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INFN - Roma Tre

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Exclusive semileptonic B-meson decays using lattice QCD and unitarity

in collaboration with:

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outline of the talk

* **the Dispersion Matrix approach:** an attractive way to implement **unitarity** and **Lattice QCD** calculations in the analysis of exclusive semileptonic decays of hadrons [PRD ’21 (2105.02497), PRD ’21 (2105.07851), PRD ’22 (2105.08674)]
* results for $B_{(s)} \to D_{(s)}^{(*)} \ell \nu_\ell$ decays: extraction of $|V_{cb}|$ and theoretical determination of $R(D_{(s)}^{(*)})$ [PRD ’22, 2109.15248, 2204.05925]
* results for $B \to \pi \ell \nu_\ell$ and $B_s \to K \ell \nu_\ell$ decays: extraction of $|V_{ub}|$ [2202.10285, to appear in JHEP]

motivations

* two critical issues in semileptonic $B \to D^{(*)} \ell \nu_\ell$ decays

- **exclusive/inclusive $|V_{cb}|$ puzzle:**

<u>exclusive</u> (FLAG '21): $|V_{cb}|_{excl.} \cdot 10^3 = 39.36\ (68)$

<u>inclusive</u> (HFLAV '21): $|V_{cb}|_{incl.} \cdot 10^3 = 42.19\ (78)$

~2.7 σ difference excl./incl.

(Bordone et al. 2107.00604) $|V_{cb}|_{incl.} \cdot 10^3 = 42.16\ (50)$

- **$R(D^{(*)})$ anomalies:**

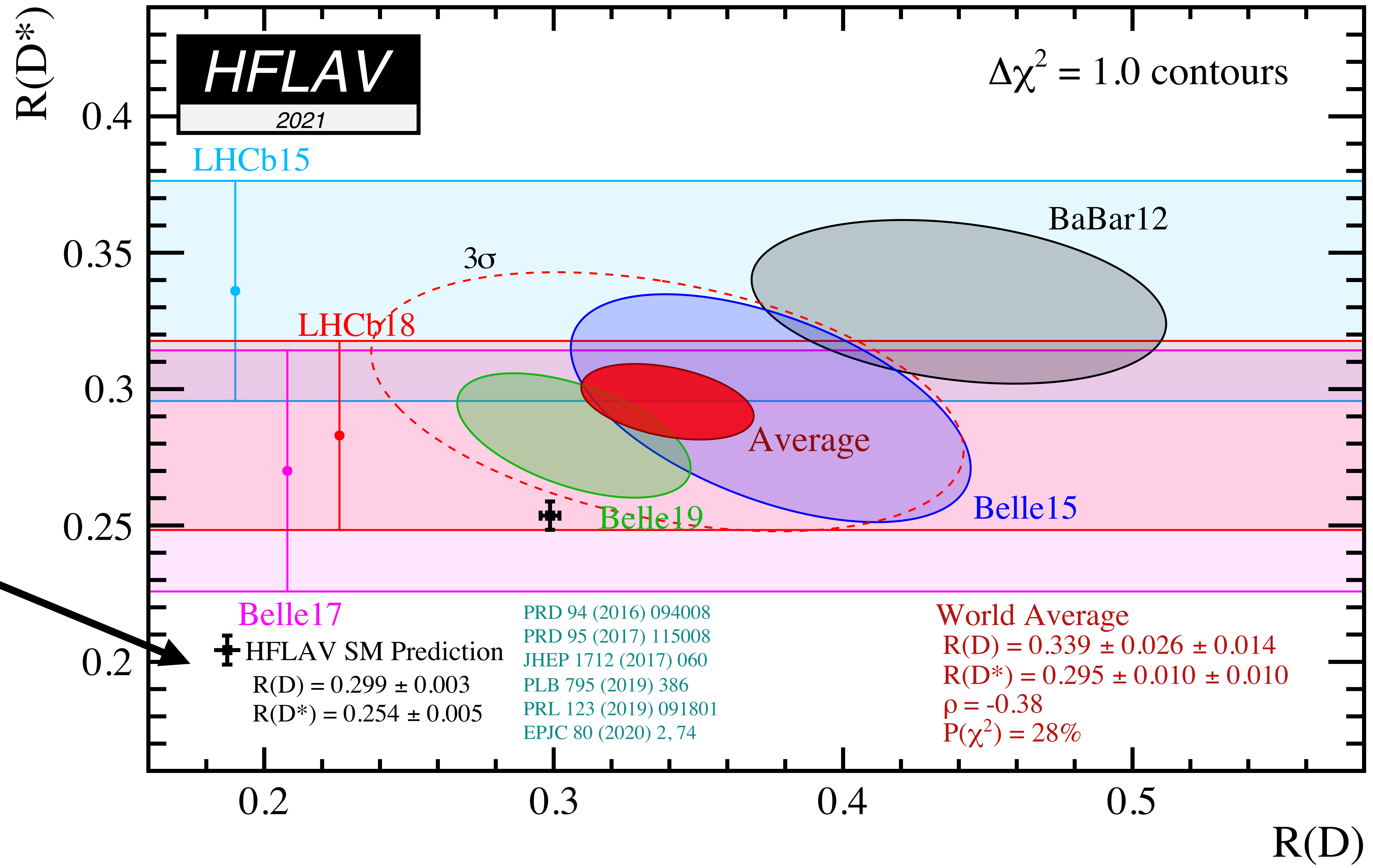
$$R(D) = \frac{\mathcal{B}(B \to D\tau\nu_\tau)}{\mathcal{B}(B \to D\ell\nu_\ell)}$$

