The HVP and HLbL contributions to $(g-2)_\mu$ from lattice QCD

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The magnetic moment

- The magnetic moment $\vec{\mu}$ determines the shift of a particle's energy in the presence of a magnetic field $\vec{B}$

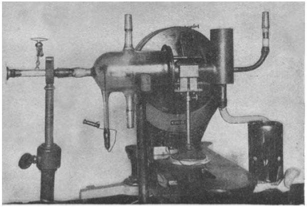
$$V = -\vec{\mu} \cdot \vec{B}$$

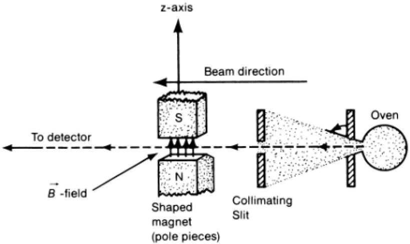
- The intrinsic spin $\vec{S}$ of a particle contributes

$$\vec{\mu} = g \left(\frac{e}{2m} \right) \vec{S}$$

with electric charge e, particle mass m, and Landé factor g.

Stern & Gerlach, 1922





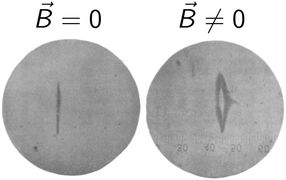


Fig. 2. Fig. 3.

- Send silver atoms through non-uniform magnetic field, $\vec{F} = -\vec{\nabla} V$
- Atoms electrically neutral $\Rightarrow$ spin effects can dominate
- Silver has single 5s electron and fully filled shells below $\Rightarrow$ observe μ of the electron
- $\vec{B} \neq 0$: two distinct lines $\Rightarrow$ quantized spin, distance of lines $\Rightarrow g_e$

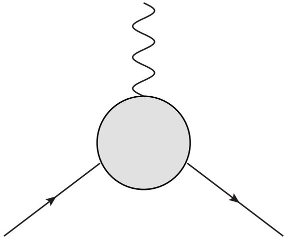
The anomalous magnetic moment

- 1924: Stern and Gerlach measured $g_e = 2.0(2)$
- 1928: Dirac shows that relativistic quantum mechanics yields $g_e = 2$
- 1947 (Phys. Rev. 72 1256, November 3): Kusch & Foley (Columbia) measure $g_e = 2.00229(8)$ in the Zeeman spectrum of gallium
- 1947 (Phys. Rev. 73 416, December 30): Schwinger calculates lowest-order radiative photon correction within quantum field theory (QFT): $g_e = 2 + \alpha/\pi = 2.00232\ldots$

  Define anomalous magnetic moment $a_e = (g_e - 2)/2$ exhibiting effects of QFT

The anomalous magnetic moment

- In QFT a can be expressed in terms of scattering of particle off a classical photon background



For external photon index μ with momentum q the scattering amplitude can be generally written as

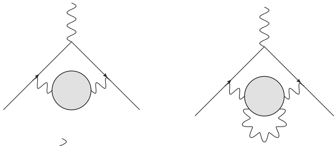
$$(-ie)\left[\gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q^\nu}{2m}F_2(q^2)\right]$$

with $F_2(0) = a$.

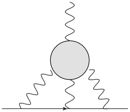
There is a tension of 3.7σ for the muon

$$a_\mu^{\mathrm{E821}} - a_\mu^{\mathrm{SM}} = 27.4\,\underbrace{(2.7)}_{\mathrm{HVP}}\,\underbrace{(2.6)}_{\mathrm{HLbL}}\,\underbrace{(0.1)}_{\mathrm{other}}\,\underbrace{(6.3)}_{\mathrm{E821}} \times 10^{-10}$$

Hadronic Vacuum Polarization (HVP)



Hadronic Light-by-Light (HLbL)



There is also a -2.4σ tension for the electron, topic for another talk

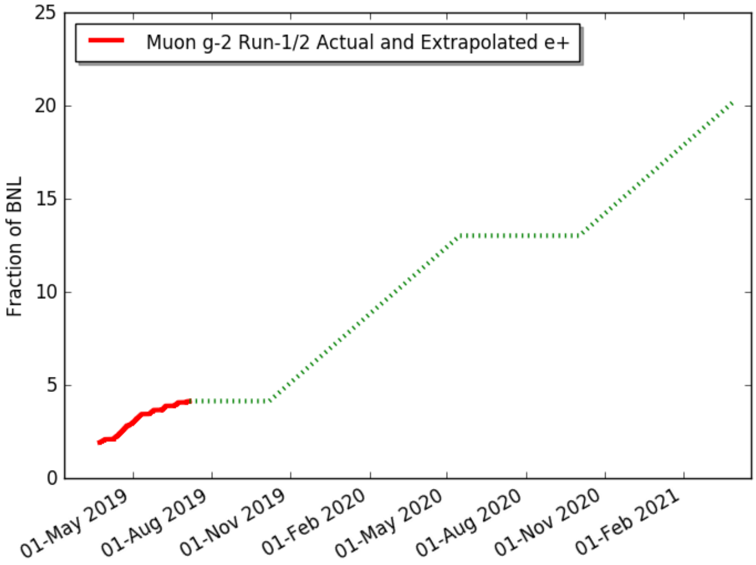
New experiment: Fermilab E989

$$a_\mu^{\text{E821}} - a_\mu^{\text{SM}} = 27.4\,\underbrace{(2.7)}_{\text{HVP}}\,\underbrace{(2.6)}_{\text{HLbL}}\,\underbrace{(0.1)}_{\text{other}}\,\underbrace{(6.3)}_{\text{E821}} \times 10^{-10}$$

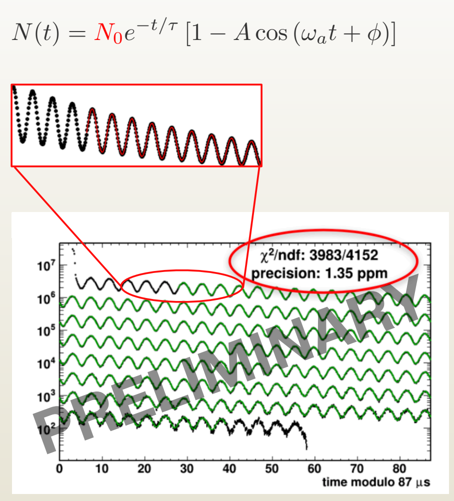
$$\delta a_\mu^{\text{E989, } \mathbf{2019}} = 4.5 \times 10^{-10}\,, \qquad \delta a_\mu^{\text{E989, 2021}} = 1.6 \times 10^{-10}$$

Need to improve uncertainties on HVP and HLbL contributions

Statistics Run 1 in 2018 and Run 2 in 2019 and projection (talk by C. Ferrari in Saclay 2019):



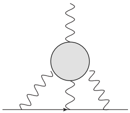
Run 1 fit (talk by N. Tran at FPCP 2019):



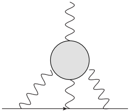
Relative unblinding of 6 groups for a data subset ("60 hours dataset") successful.

First results will possibly be published in second quarter of 2020

HLbL contribution



Current HLbL value is model estimate

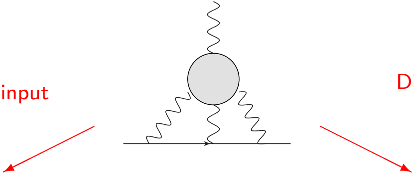


Contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{10}$

	PdRV09	JN09	FJ17
π^0, η, η'	11.4(1.3)	9.9(1.6)	9.5(1.2)
π, K loops	-1.9(1.9)	-1.9(1.3)	-2.0(5)
axial-vector	1.5(1.0)	2.2(5)	0.8(3)
scalar	-0.7(7)	-0.7(2)	-0.6(1)
quark loops	0.2 (charm)	2.1(3)	2.2(4)
tensor			0.1(0)
NLO			0.3(2)
Total	10.5(4.9)	11.6(3.9)	10.3(2.9)
	10.5(2.6) (quadrature)		

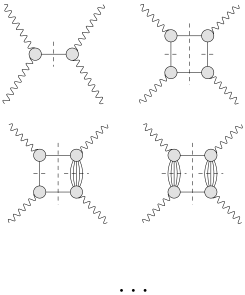
Potential double-counting and ad-hoc uncertainties

Two new avenues for a model-independent value for the HLbL

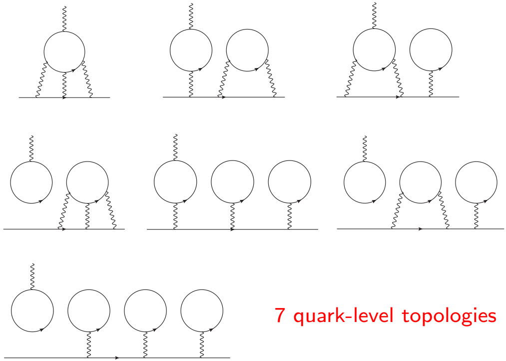


Dispersive analysis +
Experimental/lattice input

Direct lattice calculation



How to estimate uncertainty of truncation of cuts/states?



