# $U(1)_A$ axial anomaly, $\eta'$, and topological susceptibility in the holographic soft-wall model

Floriana Giannuzzi
INFN Bari, Italy

QCD@Work 2022, International Workshop on QCD, theory and experiment

June 29th 2022

based on PRD 104 014021 (2021)

# Outline

AdS/QCD for pseudoscalar mesons and glueballs

- Ps glueballs in pure gluondynamics
- Review nonsinglet ps mesons
- Mass of $\eta'$ from mixing between ps glueballs and singlet ps mesons
- Reproduce anomaly equation
- Topological susceptibility as a function of $m_q$

# Soft-wall model [Karch *et al.*, PRD 74, 015005 (2006)]

- Bottom-up approach: build a $5d$ effective theory motivated by QCD in AdS space
- break conformal invariance through "dilaton" $\mathrm{e}^{-c^2z^2}$ in the action, $ds^2 = \frac{R^2}{z^2}(dt^2 - d\bar{x}^2 - dz^2)$
- relate bulk fields and operators on the dual boundary field theory [Witten, ATMP 2, 253 (1998); Gubser *et al.*, PLB 428, 105 (1998)]

1.

| $4d$ | $5d$ |
|---|---|
| operator $\mathcal{O}(x)$ | field $\phi(x,z)$ |
| $\Delta$ | $m_5^2R^2 = (\Delta - p)(\Delta + p - 4)$ |

2. Boundary value of field is the source of operator $\phi_0(x)$
   in Fourier space $\phi(q,z) = K(q,z)\ \phi_0(q)$ with $K(q,z)$ bulk-to-boundary propagator

3. $\langle \mathrm{e}^{\int d^dx\, \phi_0(x)\mathcal{O}(x)} \rangle = Z_S[\phi_0(x)] \approx \mathrm{e}^{i\mathcal{S}_{OS}}$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{\delta^2\mathcal{S}_{OS}}{\delta\phi_0(x_1)\delta\phi_0(x_2)}\bigg|_{\phi_0=0}$$

# Soft-wall model for ps mesons

| QCD operators | fields in soft wall |
| --- | --- |
| $\bar{q}_{L/R}\gamma^{\mu}T^{A}q_{L/R}$ | $A^{M}_{L/R}(x,z)$ |
| $\bar{q}_{R}q_{L}$ | $X(x,z)=e^{i\eta^{A}(x,z)T^{A}}X_0(z)e^{i\eta^{A}(x,z)T^{A}}$ |
| $G^2$ | $Y(x,z)=Y_0(z)e^{2ia(x,z)}$ |

- global $U(n_f)_L \times U(n_f)_R$ in $4d$ becomes a local symmetry in $5d$
- axial field $A=(A_L-A_R)/2$, gauge condition $A_5=0$
- transverse and longitudinal fields: $A^{A}_{\mu}=A^{A}_{\perp\,\mu}+\partial_\mu\varphi^{A}$
- $X_0(z)=\sqrt{2}v_q(z)\,I_{n_f}$ convenient redefinition, with $v_q(z)=m_q\,z+\sigma\,z^3$
- $Y_0(z)=\dfrac{y_0}{R}+\dfrac{2\,y_1}{R\,c^4}(e^{c^2z^2}(-1+c^2z^2)+1)$
- $a(x,z)=a_{PG}(x,z)+V_a(z)\,a_f(x,z)$ with $V_a(z)=e^{-v_q(z)^2}$

Lagrangian:

$$\mathcal{L} = \frac{1}{k}\sqrt{g}\,e^{-\phi}\left[\text{Tr}\left\{-\frac{1}{4g_5^2}(F_L^2 + F_R^2) + |DX|^2 - m_X^2|X|^2\right\} + \frac{1}{2}\mathcal{K}_a\right]$$

$$\mathcal{K}_a = |\partial_M Y_0(z) + 2iY_0(\partial_M a(x,z) - \eta^0(x,z)\partial_M V_a(z) - A_M^0(x,z)V_a(z))|^2$$

Transformation rules of the 5$d$ fields under $U(1)_A$:

$$\eta^0 \to \eta^0 - \alpha$$
$$\varphi^0 \to \varphi^0 - \alpha$$
$$a \to a - V_a\alpha$$

