Gravitational coupling of QED and QCD: 3- and 4-point functions in momentum space

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Abstract

Conformal symmetry has important consequences for strong interactions at short distances and provides powerful tools for practical calculations. Even if the Lagrangians of Quantum Chromodynamics (QCD) and Electrodynamics (QED) are invariant under conformal transformations, this symmetry is broken by quantum corrections. The signature of the symmetry breaking is encoded in the presence of massless poles in correlators involving stress-energy tensors. We present a general study of the correlation functions TJJ and $TTJJ$ of conformal field theory (CFT) in the flat background limit in momentum space, following a reconstruction method for tensor correlators. Furthermore, we discuss the dimensional degeneracies of the tensor structures related to these correlators, and we present the perturbative realizations of 3- and 4-point functions in momentum space for QED and QCD.

TJJ Reconstruction

Any correlation function can be decomposed into a **transverse traceless** part and a **longitudinal** one in momentum space. For instance, the decomposition of the TJJ correlator is given as

$$\begin{aligned}\langle T^{\mu_1\nu_1}\,J^{\mu_2}\,J^{\mu_3}\rangle &= \langle t^{\mu_1\nu_1}\,j^{\mu_2}\,j^{\mu_3}\rangle + \langle T^{\mu_1\nu_1}\,J^{\mu_2}\,j^{\mu_3}_{loc}\rangle + \langle T^{\mu_1\nu_1}\,j^{\mu_2}_{loc}\,J^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}_{loc}\,J^{\mu_2}\,J^{\mu_3}\rangle\\ &\quad - \langle T^{\mu_1\nu_1}\,j^{\mu_2}_{loc}\,j^{\mu_3}_{loc}\rangle - \langle t^{\mu_1\nu_1}_{loc}\,j^{\mu_2}_{loc}\,J^{\mu_3}\rangle - \langle t^{\mu_1\nu_1}_{loc}\,J^{\mu_2}\,j^{\mu_3}_{loc}\rangle + \langle t^{\mu_1\nu_1}_{loc}\,j^{\mu_2}_{loc}\,j^{\mu_3}_{loc}\rangle\,,\end{aligned}$$

where for the operator J we define

$$j^\mu(p) = \pi^\mu_\alpha(p)\,J^\alpha(p), \qquad j^\mu_{loc}(p) = \frac{p^\mu p_\alpha}{p^2}J^\alpha(p), \qquad \pi^\mu_\alpha = \delta^\mu_\alpha - \frac{p^\mu p_\alpha}{p^2},$$

and for the operator T we have

$$t^{\mu\nu}(p) = \Pi^{\mu\nu}_{\alpha\beta}(p)\,T^{\alpha\beta}(p), \qquad \Pi^{\mu\nu}_{\alpha\beta} = \pi^{(\mu}_\alpha\pi^{\nu)}_\beta - \frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta},$$

$$t^{\mu\nu}_{loc}(p) = \left(I^{\mu\nu}_{\alpha\beta} + \frac{1}{d-1}\pi^{\mu\nu}\delta_{\alpha\beta}\right)T^{\alpha\beta}(p), \qquad I^{\mu\nu}_{\alpha\beta} = \frac{p_\beta}{p^2}\left[2p^{(\mu}\delta^{\nu)}_\alpha - \frac{p_\alpha}{d-1}\left(\delta^{\mu\nu} + (d-2)\frac{p^\mu p^\nu}{p^2}\right)\right].$$

From this decomposition, the longitudinal part is completely fixed using **canonical Ward Identities** and only depends on lower correlation functions. The symmetries and properties of the projectors allow writing the transverse traceless part as a combination of **independent form factors** times **independent tensorial structures** constructed by using the momenta and the metric deltas as

$$\begin{aligned}\langle t^{\mu_1\nu_1}(p_1)j^{\mu_2\,a_2}(p_2)j^{\mu_3\,a_3}(p_3)\rangle &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\,\pi^{\mu_2}_{\alpha_2}(p_2)\,\pi^{\mu_3}_{\alpha_3}(p_3)\,\big(A_1^{a_2\,a_3}\,p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_1^{\alpha_3} + A_2^{a_2\,a_3}\,\delta^{\alpha_2\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}\\ &\quad + A_3^{a_2\,a_3}\,\delta^{\alpha_1\alpha_2}p_2^{\beta_1}p_1^{\alpha_3} + A_3^{a_2\,a_3}(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}p_2^{\beta_1}p_3^{\alpha_2} + A_4^{a_2\,a_3}\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\big)\end{aligned}$$