7 quark-level topologies

Dispersive analysis - recent results

- JHEP1704(2017)161 (Colangelo et al.): Pion-box plus S-wave rescattering
  $a_\mu^{\pi-box} + a_\mu^{\pi\pi,\pi-pole\ LHC,J=0} = -2.4(1) \times 10^{-10}$
- PRL121(2018)112002 (Hoferichter et al.); 1808.04823: Pion-pole contribution
  $a_\mu^{\pi-pole} = 6.26(30) \times 10^{-10}$ reconstructing $\pi \to \gamma^*\gamma^*$ form factor from $e^+e^- \to 3\pi$, $e^+e^-\pi^0$ and $\pi^0 \to \gamma\gamma$ width
- PRD100(2019)034520 (Mainz): Pion-pole contribution
  $a_\mu^{\pi-pole} = 6.23(23) \times 10^{-10}$ (Lattice+Dispersive FF normalization by PrimEx)

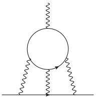
Combining these results one finds: $a_\mu^{\pi-pole} + a_\mu^{\pi-box} + a_\mu^{\pi\pi} = 3.9(3) \times 10^{-10}$

Further estimates: $a_\mu^{\eta,\eta'} \approx 3 \times 10^{-10}$, $a_\mu^{\text{axial vector}} \approx 1 \times 10^{-10}$, $a_\mu^{\text{short distance}} \approx 1 \times 10^{-10}$

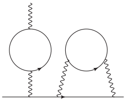
Control of truncation error very important. Best current approach using large-Nc Regge models: arXiv:1910.13432, Colangelo et al.

7 quark-level topologies of direct lattice calculation

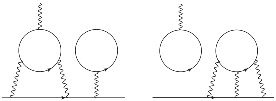
Hierarchy imposed by QED charges of dominant up- and down-quark contribution



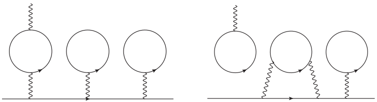
$Q_u^4 + Q_d^4 = 17/81$



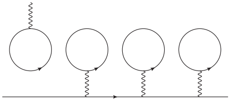
$(Q_u^2 + Q_d^2)^2 = 25/81$



$(Q_u^3 + Q_d^3)(Q_u + Q_d) = 9/81$



$(Q_u^2 + Q_d^2)(Q_u + Q_d)^2 = 5/81$

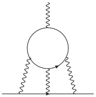


$(Q_u + Q_d)^4 = 1/81$

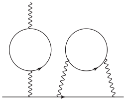
Further insight for magnitude of individual topologies can be gained by studying long-distance behavior of QCD correlation functions (Bijnens, RBC, ...)

7 quark-level topologies of direct lattice calculation

Hierarchy imposed by QED charges of dominant up- and down-quark contribution

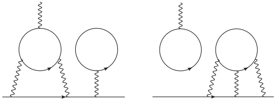


$Q_u^4 + Q_d^4 = 17/81$

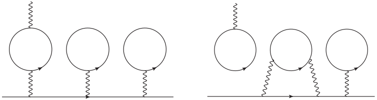


$(Q_u^2 + Q_d^2)^2 = 25/81$

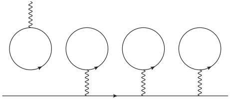
Dominant diagrams in top row: connected and leading disconnected diagram



$(Q_u^3 + Q_d^3)(Q_u + Q_d) = 9/81$



$(Q_u^2 + Q_d^2)(Q_u + Q_d)^2 = 5/81$

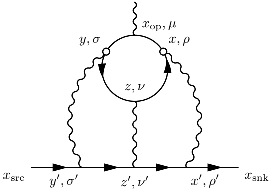


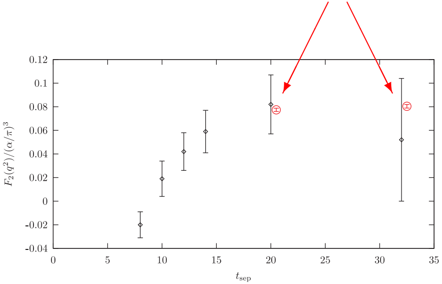
$(Q_u + Q_d)^4 = 1/81$

Further insight for magnitude of individual topologies can be gained by studying long-distance behavior of QCD correlation functions (Bijnens, RBC, ...)

PRD93(2015)014503 (Blum, Christ, Hayakawa, Izubuchi, Jin, and CL):

New sampling strategy with 10x reduced noise for same cost (red versus black):





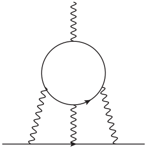
Stochastically evaluate the sum over vertices x and y:

- Pick random point x on lattice
- Sample all points y up to a specific distance $r = |x - y|$
- Pick y following a distribution $P(|x - y|)$ that is peaked at short distances

PRL118(2016)022005 (Blum, Christ, Hayakawa, Izubuchi, Jin, Jung, and CL):

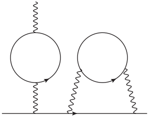
- Calculation at physical pion mass with finite-volume QED prescription ($\mathrm{QED_L}$) at single lattice cutoff of $a^{-1} = 1.73$ GeV and lattice size $L = 5.5$ fm.

- Connected diagram:



$$a_\mu^{\mathrm{cHLbL}} = 11.6(0.96) \times 10^{-10}$$

- Leading disconnected diagram:



$$a_\mu^{\mathrm{dHLbL}} = -6.25(0.80) \times 10^{-10}$$

- Large cancellation expected from pion-pole-dominance considerations is realized: $a_\mu^{\mathrm{HLbL}} = a_\mu^{\mathrm{cHLbL}} + a_\mu^{\mathrm{dHLbL}} = 5.35(1.35) \times 10^{-10}$

Potentially large systematics due to finite-volume QED!

First lattice HLbL calculation with controlled systematics: arXiv:1911.08123

The hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD

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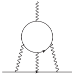
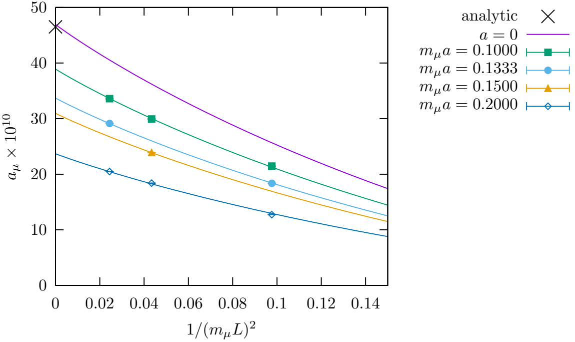
We report the first result for the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment with all errors systematically controlled. Several ensembles using 2+1 flavors of physical mass Möbius domain-wall fermions, generated by the RBC/UKQCD collaborations, are employed to take the continuum and infinite volume limits of finite volume lattice QED+QCD. We find $a_\mu^{\mathrm{HLbL}} = 7.20(3.98)_{\mathrm{stat}}(1.65)_{\mathrm{sys}} \times 10^{-10}$. Our value is consistent with previous model results and leaves little room for this notoriously difficult hadronic contribution to explain the difference between the Standard Model and the BNL experiment.

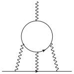
Lattice QCD ensembles at physical pion mass:

	48I	64I	24D	32D	48D	32Dfine
a^{-1} (GeV)	1.730	2.359	1.015	1.015	1.015	1.378
a (fm)	0.114	0.084	0.194	0.194	0.194	0.143
L (fm)	5.47	5.38	4.67	6.22	9.33	4.58
L_s	48	64	24	24	24	32
m_π (MeV)	139	135	142	142	142	144
m_μ (MeV)	106	106	106	106	106	106
# meas con	65	43	157	70	8	55
# meas discon	104	44	156	69	0	55

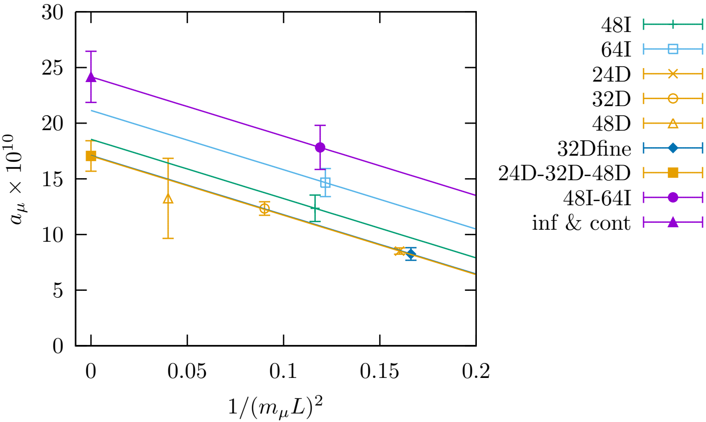
QED test (replace quark loop by lepton loop): 

$$a_\mu(L,a) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_3}{(m_\mu L)^3}\Big)$$
$$\times\Big(1 - c_1(m_\mu a)^2 + c_2(m_\mu a)^4\Big)$$