For the fields $\varphi$, $\eta$, $a$:

$$\begin{aligned}\mathcal{L} \; = \; & \frac{R}{k}e^{-\phi}\left[\frac{1}{4n_f g_5^2 z}(\partial_z\partial_\nu\varphi^0)^2 + \frac{1}{2g_5^2 z}(\partial_z\partial^\nu\varphi^8)^2 + \right.\\ & -\frac{2R^2\,v_q^2}{n_f z^3}(\partial_z\eta^0)^2 + \frac{2R^2\,v_q^2}{n_f z^3}(\partial_\nu\eta^0 - \partial_\nu\varphi^0)^2 - \frac{4\,R^2\,v_q^2}{z^3}(\partial_z\eta^8)^2 + \frac{4\,R^2\,v_q^2}{z^3}(\partial_\nu\eta^8 - \partial_\nu\varphi^8)^2 + \\ & \left. -\frac{2R^2}{z^3}Y_0^2(\partial_z a - \eta^0\partial_z V_a)^2 + \frac{2R^2}{z^3}Y_0^2(\partial_\nu a - V_a\partial_\nu\varphi^0)^2\right]\end{aligned}$$

Known parameters:

| | |
|---|---|
| $R/k = N_c/16\pi^2$ | two-point function of vector mesons |
| $g_5^2 = 3/4$ | two-point function of scalar mesons |
| $\sigma = -\frac{2\pi^2}{N_c}\langle \bar{q}q \rangle$ | derivative of os action wrt $m_q$ |
| $c = 388$ MeV | $\rho$ mass |

Need to fix $m_q$, $\langle \bar{q}q \rangle$, $y_0$, $y_1$

# Pure gauge

Action with only $a$ field for ps glueballs

$$\mathcal{S}_{PG} = \frac{R}{k}\int d^5x\, e^{-\phi}\left[-\frac{2R^2}{z^3}Y_0^2(\partial_z a_P)^2 + \frac{2R^2}{z^3}Y_0^2(\partial_\mu a_P)^2\right]$$

Two-point function

$$\Pi_{aa}(q^2) = \left.\frac{\partial^2\mathcal{S}_{os}}{\partial a_0 \partial a_0}\right|_{a_0\to 0} = \left.\frac{R}{k}\frac{4Y_0(z)^2 e^{-c^2z^2}}{z^3}\tilde{a}'_{PG}(z)\tilde{a}_{PG}(z)\right|_{z\to 0}$$

where $\tilde{a}_{PG}$ is the BTBP

- eigenvalues $m_n^2 = 4c^2(n+2)$ from position of poles of $\Pi_{aa}(q^2)$
  - the lightest state has mass $m_{G\tilde{G},0} = 1.1$ GeV
  - $\eta(1405)$ compatible with the first radial excitation, with $m_{G\tilde{G},1} = 1.34$ GeV

- decay constants from residues$=m_n^4 f_n^2$:

$$f_n^2 = \frac{R}{k} 2c^2 y_0^2 \frac{(n+1)}{(n+2)}$$

- high-$Q^2$ expansion ($Q^2 = -q^2$)

$$\Pi_{aa}(Q^2) = Q^4 \left( -\frac{N_c}{32\pi^2} y_0^2 \log Q^2 + \mathcal{O}(1) \right)$$

Matching to QCD we get $y_0 = \frac{1}{\sqrt{N_c}} \frac{\alpha_s}{\pi}$

$\Rightarrow Y$ field appears in the full Lagrangian at a lower order ($\mathcal{O}(1/N_c)$) in the large $N_c$ expansion with respect to the other fields, as expected

With $y_0 = \frac{1}{\sqrt{N_c}\pi}$, ground-state decay constant is $f_{G\tilde{G},0} = 9.8$ MeV

- compute the topological susceptibility

  In AdS/QCD, it is given by:

$$\chi_t = -\lim_{q^2 \to 0} \Pi_{aa}(q^2) = \frac{1}{\pi^2} \frac{\alpha_s}{\pi} y_1$$