$$R(D^*) = \frac{\mathcal{B}(B \to D^*\tau\nu_\tau)}{\mathcal{B}(B \to D^*\ell\nu_\ell)} \qquad \ell = e, \mu$$

~3.4σ differences between exp.'s and "SM"

" SM" = mix of theoretical calculations and experimental data to constrain the shape of the hadronic form factors (FFs)

pure SM predictions ?

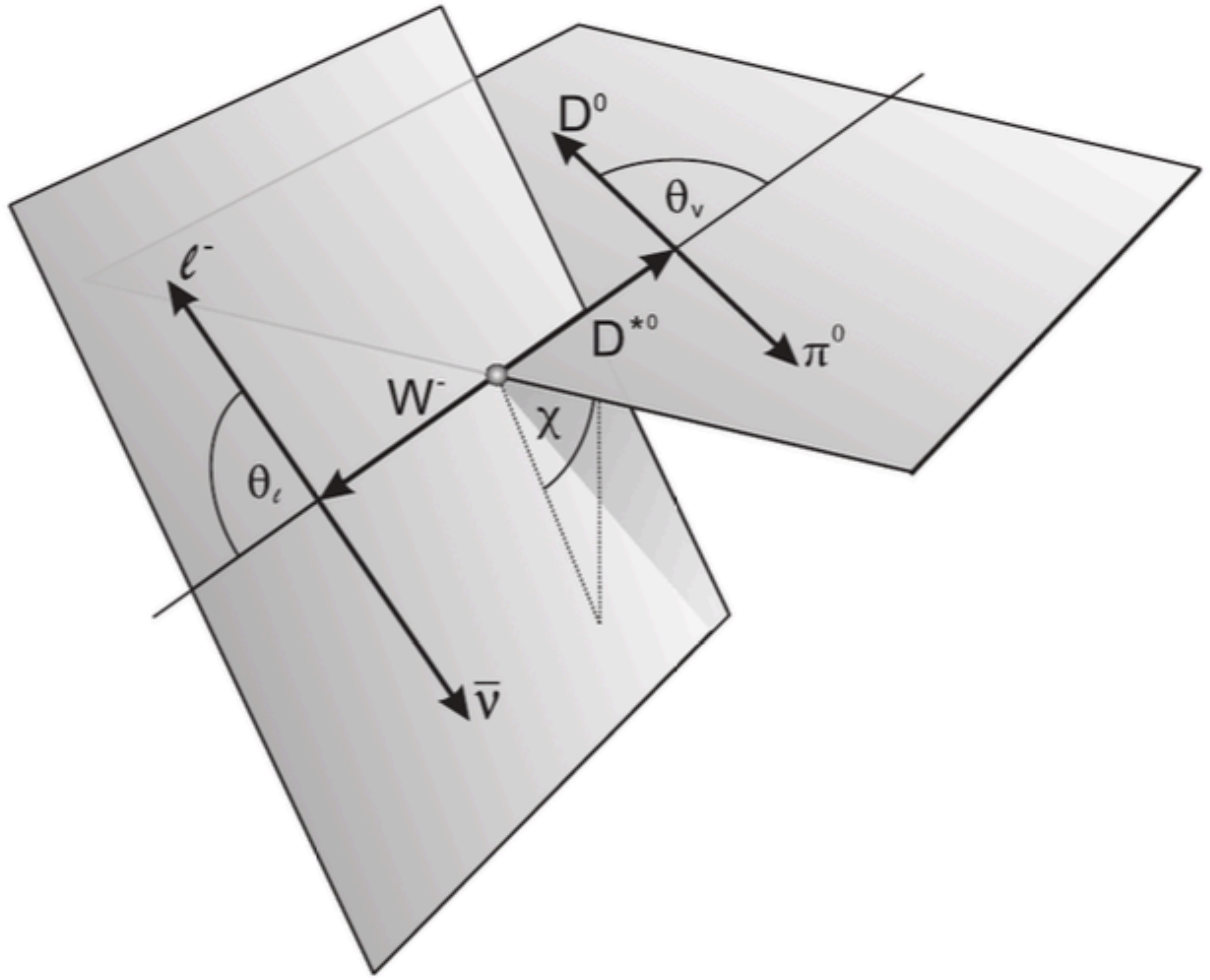


hadronic form factors in semileptonic $B \to D^{(*)}\ell\nu_\ell$ decays

$$\frac{d\Gamma(B \to D\ell\nu_\ell)}{dq^2} = \frac{G_F^2}{24\pi^3}\eta_{EW}^2 |V_{cb}|^2 p_D^3 \, f_+^2(q^2)$$

for massless leptons ($\ell = e, \mu$)

$$\frac{d^4\Gamma(B \to D^*\ell\nu_\ell)}{dw \, d\cos\theta_v \, d\cos\theta_\ell \, d\chi} = \frac{3}{4}\frac{G_F^2}{(4\pi)^4}\eta_{EW}^2 |V_{cb}|^2 m_B^3 \, r^2 \sqrt{w^2-1}(1+r^2-2rw)$$
$$\cdot \Big\{ H_+^2(w) \sin^2\theta_v \, (1-\cos\theta_\ell)^2 + H_-^2(w) \sin^2\theta_v \, (1+\cos\theta_\ell)^2$$
$$+ \; 4 \, H_0^2(w) \cos^2\theta_v \, \sin^2\theta_\ell - 2 \, H_-(w)H_+(w) \sin^2\theta_v \, \sin^2\theta_\ell \, \cos 2\chi$$
$$- \; 2 \, H_+(w)H_0(w) \cos 2\theta_v \, \sin\theta_\ell \, (1-\cos\theta_\ell) \cos\chi$$
$$+ \; 2 \, H_-(w)H_0(w) \cos 2\theta_v \, \sin\theta_\ell \, (1+\cos\theta_\ell) \cos\chi \Big\}$$



$$g(w) = \frac{1}{r\sqrt{w^2-1}} \frac{H_+(w) - H_-(w)}{2m_B^2} \qquad r = \frac{m_{D^*}}{m_B} \qquad w = \frac{1+r^2-q^2/m_B^2}{2r}$$