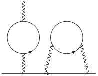


Connected diagram (QCD+QED): 

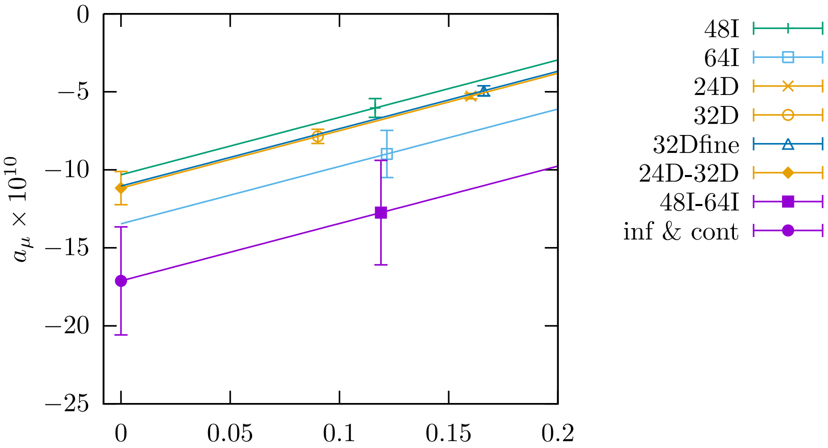
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2}$$
$$- c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

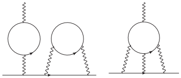


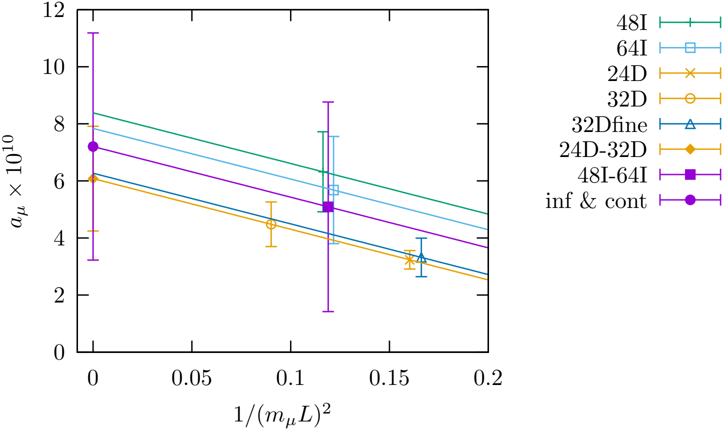
Hybrid method: Already used for 2018 HVP, for very noisy long-distance contribution in connected diagram fit to constant instead of $c_0 + a^2 c_1$ for continuum limit

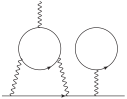
Leading disconnected diagram (QCD+QED): 

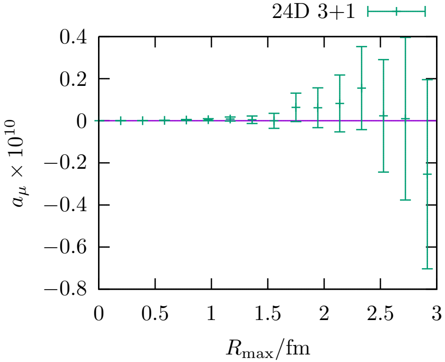
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$



Connected plus leading disconnected (QCD+QED): 



Subleading diagrams: 



Systematic errors estimated by difference of a_μ result from

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

to

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_2}{(m_\mu L)^3} - c_1^{\mathrm{I}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

for $O(1/L^3)$ and the maximum difference to either

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2(a\ \mathrm{GeV})^4\Big)$$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1(a\ \mathrm{GeV})^2 + c_2^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^4 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

for $O(a^4)$.

Similarly the difference from

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} \\ -c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

to

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} \\ -\Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big) \\ \times\Big(1 - \frac{\alpha_S}{\pi}\log\big((a\ \mathrm{GeV})^2\big)\Big)\Big)$$

for $O(a^2 \log a^2)$ and the maximum difference to either

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Bigg(1 - \frac{b_2}{(m_\mu L)^2} \\ -\Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big) \\ \times\Big(1 - \frac{1}{m_\mu L}\Big)\Bigg)$$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2}\Big) \\ \times\Big(1 - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 \\ +c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

form $O(a^2/L)$ (non-locality of $\mathrm{QED_L}$ could increase these).

Summary of results:

	con	discon	tot
a_μ	24.16(2.30)	-17.12(3.46)	7.20(3.98)
sys hybrid $\mathcal{O}(a^2)$	0.20(0.45)	0	0.20(0.45)
sys $\mathcal{O}(1/L^3)$	2.34(0.41)	1.72(0.32)	0.83(0.56)
sys $\mathcal{O}(a^4)$	0.93(0.32)	0.83(0.46)	1.07(0.97)
sys $\mathcal{O}(a^2 \log(a^2))$	0.23(0.08)	0.05(0.16)	0.05(0.16)
sys $\mathcal{O}(a^2/L)$	4.43(1.38)	3.93(2.30)	0.72(2.06)
sys strange con	0.30	0	0.30
sys sub-discon	0	0.50	0.50
sys all	5.12(1.32)	4.41(2.15)	1.65(1.13)

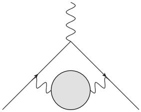
$$\boxed{a_\mu^{\mathrm{tot}} = 7.20(3.98)_{\mathrm{stat}}(1.65)_{\mathrm{sys}} \times 10^{-10}}$$

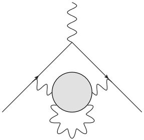
Next steps in first-principles calculation of HLbL

- Further reduce statistical and finite-volume errors
- Take infinite-volume limit also with finite-volume QCD+infinite-volume QED mixed approach PRD96(2017)034515 (Blum, Christ, Hayakawa, Izubuchi, Jin, Jung, and CL)
- We anticipate most of the running for these updates to be completed by the end of the first quarter of 2020

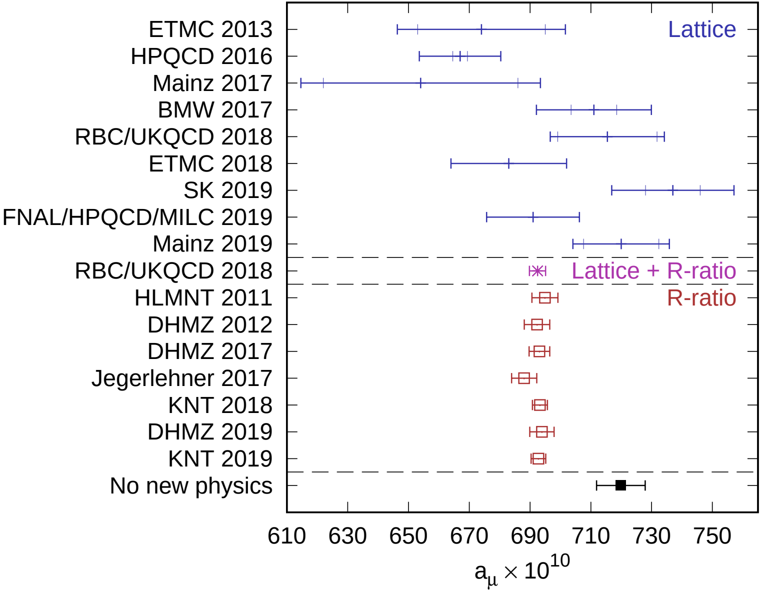
Continued effort using these methods to reduce HLbL uncertainty over next years to $\delta a_\mu^{\mathrm{HLbL}} \sim 1 \times 10^{-10}$, below Fermilab E989 uncertainty

HVP contribution





Status of HVP determinations



RBC/UKQCD status 2018

PHYSICAL REVIEW LETTERS **121,** 022003 (2018)

Editors' Suggestion

Calculation of the Hadronic Vacuum Polarization Contribution to the Muon Anomalous Magnetic Moment

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We present a first-principles lattice QCD + QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_\mu^{\mathrm{HVP\ LO}} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with R-ratio data, we significantly improve the precision to $a_\mu^{\mathrm{HVP\ LO}} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_\mu^{\mathrm{HVP\ LO}}$.

Pure lattice result and dispersive result with reduced $\pi\pi$ dependence (window method)

Aaron Meyer (BNL) & Mattia Bruno (BNL $\rightarrow$ CERN) joined since this 2018 paper

Lattice QCD – Time-Moment Representation

Starting from the vector current $J_\mu(x) = i \sum_f Q_f \overline{\Psi}_f(x)\gamma_\mu \Psi_f(x)$ we may write

$$a_\mu^{\mathrm{HVP\ LO}} = \sum_{t=0}^{\infty} w_t C(t)$$

with

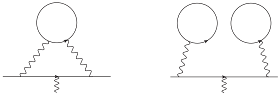
$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle$$

and w_t capturing the photon and muon part of the HVP diagrams (Bernecker-Meyer 2011).