  $\Rightarrow$ from $\chi_{PG} \sim (191\ \text{MeV})^4$ fix $y_1 = 0.041\ \text{GeV}^4$

# Nonsinglet ps mesons

Equations of motion:

$$\partial_z \left( \frac{v_q^2 e^{-\phi}}{z^3} \partial_z \eta^8 \right) + q^2 \frac{v_q^2 e^{-\phi}}{z^3} (\eta^8 - \varphi^8) = 0$$

$$\partial_z \left( \frac{e^{-\phi}}{g_5^2 z} \partial_z \varphi^8 \right) + \frac{8 v_q^2 e^{-\phi}}{z^3} (\eta^8 - \varphi^8) = 0$$

Combining the two equations, integrating and using $\partial_z \varphi^8 = 0$ and $\partial_z \eta^8 = 0$ at $z \to \infty$ :

$$\frac{q^2}{g_5^2 z} \partial_z \varphi^8 - \frac{8 v_q^2}{z^3} \partial_z \eta^8 = 0 \qquad \text{constraint equation}$$

Operator mixing on the boundary ⇒ fields in the bulk are given by a linear combination of the sources in Fourier space

In matrix formalism [Kaminski:2009dh]

$$\Phi = \begin{pmatrix} \varphi^8 \\ \eta^8 \end{pmatrix} = F\,\Phi_0$$

where $F$ is a matrix and $\Phi_0(q^2) = (\varphi_0^8(q^2), -\eta_0^8(q^2))^T$ is the vector of the sources of the two operators.

On-shell action:

$$\mathcal{S}_{os} = -\lim_{z\to\varepsilon} \frac{R}{k} \int d^4k\; e^{-c^2 z^2}\, \Phi_0^\dagger F^\dagger B F' \Phi_0$$

with $B = \begin{pmatrix} \dfrac{q^2}{2g_5^2 z} & 0 \\ 0 & -\dfrac{4v_q^2 R^2}{z^3} \end{pmatrix}$

One-point functions:

$$J^8 = \frac{\partial \mathcal{S}_{os}}{\partial \Phi_0}\bigg|_{z\to\varepsilon} = \begin{pmatrix} \langle J^8_\varphi \rangle \\ \langle J^8_\eta \rangle \end{pmatrix} = \begin{pmatrix} -\frac{R}{k}\frac{e^{-\phi}q^2}{2g_5^2 z}(\varphi^8)'\Big|_{z\to\varepsilon} \\ -\frac{R}{k}\frac{e^{-\phi}4v_q^2R^2}{z^3}(\eta^8)'\Big|_{z\to\varepsilon} \end{pmatrix}$$

$\langle J^8_\varphi \rangle = \langle \partial_\mu \bar\psi \gamma_5 \gamma^\mu T^8 \psi \rangle$ one-point function of the nonsinglet longitudinal axial current

$\langle J^8_\eta \rangle = \langle 2m_q \bar\psi \gamma_5 T^8 \psi \rangle$ one-point function of the nonsinglet pseudoscalar current

Using constraint equation $\frac{q^2}{g_5^2 z}\partial_z \varphi^8 - \frac{8v_q^2}{z^3}\partial_z \eta^8 = 0$

$$\Rightarrow \qquad \langle J^8_\varphi \rangle = \langle J^8_\eta \rangle$$

establishes partial conservation of axial current

Two-point functions:

$$\Pi^{88} = \frac{\partial^2 \mathcal{S}_{os}}{\partial \Phi_0 \partial \Phi_0}\bigg|_{z\to\varepsilon} = \begin{pmatrix} \Pi^{88}_{\varphi\varphi} & \Pi^{88}_{\varphi\eta} \\ \Pi^{88}_{\eta\varphi} & \Pi^{88}_{\eta\eta} \end{pmatrix}$$

Large $Q^2 = -q^2$ expansion: $\Pi^{88}_{\varphi\varphi}(Q^2) \xrightarrow[Q^2\to\infty]{} \frac{3}{16\pi^2} 4m_q^2 Q^2 \log Q^2 + \mathcal{O}(Q^2)$

Fix last two parameters:

- $|\langle q\bar{q}\rangle| = (0.281\ \mathrm{GeV})^3$ from pion-decay constant at $m_q = 0$:

$$f_\pi^2\big|_{m_q=0} = -\frac{R}{k}\frac{e^{-\phi}}{g_5^2 z}\partial_z F_{00}\bigg|_{\substack{q^2=0\\ z\to\varepsilon}} = (91.6\ \text{MeV})^2$$

- $m_q = m_u = 3.7\ \mathrm{MeV}$ from $\pi$ mass $m_{\eta_8} = m_\pi = 139\ \mathrm{MeV}$, so $f_{0,u} = 92.3$ MeV
- $m_q = m_s = 59.5\ \mathrm{MeV}$ from $\eta$ mass $m_{\eta_8} = m_\eta = 548\ \mathrm{MeV}$, so $f_{0,s} = 103$ MeV
- if $m_q = 0$, then $m_{\eta_8} = 0$

# Singlet ps mesons

Equations of motion for $a, \varphi^0, \eta^0$:

$$\partial_z \left( \frac{v_q^2 e^{-\phi}}{n_f z^3} \partial_z \eta^0 \right) + q^2 \frac{v_q^2 e^{-\phi}}{n_f z^3} (\eta^0 - \varphi^0) + \frac{Y_0^2 e^{-\phi}}{z^3} (\partial_z V_a)(\partial_z a - \eta^0 \partial_z V_a) = 0$$

$$\partial_z \left( \frac{e^{-\phi}}{2 n_f g_5^2 z} \partial_z \varphi^0 \right) + \frac{4 v_q^2 R^2 e^{-\phi}}{n_f z^3} (\eta^0 - \varphi^0) + \frac{4 Y_0^2 R^2 e^{-\phi}}{z^3} V_a (a - \varphi^0 V_a) = 0$$

$$\partial_z \left( \frac{Y_0^2 e^{-\phi}}{z^3} (\partial_z a - \eta^0 \, \partial_z V_a) \right) + q^2 \frac{Y_0^2 e^{-\phi}}{z^3} (a - \varphi^0 V_a) = 0$$

A combination of the three equations gives (after integrating):

$$\frac{4 v_q^2 R^2}{n_f z^3} \partial_z \eta^0 - \frac{q^2}{2 n_f g_5^2 z} \partial_z \varphi^0 + \frac{4 Y_0^2 R^2}{z^3} V_a (\partial_z a - \eta^0 \partial_z V_a) = 0 \qquad \text{constraint equation}$$

where the integration constant is zero since $\partial_z \varphi^0$, $\partial_z \eta^0$, $a$, and $Y_0^2 V_a$ vanish as $z \to \infty$ due to boundary conditions.

Matrix formalism:

$$\Psi = \begin{pmatrix} \varphi^0 \\ \eta^0 \\ a \end{pmatrix} = H\,\Psi_0$$

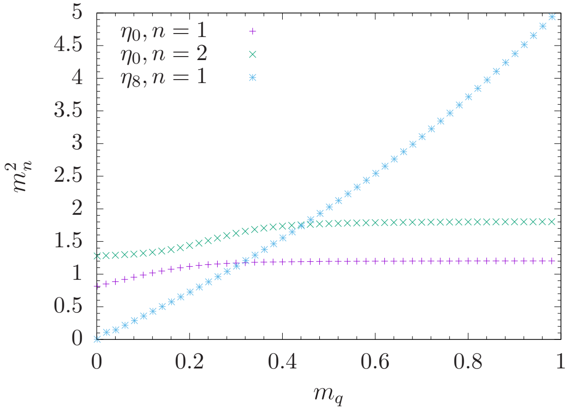
$\Psi_0(q^2) = (\varphi_0^0(q^2), -\eta_0^0(q^2), a_0(q^2))^T$ is the vector containing the sources of the three operators

The two-point functions are:

$$\Pi^{00} = \left.\frac{\partial^2 \mathcal{S}_{os}}{\partial \Psi_0 \partial \Psi_0}\right|_{z\to\varepsilon} = \begin{pmatrix} \Pi^{00}_{\varphi\varphi} & \Pi^{00}_{\varphi\eta} & \Pi_{\varphi a} \\ \Pi^{00}_{\eta\varphi} & \Pi^{00}_{\eta\eta} & \Pi_{\eta a} \\ \Pi_{a\varphi} & \Pi_{a\eta} & \Pi_{aa} \end{pmatrix}$$

