Tsallis statistics and QCD thermodynamics

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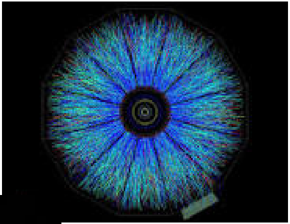
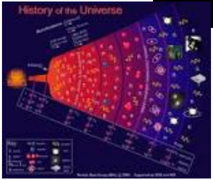
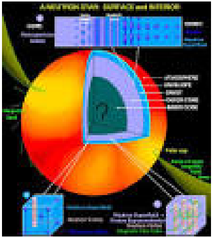
Other references: E.M. D.P.Menezes, A.Deppman, Physica A421 (2015) 15; A.Deppman, PRD93 (2016) 054001; A.Deppman, E.M., D.P.Menezes, T.Frederico, Entropy 20 (2018) 633.

QGP: QCD and its applications







Thermodynamical approach

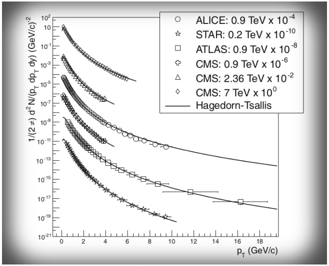
R. Hagedorn: thermodynamical approach to High Energy Collisions

exponential distributions of energy and momentum

exponential hadron mass spectrum

Hadron Resonance Gas models, conf./deconf. phase-transition

- but disagrees from experimental data ($\frac{d^2N}{dp_\perp dy} \not\approx e^{-p_\perp/T}$)
- when using Tsallis statistics $\longrightarrow$ power-law distribution $\longrightarrow$ $\longrightarrow$ the agreement is perfect (in many orders of magnitude)



Tsallis Statistics

- Tsallis statistics ➞ generalization of Boltzmann-Gibbs (BG) statistics

$$S_q \equiv -k_B \sum_i p_i^q \ln_q p_i \,, \qquad \ln_q(p) \equiv \frac{p^{1-q}-1}{1-q} \quad \left[\underset{q\to 1}{\longrightarrow} \log(p) \right] .$$

- A consequence is that the entropy of the system is non-additive. For two independent systems A and B **[C.Tsallis, J.Stat.Phys. 52 '98]**.

$$S_{A+B} = S_A + S_B + k_B^{-1}(1-q) S_A S_B \,,$$

where the entropic index q measures the degree of non-extensivity.

- q-expotential and q-logarithm functions:

$$e_q^{(\pm)}(x) = [1 \pm (q-1)x]^{\pm 1/(q-1)} \,, \quad \log_q^{(\pm)}(x) = \pm (x^{\pm(q-1)} - 1)/(q-1) \,,$$

- Occupation number:

$$n_q^{(\pm)}(x) = \frac{1}{\left(e_q^{(\pm)}(\beta(\varepsilon_p - \mu)) - \xi \right)^{1 \pm (q-1)}} \qquad \left[\underset{q\to 1}{\longrightarrow} \frac{1}{e^{\beta(\varepsilon_p - \mu)} - \xi} \right] ,$$

and $x = \beta(\varepsilon_p - \mu)$, $\beta \equiv (k_B T)^{-1}$, particle energy $\varepsilon_p = \sqrt{p^2 + m^2}$, with μ the chemical potential, $\xi = \pm 1$ for bosons/fermions.