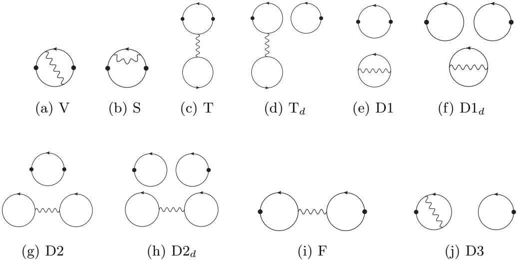
The correlator $C(t)$ is computed in lattice QCD+QED at physical pion mass with non-degenerate up and down quark masses including up, down, strange, and charm quark contributions. The missing bottom quark contributions are computed in pQCD.

Diagrams

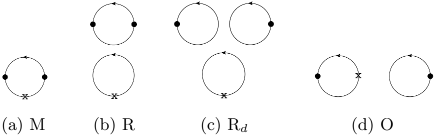
Isospin limit



QED corrections



Strong isospin breaking



Window method (implemented in RBC/UKQCD 2018)

We therefore also consider a window method. Following Meyer-Bernecker 2011 and smearing over t to define the continuum limit we write

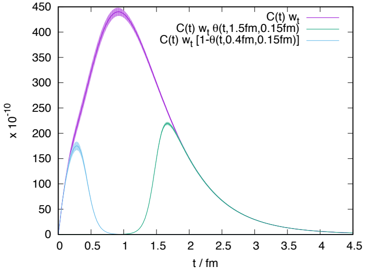
$$a_\mu = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}}$$

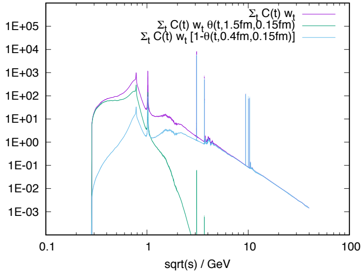
with

$$\begin{aligned} a_\mu^{\mathrm{SD}} &= \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)] \,, \\ a_\mu^{\mathrm{W}} &= \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)] \,, \\ a_\mu^{\mathrm{LD}} &= \sum_t C(t) w_t \Theta(t, t_1, \Delta) \,, \\ \Theta(t, t', \Delta) &= [1 + \tanh [(t - t')/\Delta]] /2 \,. \end{aligned}$$

In this version of the calculation, we use $C(t) = \frac{1}{12\pi^2} \int_0^\infty d(\sqrt{s}) R(s) s e^{-\sqrt{s}t}$ with $R(s) = \frac{3s}{4\pi\alpha^2} \sigma(s, e^+e^- \to \mathrm{had})$ to compute a_μ^{SD} and a_μ^{LD}.

How does this translate to the time-like region?





Most of $\pi\pi$ peak is captured by window from $t_0 = 0.4$ fm to $t_1 = 1.5$ fm, so replacing this region with lattice data reduces the dependence on BaBar versus KLOE data sets.

The pure lattice calculation of RBC/UKQCD 2018:

$$
\begin{aligned}
10^{10} \times a_\mu^{\mathrm{HVP\ LO}} &= 715.4(18.7) \\
&= 715.4(16.3)_{\mathrm{S}}(7.8)_{\mathrm{V}}(3.0)_{\mathrm{C}}(1.9)_{\mathrm{A}}(3.2)_{\mathrm{other}}
\end{aligned}
$$

(S) statistics, (V) finite-volume errors, (C) the continuum limit extrapolation, (A) scale setting uncertainty;

other $\supset$ neglected diagrams for QED and SIB, estimate of bottom quark contribution

Statistical noise mostly from isospin symmetric light quark connected (14.2) and disconnected (3.3), QED (5.7), SIB (4.3)

RBC/UKQCD 2019 update (in preparation):

- Improved methodology
- A lot of new data

Improved methodology

Improved statistics and systematics – Bounding Method

BMW/RBC/UKQCD 2016

The correlator in finite volume

$$C(t) = \sum_n |\langle 0|V|n\rangle|^2 e^{-E_n t} .$$

We can bound this correlator at each t from above and below by the correlators

$$\tilde{C}(t; T, \tilde{E}) = \begin{cases} C(t) & t < T , \\ C(T)e^{-(t-T)\tilde{E}} & t \geq T \end{cases}$$

for proper choice of $\tilde{E}$. We can chose $\tilde{E} = E_0$ (assuming $E_0 < E_1 < \ldots$) to create a strict upper bound and any $\tilde{E}$ larger than the local effective mass to define a strict lower bound.

Therefore if we had precise knowledge of the lowest $n = 0, \ldots, N$ values of $|\langle 0|V|n\rangle|$ and E_n, we could define a new correlator

$$C^N(t) = C(t) - \sum_{n=0}^{N} |\langle 0|V|n\rangle|^2 e^{-E_n t}$$

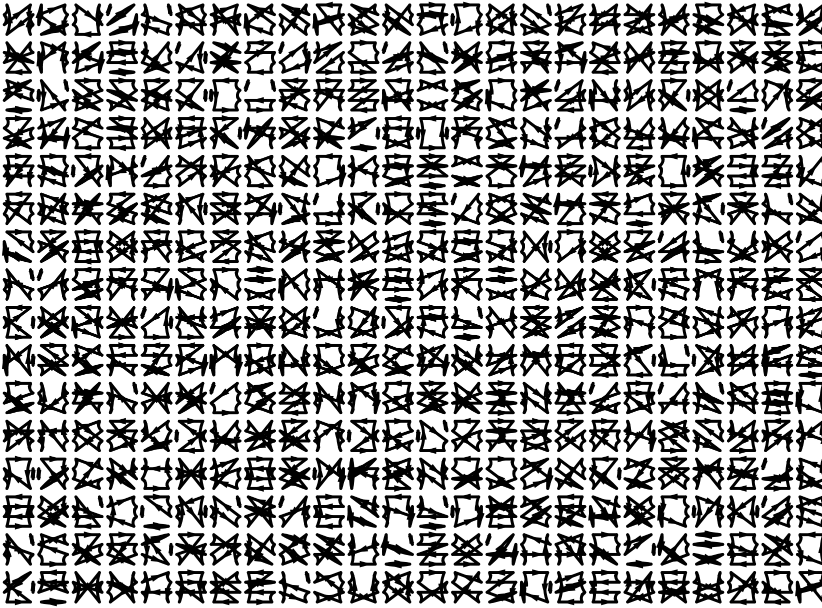
which we could bound much more strongly through the larger lowest energy $E_{N+1} \gg E_0$. New method: do a GEVP study of FV spectrum to perform this subtraction.

GEVP operators ($I = 1$, $I_3 = 0$, $p_{\rm tot} = \vec{0}$, T_1^-): 2pi (1,2,3,4 units of momentum), 4pi (two different), local and smeared vector currents

- Two data sets: n-pi operators made out of improved ($\gamma_5, \gamma_5\gamma_t, \gamma_5\gamma_i e^{i\vec{p}\vec{x}}$) or unimproved ($\gamma_5$) pion operators
- Operators automatically generated for given representation: `https://github.com/asmeyer2012/wickop`
- Automatically contract operators to diagrams: `https://github.com/lehner/Wick`
- Automatically evaluate diagrams using A2A/Distillation data: `https://github.com/lehner/Contractor`

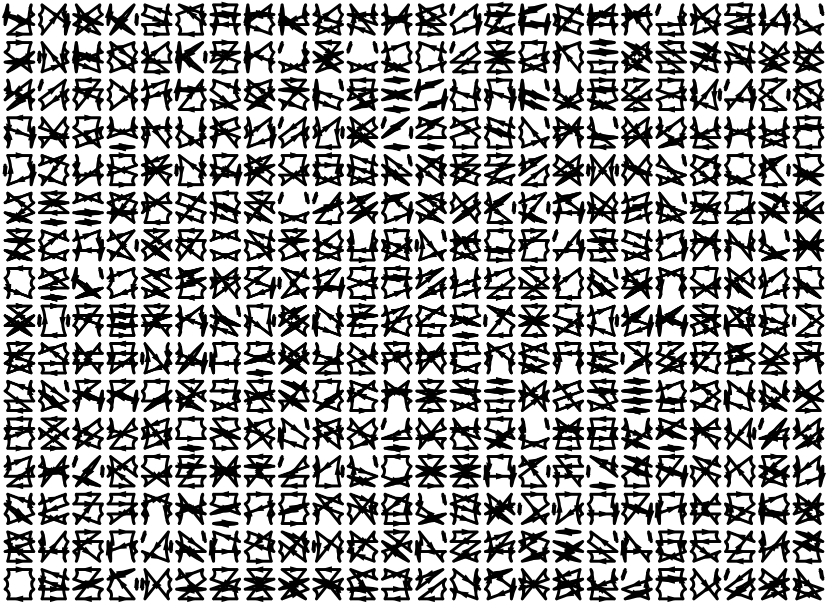
974 contractions for 4pi-4pi:

1/3:



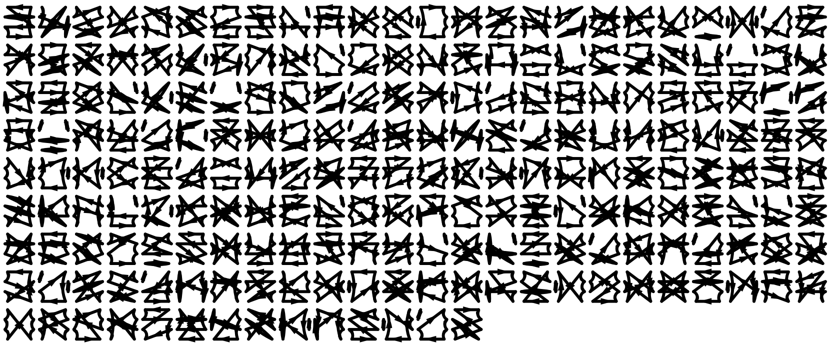
974 contractions for 4pi-4pi:

2/3:

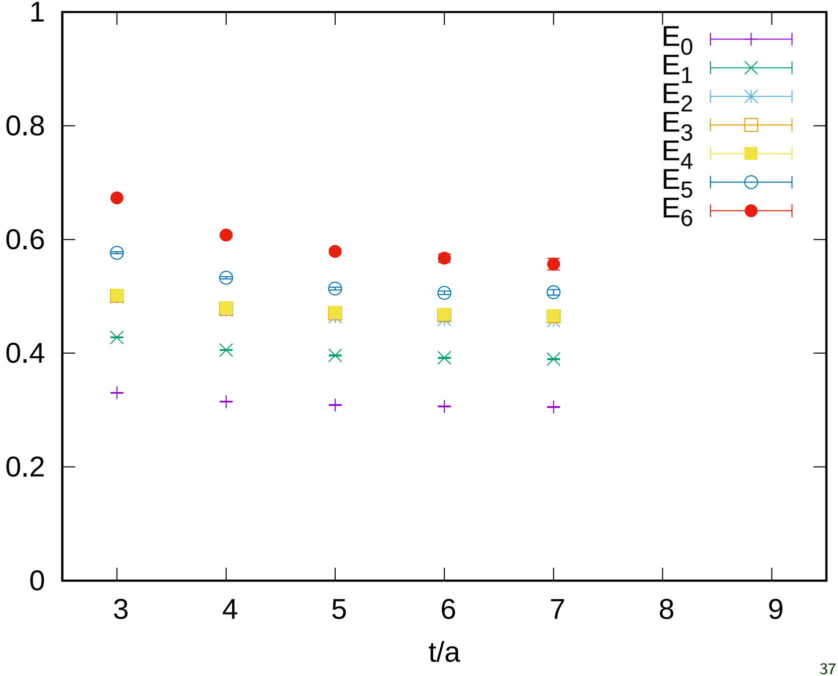


974 contractions for 4pi-4pi:

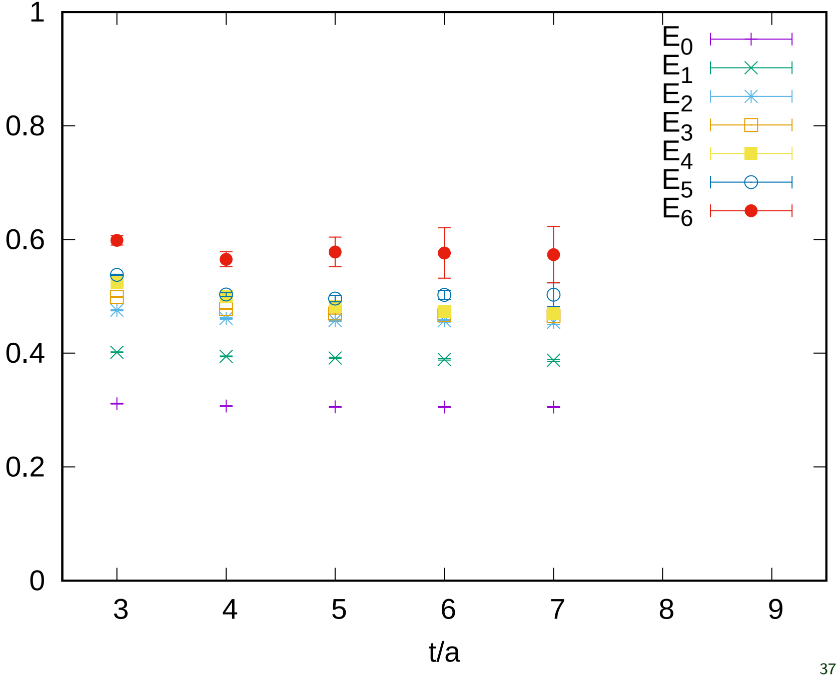
3/3:



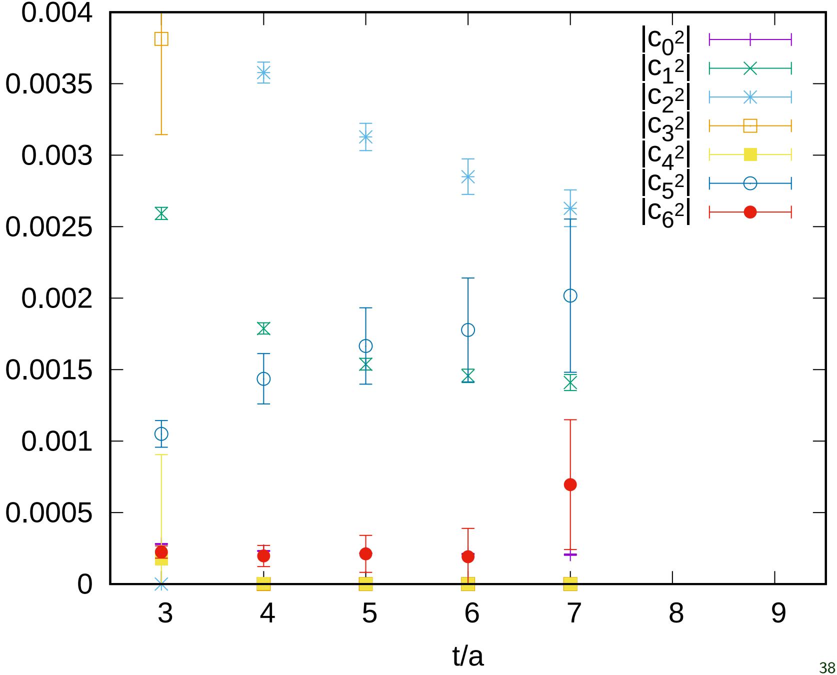
Unimproved energies, $a^{-1} = 1.73$ GeV