$$f(w) = \frac{H_+(w) + H_-(w)}{2}$$

$$F_1(w) = m_B\sqrt{1-2rw+r^2}\,H_0(w)$$

for massive leptons ($\ell = \tau$) one should add $f_0(q^2)$ for $B \to D$ and $P_1(w)$ for $B \to D^*$

the form factors $f_{+(0)}(q^2)$ for $B \to D$ and $g(w), f(w), F_1(w), P_1(w)$ for $B \to D^*$ correspond to channels with definite spin-parity

* several parameterizations of the form factors are available in the literature: CLN, BCL, BGL, BSZ, …

BGL approach

(Boyd, Grinstein and Lebed '95-'97)

$$\chi_{1^-}(q^2) = \text{transverse vector susceptibility} \equiv \frac{1}{2}\frac{\partial^2}{\partial(q^2)^2}\left[q^2\,\Pi_{1^-}(q^2)\right] \xrightarrow[\text{dispersion relation}]{} \frac{1}{\pi}\int_{t_+}^{\infty} dt \frac{t\,\mathrm{Im}\Pi_{1^-}(t)}{(t-q^2)^3} \qquad t_\pm = (m_B \pm m_D)^2$$

$$\mathrm{Im}\Pi_{1^-} \propto \frac{1}{3}\sum_{i=1}^{3} \sum\!\!\!\!\!\!\int_X d\rho_X\, \delta^4(q-p_X)\,|\langle 0|\bar{c}\gamma_i b|X\rangle|^2 \qquad X = B_c^*, B\bar{D}, B\bar{D}^*, \ldots \qquad \langle 0|\bar{c}\gamma_i b|B\bar{D}\rangle \xrightarrow[\text{crossing symmetry}]{} \langle D|\bar{c}\gamma_i b|B\rangle$$

$$\frac{1}{\pi}\int_{t_+}^{\infty} dt \frac{W_+(t)\,|f_+(t)|^2}{(t-q^2)^3} \leq \chi_{1^-}(q^2) \qquad W_+(t) = \text{computable function depending on the FF}$$