In the general case, the form factors $A_i^{a_2a_3}$ are fixed by the conformal invariance and in particular, by the Conformal Ward Identities

$$K^\kappa\,\langle T^{\mu_1\nu_1}\,J^{\mu_2}\,J^{\mu_3}\rangle = 0, \qquad D\,\langle T^{\mu_1\nu_1}\,J^{\mu_2}\,J^{\mu_3}\rangle = 0,$$

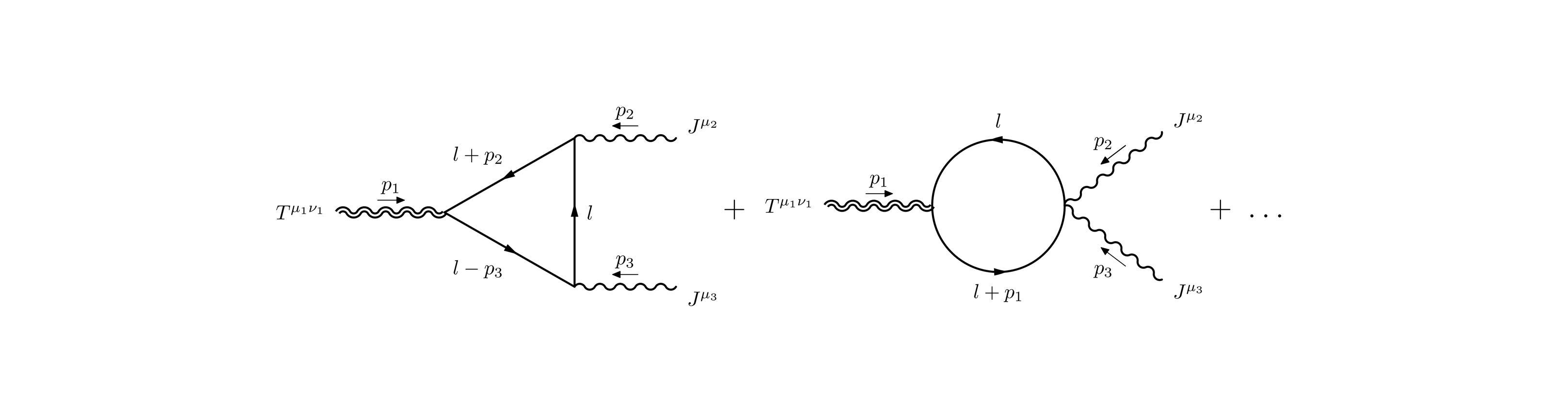
where D and K^κ are the dilatation and the special conformal operators respectively.

TJJ Perturbative Realization

The **conformal perturbative realization** for $\langle TJJ\rangle$ is contructed by using the actions

$$\begin{aligned}S_{1/2} &= \int d^dx\,V\,\left(\frac{i}{2}(\bar\psi\,\gamma^\lambda\,\partial_\lambda\psi \,-\, \partial_\gamma\bar\psi\,\gamma^\lambda\,\psi) \,-\, e\,\bar\psi\,\gamma^\lambda A_\lambda\,\psi \,-\, \frac{i}{4}\omega_{\mu ab}\,V^\mu_c\,\bar\psi\gamma^{abc}\psi\right)\\ S_0 &= \int d^dx\,\sqrt{-g}\,\left(\partial^\mu\phi^\dagger\,\partial_\mu\phi + ie\,A^\mu\,(\partial_\mu\phi^\dagger\,\phi - \phi^\dagger\,\partial_\mu\phi) \,+\, e^2A^\mu A_\mu\,\phi^\dagger\phi \,+\, \chi R\,\phi^\dagger\phi\right)\end{aligned}$$

and by computing the diagrams



From the Feynman diagrams, the form factors in the decomposition can be written explicitly in terms of master integrals B_0 and C_0. In $d=4$, this correlator manifests UV **divergences** that can be renormalized by adding the **gauge invariant counterterm**