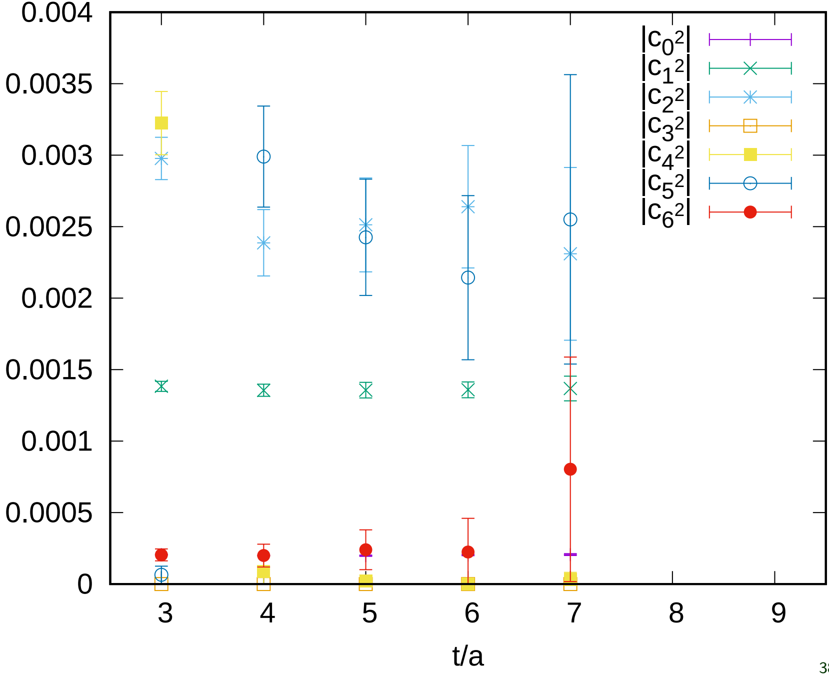
Improved energies, $a^{-1} = 1.73$ GeV

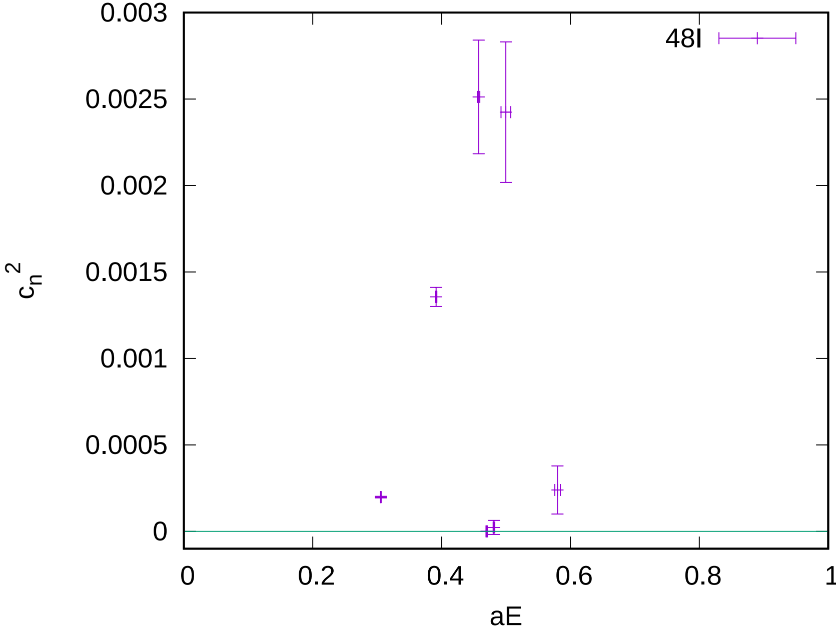


Unimproved $c_n^2 = |\langle 0|V|n\rangle|^2$

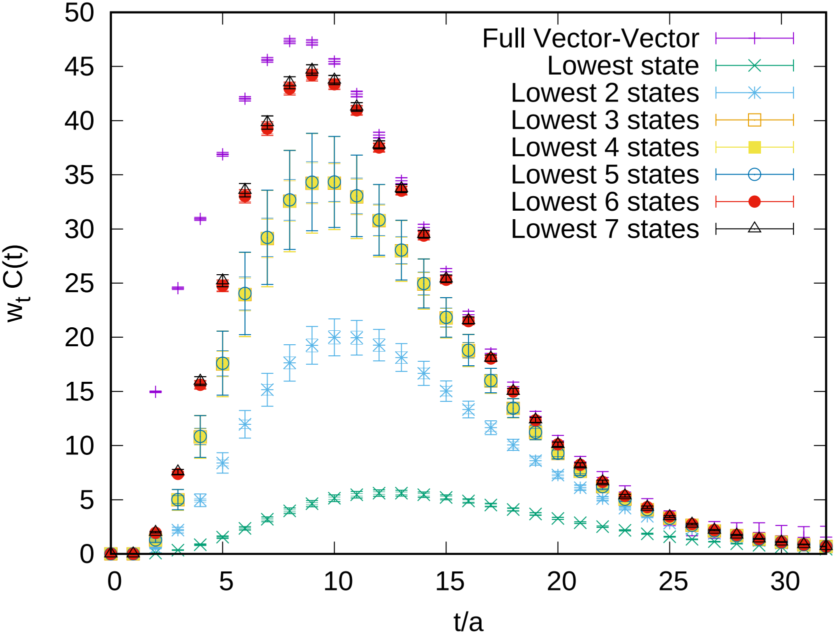


Improved $c_n^2 = |\langle 0|V|n\rangle|^2$

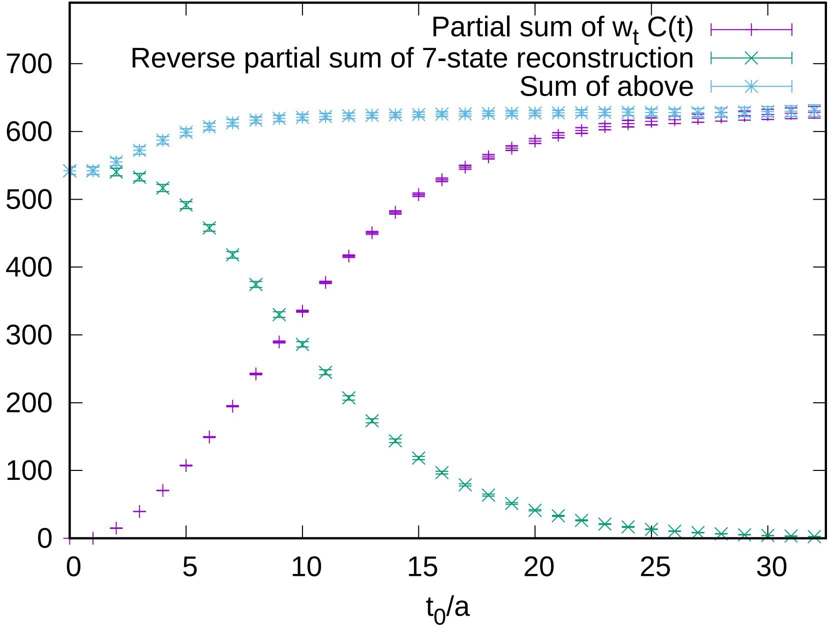




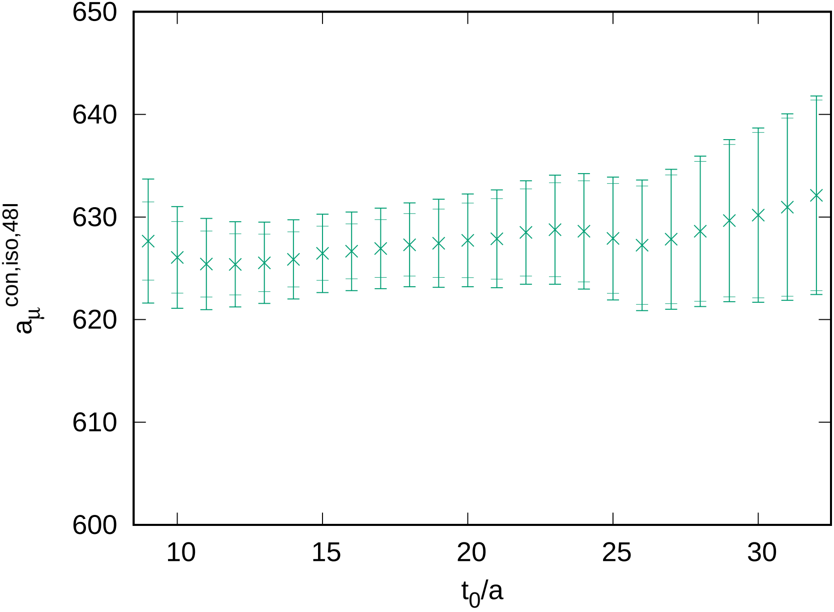
$a^{-1} = 1.73$ GeV



Full Vector-Vector
Lowest state
Lowest 2 states
Lowest 3 states
Lowest 4 states
Lowest 5 states
Lowest 6 states
Lowest 7 states
$w_t\, C(t)$
t/a
0
5
10
15
20
25
30
35
40
45
50



Partial sum of w_t C(t)
Reverse partial sum of 7-state reconstruction
Sum of above
0
100
200
300
400
500
600
700
0
5
10
15
20
25
30
t_0/a



$a_\mu^{ud,conn,iso,48I} 10^{10} = 626.5(2.6)_S(0.4)_{Z_V}(2.6)_{a^{-1}}(0.5)_{\mathrm{Bnd}}(0.5)_{\mathrm{Exc.st.}}$

Compare to previous statistical error for light-connected-iso: 14.2

Improved systematics – compute finite-volume effects from first-principles

RBC/UKQCD study of QCD at **physical pion mass** at three different volumes:

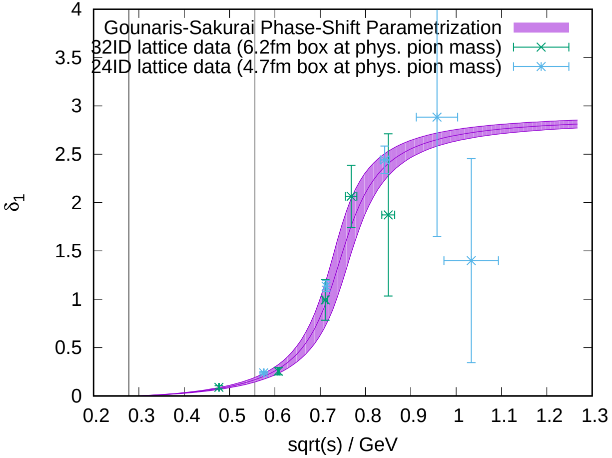
$\boxed{L = 4.66 \text{ fm},\ L = 5.47 \text{ fm},\ L = 6.22 \text{ fm}}$

Results for light-quark isospin-symmetric connected contribution:

- $a_\mu(L = 6.22 \text{ fm}) - a_\mu(L = 4.66 \text{ fm}) = 12.2 \times 10^{-10}$ (sQED), $21.6(6.3) \times 10^{-10}$ (lattice QCD)

- Need to do better than sQED in finite-volume

First constrain the p-wave phase shift from our $L = 6.22$ fm physical pion mass lattice:



$E_\rho = 0.766(21)$ GeV (PDG 0.77549(34) GeV)
$\Gamma_\rho = 0.139(18)$ GeV (PDG 0.1462(7) GeV)