* the hadronic form factors corresponding to definite spin-parity can be represented as an expansion, originating from unitarity, analyticity and crossing symmetry, in terms of the conformal variable z ($|z| \leq 1$)

$$f_+(q^2) = \frac{1}{\sqrt{\chi_{1^-}(q_0^2)}} \frac{1}{\phi_+(z(q^2), q_0^2)\; P_+(z(q^2))} \sum_{n=0}^{\infty} a_n\, z^n(q^2) \qquad z(t) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}$$

→ analytic function inside the unit circle of z

$\phi_+(z(q^2), q_0^2)$ = kinematical function (q_0^2 = auxiliary quantity)

$P_+(z(q^2))$ = Blaschke factor including resonances below the pair-production threshold $t_+ = (m_B + m_D)^2$

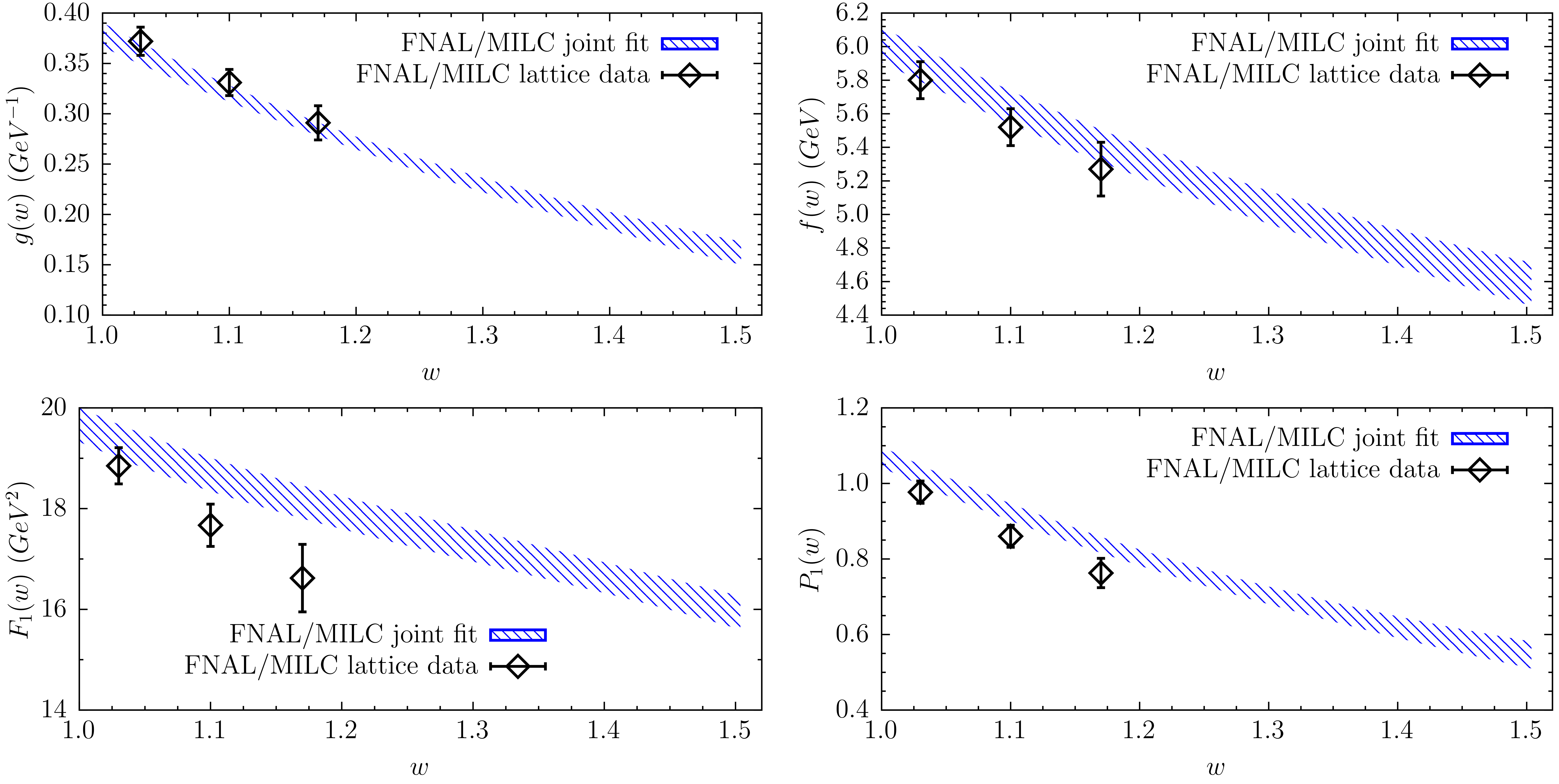
unitarity constraint: $\sum_{n=0}^{\infty} a_n^2 \leq 1$