$$S_{ct} = -\frac{c}{\epsilon}\int d^dx\,F_{\mu\nu}F^{\mu\nu}\,.$$

The renormalization procedure of the correlator induces a breaking of the conformal invariance, manifested in the presence of an **anomaly pole**. The correlator, which was classically conformal invariant, acquires, after renormalization, a trace anomaly contribution. The effective action related to the anomaly contribution can be written as

$$S_{pole} = \frac{e^2}{36\pi^2}\int d^4x\,d^4y\,\left(\Box h^\lambda_\lambda(x) - \partial_\mu\partial_\nu h^{\mu\nu}(x)\right)\Box^{-1}_{xy}\,F_{\alpha\beta}(x)F^{\alpha\beta}(y).$$

Dimensional Dependent Degeneracies

The decomposition for the $\langle TJJ\rangle$ is valid in general dimension d, and in particular for $d\geq 4$. In $d<4$, some tensor structures are redundant since **dimensional dependent tensorial relations** are to be considered. Such relations can be found by using **Lovelock's double antisymmetrization method** [3,4]. It is a rather well-known fact that given a (k,l) rank tensor S, the relation

$$S_{[\alpha_1\ldots\alpha_k}{}^{\beta_1\ldots\beta_l}\,\delta^{\beta_{l+1}}_{\alpha_{k+1}}\cdots\delta^{\beta_{l+m}}_{\alpha_{k+m}]} = 0 \qquad \text{for}\ \ k+m>d\,,$$

is trivially satisfied. However, the contraction of this identity with metric tensors can produce a new type of identity, expressed in terms of **anti-symmetric traceless (k,l) tensors**

$$T_{[\alpha_1\ldots\alpha_k}{}^{[\beta_1\ldots\beta_l}\,\delta^{\beta_{l+1}}_{\alpha_{k+1}}\cdots\delta^{\beta_{l+m}]}_{\alpha_{k+m}]} = 0 \qquad \text{for}\ \ m\geq d+1-(k+l)\,.$$

As an example for the $\langle TJJ\rangle$, we consider the tensor

$$R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = p_{1[\alpha_1}p_{2\alpha_2]}\,p_1{}^{[\beta_1}p_2{}^{\beta_2]}\,.$$

The antisymmetrization of this tensor with two metric deltas defines the traceless tensor W

$$W_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = R_{\alpha_3\alpha_4}{}^{[\alpha_3\alpha_4}\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2]}_{\alpha_2}$$

that has the same properties of the Weyl tensor. In $d\leq 3$ this tensor vanishes identically, i.e.

$$W_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = 0, \quad d\leq 3,$$

giving a constraint on some tensor structures.

The possible values of (k,l) depending on the specific dimensions that produce different identities on tensor structures in $d=3$ and $d=4$ are listed below.

$d=3$			
k	l	m	rank
2 1	0 1	≥ 2	6
3 2	0 1	≥ 1	5
4 3 2	0 1 2	≥ 0	4

$d=4$							
k	l	m	rank	k	l	m	rank
0	0	≥ 5	10	4 3 2	0 1 2	≥ 1	6
1	0	≥ 4	9				
2 1	0 1	≥ 3	8	5 4 3	0 1 2	≥ 0	5
3 2	0 1	≥ 2	7				

Since the three point function $\langle TJJ\rangle$ is a rank four tensor, the only identities that could affect the decomposition are related to rank four tensors, namely those obtained from $(4,0)$, $(3,1)$ and $(2,2)$ tensors. Taking into account such tensor degeneracies **the number of form factors in $\mathrm{d}=3$ for the $\langle\mathrm{TJJ}\rangle$ reduces to three**. In particular, the following is the identity that causes the reduction in the number of independent form factors in $d=3$ for $\langle TJJ\rangle$.

$$\begin{aligned}\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1} &= -\frac{p_1^2+p_1\cdot p_2}{p_1^2p_2^2-(p_1\cdot p_2)^2}\,\delta^{\alpha_1\alpha_2}p_1^{\mu_3}p_2^{\nu_1} - \frac{p_1^2}{p_1^2p_2^2-(p_1\cdot p_2)^2}\,p_2^{\alpha_1}p_2^{\beta_1}\delta^{\alpha_2\alpha_3}\\ &\quad + \frac{p_1\cdot p_2}{p_1^2p_2^2-(p_1\cdot p_2)^2}\,p_2^{\beta_1}p_3^{\alpha_2}\delta^{\alpha_1\alpha_3} - \frac{1}{p_1^2p_2^2-(p_1\cdot p_2)^2}\,p_1^{\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}\end{aligned}$$

