_\delta p_y + b_\delta b/2(x + b_\delta y)}{1 + b_\delta^2} \\
\bar{p}_y &= \frac{p_y - b_\delta p_x + b_\delta b/2(y - b_\delta x)}{1 + b_\delta^2}
\end{aligned}$$

$$b_\delta = \frac{b\delta}{2(1+\delta)}.$$

Better expression

- Solenoid is basically integrable
- Under construction

Formulae

$$\frac{d\beta}{ds} = -2\alpha \qquad \frac{d^2\beta}{ds^2} - 2\gamma + 2K\beta = 0$$

$$\oint \gamma ds = \oint K\beta ds$$

$$\beta = \beta_0 + \frac{s^2}{\beta_0} \qquad \alpha = -\frac{s}{\beta_0} \qquad \phi = \int_0^s \frac{ds}{\beta}$$

$$\begin{pmatrix} X \\ P_X \end{pmatrix} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

$$\begin{aligned} x &= \sqrt{2\beta J}\cos\phi \\ p_x &= -\sqrt{\frac{2J}{\beta}}(\sin\phi - \alpha\cos\phi) \end{aligned}$$

$$\sin^2\phi = \frac{1-\cos 2\phi}{2} \qquad \cos^2\phi = \frac{1+\cos 2\phi}{2}$$