GSL^2 finite-volume results compared to sQED and lattice

GSL^2 method of Meyer 2012

Results for light-quark isospin-symmetric connected contribution:

- FV difference between $a_\mu(L = 6.22 \text{ fm}) - a_\mu(L = 4.66 \text{ fm}) = 12.2 \times 10^{-10}$ (sQED), $21.6(6.3) \times 10^{-10}$ (lattice QCD), $20(3) \times 10^{-10}$ (GSL^2)
- GSL^2 prediction agrees with actual FV effect measured on the lattice, sQED is in slight tension, two-loop FV ChPT to be compared next Bijnens and Relefors 2017
- Use GSL^2 to update FV correction of Phys. Rev. Lett. 121, 022003 (2018): $a_\mu(L \to \infty) - a_\mu(L = 5.47 \text{ fm}) = 16(4) \times 10^{-10}$ (sQED), $22(1) \times 10^{-10}$ (GSL^2); sQED error estimate based on Bijnens and Relefors 2017, table 1.
- Compare also to Hansen-Patella 2019 1904.10010: $a_\mu(L \to \infty) - a_\mu(L = 5.47 \text{ fm}) \approx 14 \times 10^{-10}$, effect of neglected $e^{-\sqrt{2} m_\pi L}$ significant; currently computed

Other improvements (I) - SIB:

- Calculate both mass derivative with insertion of scalar operators as well as fit to valence pion mass dependence

Other improvements (II) - Disconnected diagrams:

- Study volume-dependence and continuum limit, much more statistics; build "tadpole" fields following our PRL116(2016)232002

Other improvements (III) - QED continuum limit and scale:

- QED analysis of 2018 paper now with continuum limit, new Z_V determination for consistency check (vector charge of pion), improved Ω mass analysis

Other improvements (IV) - QED from HLbL data:

- HLBL point-source data from HLbL work presented above
- HVP QED from re-analysis of HLbL point-source data (see also τ project and Mattia's talk last week, 1811.00508) reduces statistical noise by $\approx 10\times$ for V and S
- Infinte-volume and continuum limit also for diagram V, S, and F
- First results for T, D1, and R; other sub-leading in preparation

New data set

Ensembles at physical pion mass:

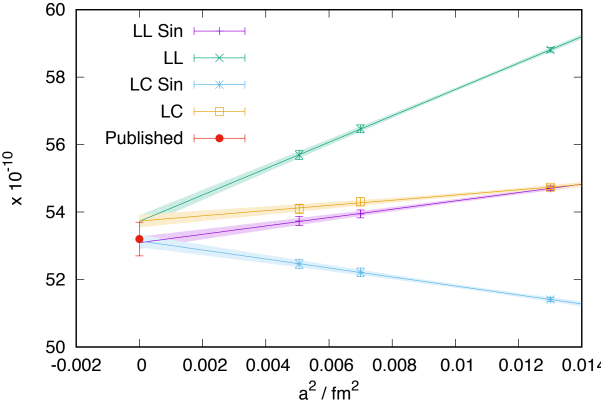
48I (1.73 GeV, 5.5fm), 64I (2.359 GeV, 5.4fm), 24ID (1 GeV, 4.7fm), 32ID (1 GeV, 6.2fm), 48ID (1 GeV, 9.3fm), 32IDf (1.37 GeV, 4.6fm)

RBC/UKQCD 2019 (data for light quarks, changes from 2018):

- A2A data for connected isospin symmetric: 48I (127 conf $\rightarrow$ 400 conf), 64I (160 conf $\rightarrow$ 250 conf), 24ID (new 130 conf, multi mass), 32ID (new 88 conf, multi mass)
- A2A data (tadpole fields) for disconnected: 48I (33 conf), 24ID (new 260 conf, multi mass), 32IDf (new 103 conf)
- QED and SIB corrections to meson and Ω masses, Z_V: 48I (30 conf) and 64I (new 30 conf)
- QED and SIB from HLbL point sources on 48I, 24ID, 32ID, 32IDf (on order of 20 conf each, 2000 points per config)
- Distillation data on 48I (33 conf), 64I (in progr.), 24ID (33 conf), 32ID (11 conf, multi-mass)
- New Ω mass operators (excited states control): 48I (130 conf)

Add $a^{-1} = 2.77$ GeV lattice spacing

- Third lattice spacing for strange data ($a^{-1} = 2.77$ GeV with $m_\pi = 234$ MeV with sea light-quark mass corrected from global fit):



- For light quark need new ensemble at physical pion mass. Started run on Summit Machine at Oak Ridge this year ($a^{-1} = 2.77$ GeV with $m_\pi = 139$ MeV).

Data is generated, now analysis/cross-checks (will have at least two independent analyses for each component)

Components:

- Light-quark conn. isospin symmetric
- QED update of diagrams V,S,F
- QED update of Z_V, $m_\pi^{+,0}$, $m_K^{+,0}$, m_Ω correction
- FV corrections (GSL^2)
- Third lattice spacing for strange (and light)
- Disconnected
- SIB connected+disconnected
- Additional systematic error estimates

Conclusions and Outlook (HVP)

- We have all ingredients to make a first-principles lattice computation of HVP with error $O(5 \times 10^{-10})$
- With lattice precision improvements, window method will be able to weigh in on BaBar/KLOE
- Data for next paper is ready and analysis is progressing

Conclusions and Outlook

- We are within months of the release of the new experimental data

- Lattice QCD+QED has matured both for HVP and HLbL and can control all systematic errors

- For both, first-principles LQCD+QED calculations can reach final experimental precision of Fermilab experiment by the end of the experiment around 2022

Backup

Dispersive method - e^+e^- status

Recent results by Keshavarzi et al. 2018, Davier et al. 2017:

Channel	This work (KNT18)	DHMZ17 [78]	Difference
Data based channels ($\sqrt{s} \leq 1.8$ GeV)			
$\pi^0\gamma$ (data + ChPT)	4.58 ± 0.10	4.29 ± 0.10	0.29
$\pi^+\pi^-$ (data + ChPT)	503.74 ± 1.96	507.14 ± 2.58	-3.40
$\pi^+\pi^-\pi^0$ (data + ChPT)	47.70 ± 0.89	46.20 ± 1.45	1.50
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31
. . .			
Total	693.3 ± 2.5	693.1 ± 3.4	0.2

Good agreement for total, individual channels disagree to some degree. **Muon g-2 Theory Initiative workshops** recently held at Fermilab, KEK, UConn, and Mainz, intend to facilitate discussions and further understanding of these tensions.

One difference: treatment of correlations, impactful in particular in case when not all experimental data agrees

Gounaris-Sakurai-Lüscher method [H. Meyer 2012, Mainz 2017]

- Produce FV spectrum and matrix elements from phase-shift study (Lüscher method for spectrum and amplitudes, GS for phase-shift parametrization)
- This allows for a prediction of FV effects beyond chiral perturbation theory given that the phase-shift parametrization captures all relevant effects (can be checked against lattice data)
- This method is now being employed by ETMC, Mainz, and RBC/UKQCD.

Dispersive method - τ status

Experiment	$a_\mu^{\mathrm{had,LO}}[\pi\pi,\tau]$ (10^{-10}) $2m_{\pi^\pm} - 0.36$ GeV	$0.36 - 1.8$ GeV
ALEPH	$9.80 \pm 0.40 \pm 0.05 \pm 0.07$	$501.2 \pm 4.5 \pm 2.7 \pm 1.9$
CLEO	$9.65 \pm 0.42 \pm 0.17 \pm 0.07$	$504.5 \pm 5.4 \pm 8.8 \pm 1.9$
OPAL	$11.31 \pm 0.76 \pm 0.15 \pm 0.07$	$515.6 \pm 9.9 \pm 6.9 \pm 1.9$
Belle	$9.74 \pm 0.28 \pm 0.15 \pm 0.07$	$503.9 \pm 1.9 \pm 7.8 \pm 1.9$
Combined	$9.82 \pm 0.13 \pm 0.04 \pm 0.07$	$506.4 \pm 1.9 \pm 2.2 \pm 1.9$

Davier et al. 2013: $a_\mu^{\mathrm{had,LO}}[\pi\pi,\tau] = 516.2(3.5) \times 10^{-10}$ $(2m_\pi^\pm - 1.8$ GeV)

Compare to e^+e^-:

- $a_\mu^{\mathrm{had,LO}}[\pi\pi, e^+e^-] = 507.1(2.6) \times 10^{-10}$ (DHMZ17, $2m_\pi^\pm - 1.8$ GeV)
- $a_\mu^{\mathrm{had,LO}}[\pi\pi, e^+e^-] = 503.7(2.0) \times 10^{-10}$ (KNT18, $2m_\pi^\pm - 1.937$ GeV)

Here treatment of isospin-breaking to relate matrix elements of $V_\mu^{I=1,I_3=1}$ to $V_\mu^{I=1,I_3=0}$ crucial. Progress towards a first-principles calculation from LQCD+QED, see 1811.00508.