$TTJJ$ Reconstruction

The decomposition method is quite general and can be extended in studying four point functions. In the case of $TTJJ$, the transverse traceless part can be constructed using three independent momenta and the metric tensor. The minimal decomposition consists of **47 independent form factors** when $d\geq 6$. Also, in this case, for $d<6$, one has to consider dimensional tensor identities that reduce the number of the independent form factors [2]. In particular, when $d=3$, these identities allow the decomposition to be extremely simplified, leading to the identification of only **5 form factors** as

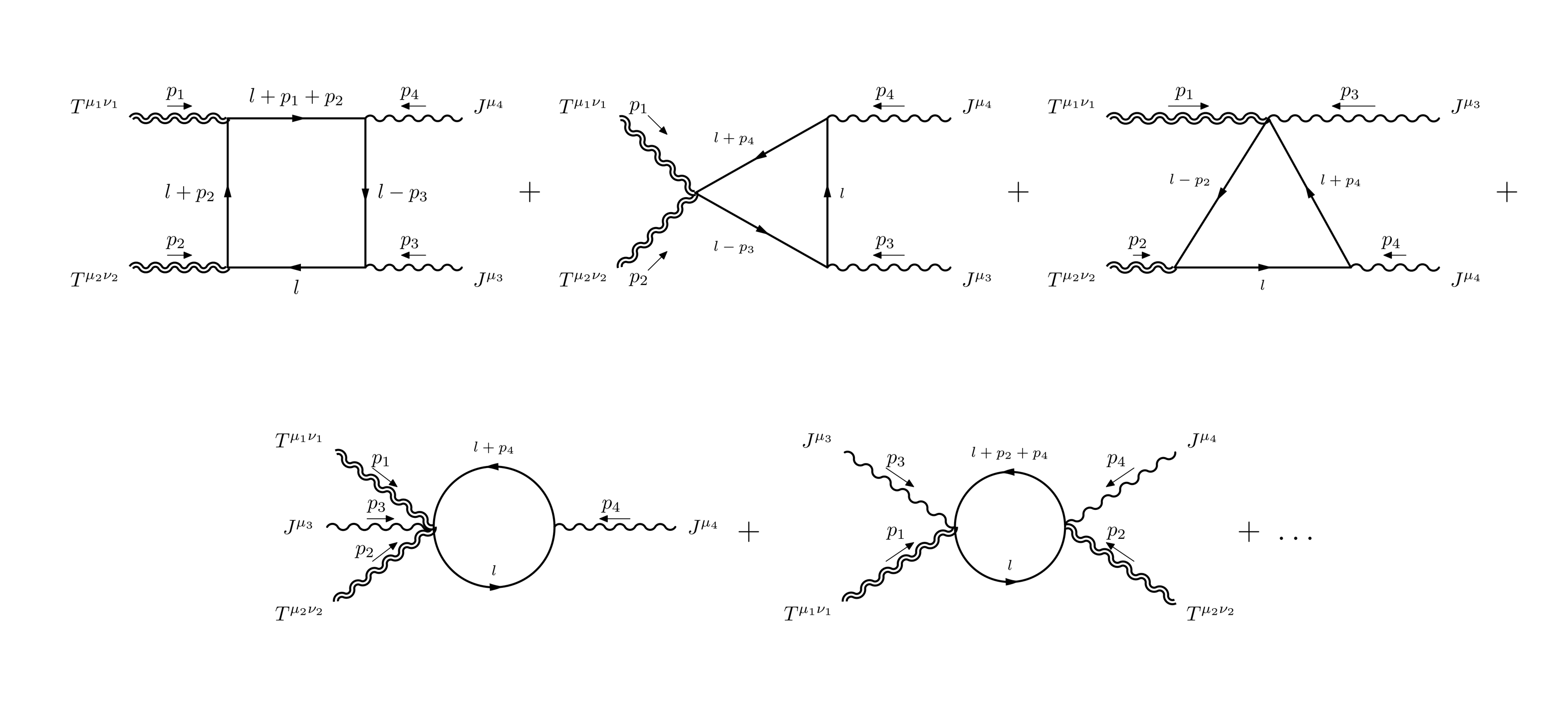
$$\begin{aligned}\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_1)j^{\mu_3\,a_3}(p_3)j^{\mu_4\,a_4}(p_4)\rangle &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\,\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\,\pi^{\mu_3}_{\alpha_3}(p_3)\,\pi^{\mu_4}_{\alpha_4}(p_4)\,\Big(A_1^{a_3a_4}\,p_1{}^{\alpha_3}p_1{}^{\alpha_4}p_3{}^{\alpha_1}p_3{}^{\alpha_2}p_3{}^{\beta_1}p_3{}^{\beta_2}\\ &\quad + A_2^{a_3a_4}\,p_1{}^{\alpha_3}p_1{}^{\alpha_4}p_3{}^{\alpha_1}p_3{}^{\beta_1}p_4{}^{\alpha_2}p_4{}^{\beta_2} + A_3^{a_3a_4}\,p_1{}^{\alpha_3}p_2{}^{\alpha_4}p_3{}^{\alpha_1}p_3{}^{\alpha_2}p_3{}^{\beta_1}p_3{}^{\beta_2}\\ &\quad + A_4^{a_3a_4}\,p_1{}^{\alpha_3}p_2{}^{\alpha_4}p_3{}^{\alpha_1}p_3{}^{\beta_1}p_4{}^{\alpha_2}p_4{}^{\beta_2} + A_5^{a_3a_4}\,p_1{}^{\alpha_3}p_2{}^{\alpha_4}p_3{}^{\alpha_2}p_3{}^{\beta_2}p_4{}^{\alpha_1}p_4{}^{\beta_1}\Big).\end{aligned}$$