Hadron production in heavy ion collisions

[A. Deppman, E.M., D.P.Menezes, T. Nunes, in preparation]

- Occupation number of a single particle:

$$n_q^{(\pm)}(x) = \frac{1}{\left(e_q^{(\pm)}(\beta(\varepsilon-\mu))-\xi\right)^{1\pm(q-1)}} \qquad \left[\underset{q\to 1}{\longrightarrow} \frac{1}{e^{\beta(\varepsilon-\mu)}-\xi}\right] .$$

- Density of hadron species in an ideal gas of massive particles:

$$\rho_i \equiv \frac{\langle N_i \rangle}{V} = \frac{g_i}{2\pi^2}\left[\int_0^{p_{i\star}} dp\, p^2 n_q^{(-)}(x_i) + \int_{p_{i\star}}^{\infty} dp\, p^2 n_q^{(+)}(x_i)\right]$$

where

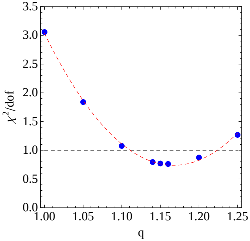
$$x_i \equiv \beta\left(\varepsilon_i - \sum_a \mu_a q_{ai}\right) , \quad \varepsilon_i \equiv \sqrt{p^2+m_i^2} , \quad \sum_q \mu_a q_{ai} \equiv \mu_u u_i + \mu_d d_i + \mu_s s_i .$$

and $p_{i\star} > 0$ if $\sum_a \mu_a q_{ai} > m_i$.

- In Au+Au collisions: $B = 2\times 197 , \quad I_3 = -39 , \quad S = 0$.
- Fit to ratios of particle abundances. Experimental values (STAR '07'):

$$\bar{p}/p = 0.65\pm 0.07 , \;\; \bar{p}/\pi^- = 0.08\pm 0.01 , \;\; \pi^-/\pi^+ = 1.00\pm 0.02 , \cdots$$

Hadron production in heavy ion collisions



Best fit in (q, μ_B, T) → $q \simeq 1.16$

$\chi^2/\text{dof} = 0.76$

$\mu_B = 52\,\text{MeV}, \; T = 58\,\text{MeV}$

Radius $= 49.6\,\text{fm}^3$

See also [D.P.Menezes et al., PRC76 '07] for $q = 1$.

Ratio	Model	Experimental (STAR '2007')
$\bar{p}/p$	0.65473	0.65(7)
$\bar{p}/\pi^-$	0.06393	0.08(1)
π^-/π^+	1.02009	1.00(2)
K^-/K^+	0.82771	0.88(5)
K^-/π^-	0.16900	0.149(20)
$\overline{\Lambda}^0/\Lambda^0$	0.76906	0.77(7)
$\overline{\Xi}^-/\Xi^-$	0.87540	0.82(8)
$K^{0\,*}/h^-$	0.07145	0.060(17)
$\overline{K}^{0\,*}/h^-$	0.06217	0.058(17)

p_T distribution in pp collisions

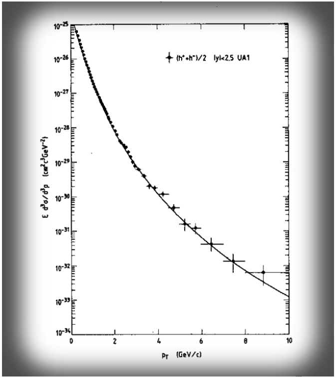
Extended Hagedorn theory to non extensive statistics: A.Deppman, Physica A 391 '12

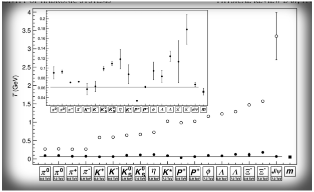
Use of Tsallis statistics: $\frac{d^2N}{dp_\perp dy} = gV \frac{p_\perp m_\perp}{(2\pi)^2} e_q^{(+)}\left(-\frac{m_\perp}{T}\right)$

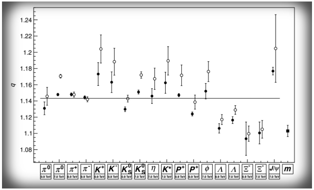
L.Marques, E.Andrade-II, A.Deppman, PRD 87 (2013) 114022

L.Marques, J.Cleymans, A.Deppman, PRD 91 (2015) 054025

Experimental value $q = 1.14 \pm 0.01$







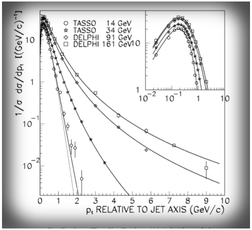
Effective hadron mass spectrum

Density of hadrons:

$\rho(m) = \rho_o \cdot [1 + (q-1)m/T_H]^{1/(q-1)}$

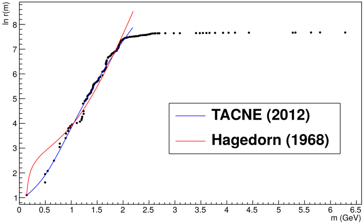
Obtained in Non-Extensive Self-Consistent Thermodynamics.

A.Deppman, Physica A 391 (2012) 6380



Power-law distributions

Cumulative number: $N(m) = \int_0^m d\tilde{m}\rho(\tilde{m})$



Non extensive mass spectrum

L.Marques, E.Andrade-II, A.Deppman, PRD 87 (2013) 114022

Hagedorn 1968: $\rho(m) = \rho_o \cdot e^{m/T_H}$

Tsallis Statistics and QCD Thermodynamics

- Grand-canonical partition function for a non-extensive ideal quantum gas is **[EM, A.Deppman, D.P.Menezes, Physica A421 '15]**

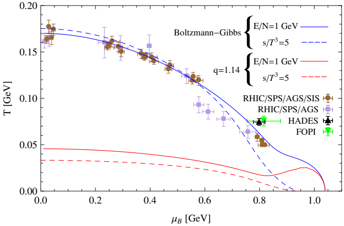
$$\log Z_q(V,T,\mu) = -\xi V \int \frac{d^3p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \log_q^{(-r)} \left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)} \right),$$

- $e_q^{(\pm)}(x) \underset{q\to 1}{\longrightarrow} \exp(x)$ and $\log_q^{(\pm)}(x) \underset{q\to 1}{\longrightarrow} \log(x)$ ➡ This result reduces to the BG statistics in the limit $q \to 1$.
- The thermodynamics of QCD in the confined phase can be studied within the Hadron Resonance Gas approach ➡ Physical observables in terms of hadronic states **[Hagedorn, Lec.Not.Phys.221 '85]**.
- Partition function given by

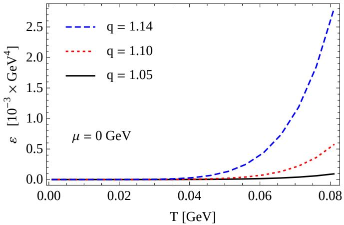
$$\log Z_q(V,T,\{\mu\}) = \sum_{i\in \text{hadrons}} \log Z_q(V,T,\mu_i) \,.$$

QCD Thermodynamics

- Energy density ➞ $\varepsilon \equiv \frac{\langle E\rangle}{V} = -\frac{1}{V}\frac{\partial}{\partial\beta}\log Z_q\Big|_{\mu_B} + \frac{1}{V}\frac{\mu_B}{\beta}\frac{\partial}{\partial\mu_B}\log Z_q\Big|_{\beta}$.
- Pressure ➞ $P = \frac{1}{\beta}\frac{\partial}{\partial V}\log Z_q$.
- Entropy ➞ $S = -\beta^2\frac{\partial}{\partial\beta}\left(\frac{\log Z_q}{\beta}\right)\Big|_{\mu_B}$.
- Baryon density ➞ $\rho_B \equiv \frac{\langle B\rangle}{V} = \frac{1}{3V}\left(\langle N_{\mathrm{quarks}}\rangle - \langle N_{\mathrm{antiquarks}}\rangle\right)$.



Chemical freeze-out line $T = T(\mu_B)$.



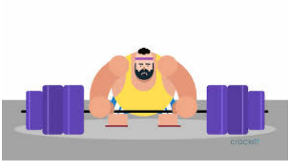
Energy density.