Poles of the two-point functions:

$m_{\eta'} = 958$ MeV for $m_q = m_s = 59.5$ MeV
in the chiral limit $m_{\eta'} = 903$ MeV



One-point functions:

$$J^0 = \frac{\partial \mathcal{S}_{os}}{\partial \Psi_0}\bigg|_{z\to\varepsilon} = \begin{pmatrix} \langle J^0_\varphi\rangle \\ \langle J^0_\eta\rangle \\ \langle J^0_a\rangle \end{pmatrix} = \begin{pmatrix} -\frac{R}{k}\frac{e^{-\phi}q^2}{4n_f g_5^2 z}\varphi_0'\Big|_{z\to\varepsilon} \\ -\frac{R}{k}\frac{e^{-\phi}2v_q^2R^2}{n_f z^3}\eta_0'\Big|_{z\to\varepsilon} \\ \frac{R}{k}\frac{e^{-\phi}2Y_0^2R^2}{z^3}a'\Big|_{z\to\varepsilon} \end{pmatrix}$$

$\langle J^0_\varphi\rangle = \langle \partial_\mu \bar\psi\gamma_5\gamma^\mu T^0\psi\rangle$ one-point function of the singlet longitudinal axial current

$\langle J^0_\eta\rangle = \langle 2m_q\bar\psi\gamma_5 T^0\psi\rangle$ one-point function of the singlet pseudoscalar current

$\langle J^0_a\rangle = \langle \frac{\alpha_s}{8\pi}G_{\mu\nu}\tilde G^{\mu\nu}\rangle$ one-point function of the topological charge density

From constraint equation $\frac{4v_q^2R^2}{n_f z^3}\partial_z\eta^0 - \frac{q^2}{2n_f g_5^2 z}\partial_z\varphi^0 + \frac{4Y_0^2R^2}{z^3}V_a(\partial_z a - \eta^0\partial_z V_a) = 0$

$z\to 0 \qquad \Rightarrow \qquad -\langle J^0_\eta\rangle + \langle J^0_\varphi\rangle + \langle J^0_a\rangle = 0$

holographic representation of the QCD anomaly equation: $\partial_\mu J^\mu_A = 2m_q\bar\psi\gamma_5 T^0\psi - \frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde G^{\mu\nu}\,\mathrm{tr}T^0$

Topological susceptibility

$$\chi_t = -\Pi_{aa}(q^2=0) = -\left.\frac{R}{k}\frac{4Y_0^2 R^2 e^{-\phi}}{z^3} H_{22}' H_{22}\right|_{\substack{q^2\to 0\\ z\to 0}}$$

In the chiral limit plus $N_c \to \infty$

$R/k \sim \mathcal{O}(N_c)$, $m_{\eta'} \sim \mathcal{O}(1/\sqrt{N_c})$, $Y_0 \sim \mathcal{O}(1/\sqrt{N_c})$ other parameters are $\mathcal{O}(N_c^0)$

Expanding eoms of fields in $1/N_c$ for $q^2 = m_{\eta'}^2$ one finds at lowest order:

$$\frac{m_{\eta'}^2}{2n_f}\frac{1}{g_5^2 z}\partial_z(\varphi^0)_0 = \frac{4Y_0^2 R^2}{z^3} V_a(\partial_z(a)_0 - (\eta^0)_0 V_a')$$

giving for $z \to 0$ the Witten-Veneziano relation $\dfrac{m_{\eta'}^2 f_\pi^2}{2n_f} = \chi_{PG}$

Topological susceptibility as a function of quark mass

Eom at $q^2 = 0$ can be integrated:

$$e^{-\phi}\frac{4Y_0^2 R^2}{z^3}(H_{22}' - V_a' H_{12}) = A_1$$
$$e^{-\phi}\frac{4v_q^2 R^2}{n_f z^3} H_{12}' + V_a A_1 = 0$$