In $d=4$ other dimensional dependent tensor identities are to be considered. Furthermore, the study of tensor degeneracies is fundamental for the construction of the trace anomaly of this correlator. In the table below, we summarize the number of independent form factors for three and four point functions involving energy momentum tensor T and currents J in $d=3$ and $d=4$.

	$d=3$	$d=4$	$d\geq 6$
tjj	3	4	4
$ttjj$	5	34	47

$TTJJ$ Perturbative Realization

The perturbative realization of the $\langle TTJJ\rangle$, in the Abelian and non-Abelian cases, is the sum of diagrams with the topologies



In both cases, when $d=4$, the procedure of renormalizing the correlator, due to the presence of divergences, breaks the conformal invariance, which is reflected in anomaly massless poles. We will present the form of the anomaly effective action for the $TTJJ$ correlator studying its implications [2]. In addition, a detailed study on tensor degeneracies for the $TTJJ$ decomposition will be investigated.

Conclusions

We have presented the general method to construct 3- and 4-point correlation functions using the decomposition into a longitudinal part and a transverse traceless one. The latter component can be written in independent form factors and tensor structures. **The number of independent form factors is related to the number of independent tensorial structures that depends on the specific dimension chosen.** Indeed, we have presented the method to consider the constraints on tensor structures in specific dimensions. These constraints change the number of independent form factors, and for the 3-point function $\langle TJJ\rangle$, we have four independent form factors in $d\geq 4$ that reduces to three in $d=3$. The 4-point function $\langle TTJJ\rangle$, in $d\geq 6$, is constructed by 47 form factors that reduce to 5 in $d=3$. Finally, in $d=4$ we have discussed how the renormalization procedure of correlators involving energy momentum tensors leads to the **presence of anomalous massless poles** and the identification of the anomaly effective action.

References

[1] Claudio Corianò, Matteo Maria Maglio, "Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex". In: *Nucl. Phys.* B 938 (2019), pp. 440–522.

[2] Claudio Corianò, Matteo Maria Maglio, Riccardo Tommasi, Work in preparation

[3] David Lovelock. “Dimensionally dependent identities”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 68.2 (1970), pp. 345–350

[4] S. Brian Edgar and A. Höglund, "Dimensionally dependent tensor identities by double antisymmetrization", In: J. Math. Phys. 43, 659 (2002)

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