- $q\uparrow$ ➞ Equation of state $P = P(\varepsilon)\uparrow$ harder ➞
  ➞ Implications for neutron stars: $M_{\max}\uparrow$

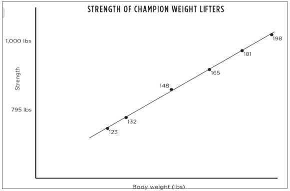
[D.P.Menezes, A.Deppman, E.M., L.B.Casto, EPJA51 '15].

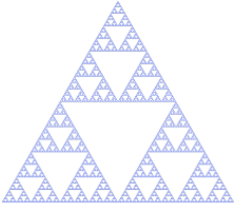
Scale and Self-Similarity

Scaling transformation ➞ changes the size of objects by a scale factor.



SCALING





SELF-SIMILARITY ➞ Self-similar object is an object which is similar to a part of itself.

(Example: Sierpiński triangle).

Tsallis Statistics and Thermofractals

- Emergence of the non-extensive behavior has been attributed to different causes: 1) long-range interactions, correlations and memory effects; 2) temperature fluctuations; 3) finite size of the system **[L.Borland, PLA 245 '98].**
- We will study a natural derivation of non-extensive statistics in terms of Thermofractals.
- Thermofractals $\equiv$ Systems in thermodynamical equilibrium presenting the following properties **[A.Deppman, PRD93 '16]**:

1. Total energy is given by:

$$U = F + E \,,$$

where $F \equiv$ kinetic energy, and $E \equiv$ internal energy of N constituent subsystems, so that $E = \sum_{i=1}^{N} \varepsilon_i^{(1)}$.

2. Constituent subsystems are thermofractals: distribution $P_{\mathrm{TF}}(E)$ is self-similar or self-affine ⟶ at level n of the subsystem hierarchy $P_{\mathrm{TF}(n)}(E)$ is equal to the distribution in the other levels:

$$P_{\mathrm{TF}(n)}(\varepsilon) \propto P_{\mathrm{TF}(n+m)}(\varepsilon) \,.$$

Tsallis Statistics and Thermofractals

- The energy distribution according to BG statistics is given by

$$P_{\mathrm{BG}}(U)dU = A\exp(-U/kT)dU\,.$$

- In thermofractals ➞ phase space must include momentum degrees of freedom of free particles as well as internal degrees of freedom. According to property 2 of self-similar thermofractals

$$P_{\mathrm{TF}(0)}(U)dU = \underbrace{A'\,F^{\frac{3N}{2}-1}\exp\left(-\frac{\alpha F}{kT}\right)dF}_{\text{Momentum d.o.f.}}\,\underbrace{\left[P_{\mathrm{TF}(1)}(\varepsilon)\right]^{\nu}d\varepsilon}_{\text{internal d.o.f.}}\,,$$

with $\alpha = 1 + \frac{\varepsilon}{kT}$ and $\frac{\varepsilon}{kT} = \frac{E}{F}$, and $\nu \equiv$ exponent to be determined.

- By imposing self-similarity

$$P_{\mathrm{TF}(0)}(U) \propto P_{\mathrm{TF}(1)}(\varepsilon)$$

one finds: $P_{\mathrm{TF}}(\varepsilon) = A\left[1+\frac{\varepsilon}{kT}\right]^{-\frac{3N}{2}\frac{1}{1-\nu}}$ ➞ $P_{\mathrm{TF}(n)}(\varepsilon) = A_{(n)}\, e_q\left[-\frac{\varepsilon}{k\tau}\right]$

➞ The distribution of thermofractals then obeys Tsallis statistics with $q-1 = \frac{2}{3N}(1-\nu)$.

Renormalization of gauge fields

- QCD phenomenology can be described by Tsallis statistics.
- Thermofractals obey Tsallis statistics.
- Question: Is it possible a thermofractal description of Yang-Mills theory?