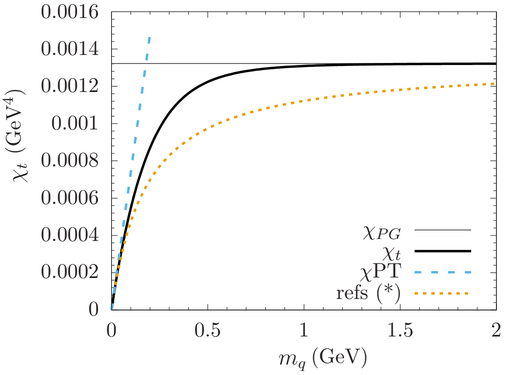
Let's write

$$-1 = \int_0^\infty dz\, H_{22}'(z) = A_1 \int_0^\infty dz\, \frac{e^{\phi(z)} z^3}{4}\left(\frac{1}{Y_0(z)^2 R^2} + n_f \left(\frac{V_a(z)}{v_q(z) R}\right)^2\right)$$

and, since $\chi_t = -\frac{R}{k} A_1$,

$$\frac{1}{\chi_t} = \frac{1}{\chi_{PG}} + \frac{1}{\chi_f} \qquad \text{with} \qquad \frac{1}{\chi_f} = \frac{k}{R} n_f \int_0^\infty dz\, \frac{e^{\phi(z)} z^3}{4}\left(\frac{V_a(z)}{v_q(z) R}\right)^2$$

$$\frac{1}{\chi_t} = \frac{1}{\chi_{PG}} + \frac{1}{\chi_f}$$




At small quark mass:

$$\chi_t \xrightarrow[m_q \to 0]{} \chi_f \sim \frac{\langle \bar{q} q \rangle}{n_f} m_q \qquad \text{as in } \chi\text{PT}$$

In the limit $m_q \to \infty$:

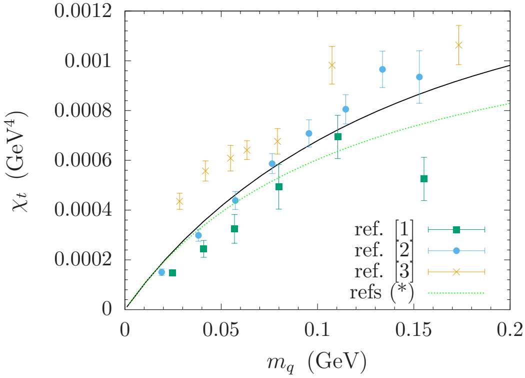
$$\chi_f \xrightarrow[m_q \to \infty]{} \frac{3}{\pi^2 n_f} m_q^4 \qquad \chi_t \xrightarrow[m_q \to \infty]{} \chi_{PG}$$

Behaviour proposed in (*):

$$\frac{1}{\chi_t} = \frac{1}{\chi_{PG}} + \frac{n_f}{m_q |\langle \bar{q} q \rangle|}$$

(*) P. Di Vecchia and G. Veneziano, Nucl. Phys. B **171**, 253-272 (1980); H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607-5632 (1992)

Comparison with lattice data with $n_f = 2$




[1] S. Aoki *et al.* [JLQCD and TWQCD], Phys. Lett. B **665**, 294-297 (2008)
[2] T. W. Chiu *et al.* [TWQCD], Phys. Lett. B **702**, 131-134 (2011)
[3] T. DeGrand, Phys. Rev. D **101**, no.11, 114509 (2020)

# Conclusions

- Masses of singlet and nonsinglet ps mesons computed in the soft-wall holographic model of QCD. Mixing among the fields dual to the axial current, pseudoscalar current and the $G\tilde{G}$ operator can explain the large mass of the $\eta'$
- Partial conservation of axial current, anomaly equation and Witten-Veneziano relation derived from the constraint equations, obtained by a combination of the equations of motion of the involved fields
- Topological susceptibility $\chi_t$ in the soft-wall model for any value of quark mass. Correction $\chi_f$ to the pure-gauge value depends linearly on the quark mass for low values of $m_q$, as in chiral perturbation theory, while other corrections arise for higher $m_q$, in particular $\chi_f$ diverges as $m_q^4$ at infinite $m_q$