Regions of precision (R-ratio data here is from **Fred Jegerlehner** 2017)

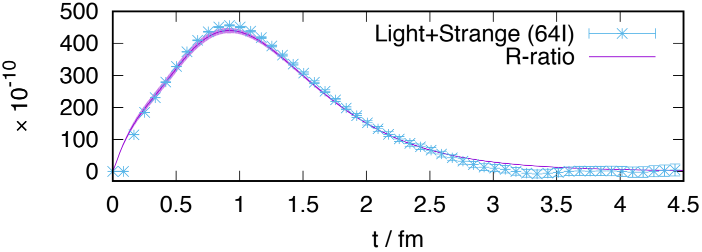


FIG. 4. Comparison of $w_t C(t)$ obtained using R-ratio data [1] and lattice data on our 64I ensemble.

The precision of lattice data deteriorates exponentially as we go to large t, however, is precise at intermediate distances. The R-ratio is very precise at long distances.

Note: in this plot a direct comparison of R-ratio and lattice data is not appropriate. Continuum limit, infinite-volume corrections, charm contributions, and IB corrections are missing from lattice data shown here.

We perform the calculation as a perturbation around an isospin-symmetric lattice QCD computation with two degenerate light quarks with mass m_{light} and a heavy quark with mass m_{heavy} tuned to produce a pion mass of 135.0 MeV and a kaon mass of 495.7 MeV.

The correlator is expanded in the fine-structure constant α as well as $\Delta m_{\mathrm{up,\ down}} = m_{\mathrm{up,\ down}} - m_{\mathrm{light}}$, and $\Delta m_{\mathrm{strange}} = m_{\mathrm{strange}} - m_{\mathrm{heavy}}$. We write

$$C(t) = C^{(0)}(t) + \alpha C^{(1)}_{\mathrm{QED}}(t) + \sum_f \Delta m_f C^{(1)}_{\Delta \mathrm{m_f}}(t) + \mathcal{O}(\alpha^2, \alpha \Delta m, \Delta m^2)\,.$$

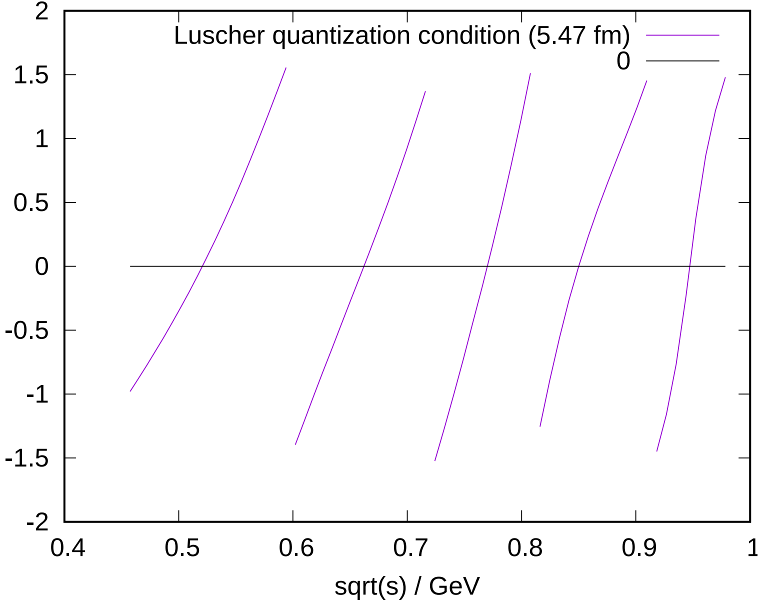
The correlators of this expansion are computed in lattice QCD with dynamical up, down, and strange quarks. We compute the missing contributions to a_μ from charm sea quarks in perturbative QCD (RHAD) by integrating the time-like region above 2 GeV and find them to be smaller than 0.3×10^{-10}.

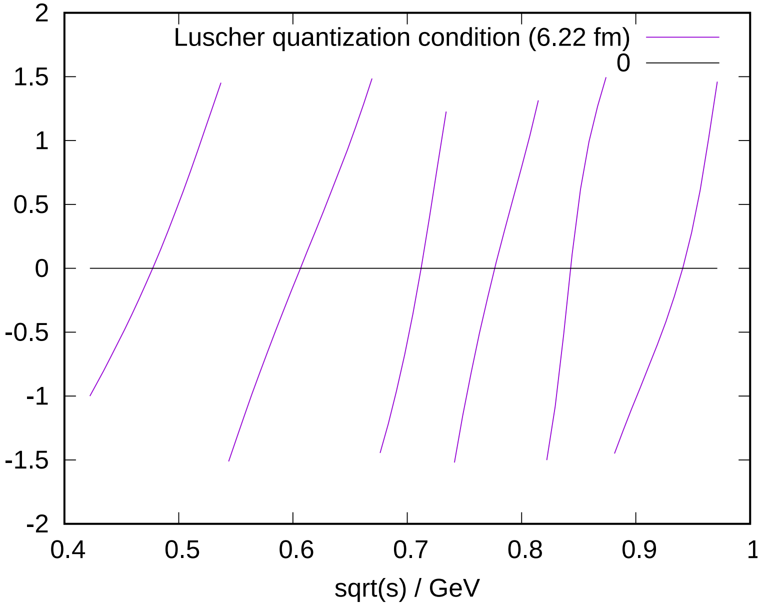
We tune the bare up, down, and strange quark masses $m_{\rm up}$, $m_{\rm down}$, and $m_{\rm strange}$ such that the π^0, π^+, K^0, and K^+ meson masses computed in our calculation agree with the respective experimental measurements. The lattice spacing is determined by setting the Ω^- mass to its experimental value.

We perform the lattice calculations for the light quark contributions using RBC/UKQCD's 48I and 64I lattice configurations with lattice cutoffs $a^{-1} = 1.730(4)$ GeV and $a^{-1} = 2.359(7)$ GeV and a larger set of ensembles with up to $a^{-1} = 2.774(10)$ GeV for the charm contribution.

From the parameter tuning procedure on the 48I we find $\Delta m_{\rm up} = -0.00050(1)$, $\Delta m_{\rm down} = 0.00050(1)$, and $\Delta m_{\rm strange} = -0.0002(2)$.

The shift of the Ω^- mass due to the QED correction is significantly smaller than the lattice spacing uncertainty and its effect on $C(t)$ is therefore not included separately.

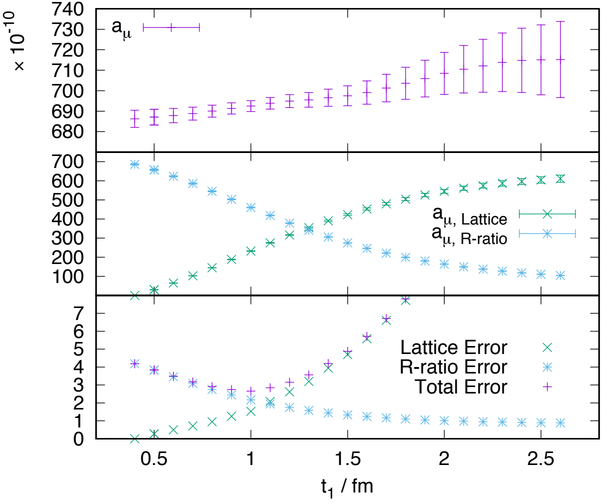

Luscher quantization condition (5.47 fm)
0
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0.4
0.5
0.6
0.7
0.8
0.9
1
sqrt(s) / GeV


Luscher quantization condition (6.22 fm)
0
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0.4
0.5
0.6
0.7
0.8
0.9
1
sqrt(s) / GeV

Consolidate continuum limit

Adding a finer lattice

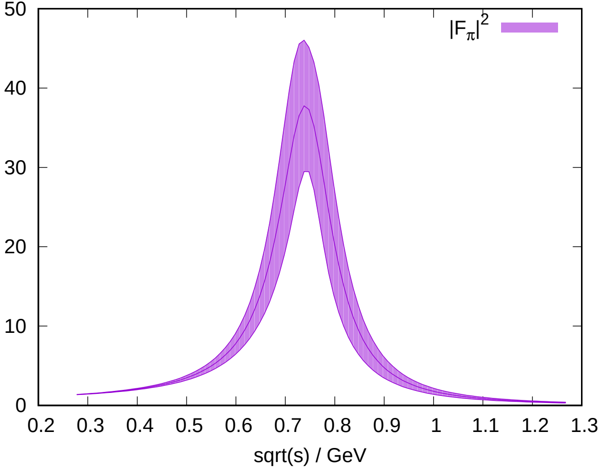
Window method with fixed $t_0 = 0.4$ fm



For $t = 1$ fm approximately 50% of uncertainty comes from lattice and 50% of uncertainty comes from the R-ratio. Is there a small slope? More in a few slides!

Can use this to check experimental data sets; see my KEK talk for more details

Predicts $|F_\pi(s)|^2$:



We can then also predict matrix elements and energies for our other lattices; successfully checked!