- Yang-Mills theory $\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\psi}_j\gamma_\mu D^\mu_{ij}\Psi_j$ is renormalizable:

$$\Gamma(p, m, g) = \lambda^{-D}\Gamma(p, \bar{m}, \bar{g})$$

F. Dyson, PR 75 (1949) 1736

M. Gell-Mann and F.E. Low, PR 95 (1954) 1300

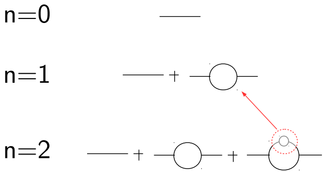
- Renormalization group equation:

$$\left[M\frac{\partial}{\partial M} + \beta_{\bar{g}}\frac{\partial}{\partial \bar{g}} + \gamma\right]\Gamma = 0$$

Callan-Symanzik Equation

C.G. Callan Jr., PRD 2 (1970) 1541

K. Symanzik, Comm. Math. Phys. 18 (1970) 227



Effective coupling constant $\bar{g}$

Effective mass $\bar{m}$

Self-similar properties of YM fields

loop in higher order is identical to a diagram in lower order.

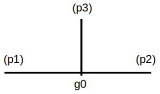
Effective thermofractal description: Calculation of q from gauge field parameters

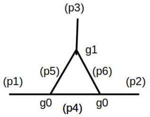
$$\bar{g}(\varepsilon) = \prod_{i=1}^{\tilde{N}} G[1-(q-1)\frac{\varepsilon_i}{\lambda}]^{\frac{1}{q-1}}$$ A.Deppman, PRD (2016)

q is related to the number of internal degrees of freedom in the fractal structure

$\bar{g}(\varepsilon)$ describes how energy flows from the initial parton to partons at higher perturbative orders.

- First order calculation of vertex function was performed for YM-theory and QCD. We can then compare what is obtained with our effective description.





- From our effective description:

$$\beta_g = \mu \frac{\partial \bar{g}}{\partial \mu} = -\frac{1}{16\pi^2}\frac{1}{q-1} g^{\tilde{N}+1}$$

with $\tilde{N} = 2$

- From QCD: $\beta_g = -\frac{1}{16\pi^2}\left[\frac{11}{3}c_1 - \frac{4}{3}c_2\right] g^3$

➡ And finally we get $q = 1.14$

[A.Deppman, EM, D.P.Menezes, PRD101 (2020) 034019]

Bose-Einstein condensation and Tsallis statistics (qBEC)

[E.M., V.S. Timóteo, A. Gammal, A. Deppman, Physica A 585 (2022)]

- Formation of Bose-Einstein condensate in hadronic systems?
  [D.Kharzeev, E.Levin, K.Tuchin, PRC75 '07]; [I.Bausista; C.Pajares, J.E.Ramirez Rev.Mex.Fis. 65 '19]; [S.Deb, D.Sahu, S.Raghunath, A.K.Pradhan, EPJA 57 '21]
- Relativistic gas of massless bosons

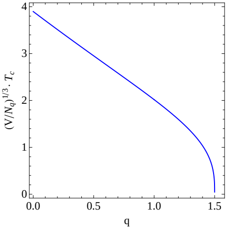
$$\langle N_q \rangle = \beta^{-1} \frac{\partial}{\partial \mu} \log Z_q \Big|_\beta , \qquad n_q^{(+)}(\varepsilon, \beta, \mu) = \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} .$$

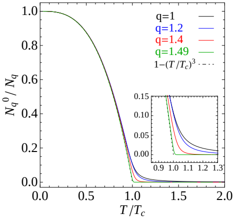
- Total number of particles [Ground state $\varepsilon_c = 0$]

$$N_q \equiv N_q^0 + N_q^\varepsilon = \left[e_q^{(+)}[\beta(\varepsilon_c - \mu)] - 1 \right]^{-q} + \frac{V}{2\pi^2} \int_0^\infty d\varepsilon \, \varepsilon^2 \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} .$$

- Critical temperature

$$N_q^\varepsilon(T) \le N_{q,\max}^\varepsilon(T_c) = \frac{VT_c^3}{\pi^2} \zeta_q(0) , \quad \text{where} \quad \zeta_q(0) = \frac{1}{2} \int_0^\infty dx \, x^2 \left[e_q^{(+)}(x) - 1 \right]^{-q} .$$

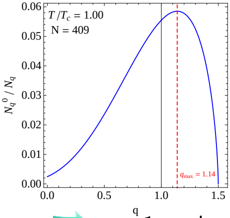


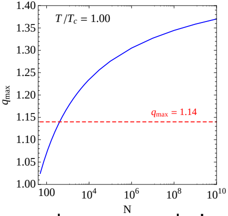


$$\frac{N_q^0}{N_q} \simeq 1 - \left(\frac{T}{T_c} \right)^3$$

Bose-Einstein condensation and Tsallis statistics (qBEC)

- Fraction of particles in the ground state:

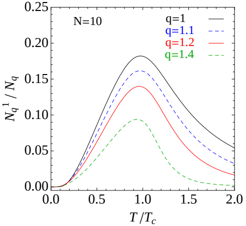


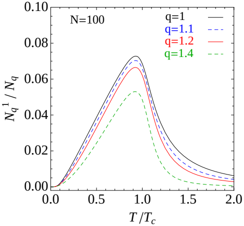


➡ $q > 1$ tends to favor the condensate population.

- Fraction of particles in the first excited state ➡
  ➡ System in a box (discretization of energy levels)

$$E_{n_x,n_y,n_z} = \frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}\,, \qquad n_x, n_y, n_z \geq 1\,.$$





Applications of Tsallis statistics

High energy collisions:

J. Cleymans; D.J. Worku. Phys. G Nucl. Part. Phys. 2012, 39, 025006
C.-Y. Wong; G. Wilk, G.; Tsallis, C. Phys. Rev. D 2015, 91, 11402
L. Marques, J. Cleymans, and A.Deppman, PRD 91 (2015) 054025

Hadron models:

P.H.G Cardoso; T.N. da Silva; A. Deppman, D.P. Menezes, EPJA 51 (2015) 155
E.Andrade II, A. Deppman, EM, D.P. Menezes, T. Nunes, PRD101 (2020) 054022

Hadron mass spectrum:

L. Marques; E. Andrade-II; A. Deppman, Phys. Rev. D 2013, 87, 114022

Hadron structure:

A. Deppman, E.M., M.J.Teixeira, V.Timóteo, in preparation.

Neutron stars:

D.P. Menezes, A. Deppman, E.M., and L.B. Castro, EPJA 51, (2015) 155

Lattice QCD:

A. Deppman JPG 41 (2014) 055108

Bose-Einstein condensation:

J.Chen, Z.Zhang, G.Su, L.Chen, Y.Shu, Physics Letter A300 (2002)
E.M., V.S. Timóteo, A. Gammal, A. Deppman, Physica A 585 (2022)

Non-extensive statistical mechanics:

E.M., A. Deppman, D.P. Menezes, Physica A 421 (2015) 15
A. Deppman, Physica A 391 (2012) 6380
A. Deppman, E.M., D.P. Menezes, T. Frederico, (2018) Entropy 20 (2017) 633

Conclusions:

- We have reviewed the non-extensive statistics in the form of Tsallis statistics of a quantum gas at finite T and μ.
  - *pp* collisions.
  - Heavy ion collisions.
  - Hadron spectrum.
  - QCD thermodynamics.
  - Bose-Einstein condensation.
- We have investigated the structure of a thermodynamical system presenting fractal properties, and shown that it naturally leads to non-extensive statistics.
- Based on the self-similar properties of thermofractals:
  - *'Field theoretical approach'* for thermofractals.
  - β function of QCD assuming a thermofractal structure.
- Self-similarity in gauge fields leads to

  Self-consistency and fractal structure

  Recursive calculations at any order

  Non extensive statistics

  Reconciles Hagedorn's theory with QCD

  Agreement with experimental data

Thank You!