important news: LQCD form factors for $B \to D^* \ell \nu_\ell$ decays from **FNAL/MILC (arXiv:2105.14019)**

joint (BGL) fit of LQCD points + Belle + BaBar exp. data: $|V_{cb}| \cdot 10^3 = 38.40 \pm 0.74$ $R(D^*) = 0.2483 \pm 0.0013$

(made by FNAL/MILC)

in tension with $|V_{cb}|_{incl.}$ and $R^{exp.}(D^*)$



to which theory do the joint-fit FFs belong ? QCD ? Are $|V_{cb}|$ and $R(D^*)$ pure SM predictions ?

our goals

*** no mixing among theoretical calculations and experimental data to describe the shape of the FFs

*** FFs entirely based on theory and on first principles (i.e. Lattice QCD)

*** susceptibilities (unitarity bounds) entirely based on first principles (i.e. Lattice QCD)

*** extrapolation of the FFs in the whole kinematical range indepedent on any assumption about the momentum dependence of the FFs (i.e. no model dependence due to an explicit parameterization or z-expansion truncation)

↓

Dispersion Matrix approach

Dispersion Matrix (DM) approach

PRD '21 (2105.02497)
PRD '21 (2105.07851)
PRD '22 (2105.08674)

* reappraisal and improvement of the method originally proposed by Bourrely et al. NPB '81 and Lellouch in NPB '96

$$\mathcal{M} = \begin{pmatrix} <\phi f|\phi f> & <\phi f|g_t> & <\phi f|g_{t_1}> & \dots & <\phi f|g_{t_N}> \\ <g_t|\phi f> & <g_t|g_t> & <g_t|g_{t_1}> & \dots & <g_t|g_{t_N}> \\ <g_{t_1}|\phi f> & <g_{t_1}|g_t> & <g_{t_1}|g_{t_1}> & \dots & <g_{t_1}|g_{t_N}> \\ \dots & \dots & \dots & \dots & \dots \\ <g_{t_N}|\phi f> & <g_{t_N}|g_t> & <g_{t_N}|g_{t_1}> & \dots & <g_{t_N}|g_{t_N}> \end{pmatrix}$$

inner product: $<g|h> \equiv \frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{z}\,\bar{g}(z)\,h(z)$

$$g_t(z) \equiv \frac{1}{1-\bar{z}(t)\,z}$$

$$<g_t|\phi f> \equiv \phi(z,q_0^2)f(z) \qquad <g_t|g_{t_m}> = \frac{1}{1-\bar{z}(t_m)\,z(t)}$$

$t_1, t_2, \dots, t_N$ are the N values of the squared 4-momentum transfer where the form factor f has been computed on the lattice and ***t is its value where we want to extrapolate/interpolate f(t)***

$$\text{unitarity bound:} \quad <\phi f|\phi f> \equiv \frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{z}\,|\phi(z,q_0^2)f(z)|^2 \le \chi(q_0^2)$$

in the case of interest $z_i \equiv z(t_i)$ and $\phi_i f_i \equiv \phi(z_i,q_0^2)f(t_i)$ are real numbers and the positivity of the inner product implies:

$$\det[\overline{\mathcal{M}}] = \begin{vmatrix} \chi(q_0^2) & \phi f & \phi_1 f_1 & \dots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-zz_1} & \dots & \frac{1}{1-zz_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \dots & \frac{1}{1-z_1 z_N} \\ \dots & \dots & \dots & \dots & \dots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \dots & \frac{1}{1-z_N^2} \end{vmatrix} \ge 0$$

* the explicit solution is ***a band of values***: $\beta - \sqrt{\gamma} \leq f(z) \leq \beta + \sqrt{\gamma}$

$$\beta = \frac{1}{d(z)\,\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_j} \qquad \gamma = \frac{1}{d^2(z)\,\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

χ, f_i : nonperturbative input quantities, $\phi(z), d(z), \phi_i, d_i$: kinematical coefficients depending on z and z_i

* unitarity is satisfied when $\gamma \geq 0$, which implies: $\chi \geq \chi_{\{f\}}^{DM} \equiv \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$

*** select only events with $\chi \geq \chi_{\{f\}}^{DM}$ ***

LQCD data do not have unitarity built-in because of uncertainties

* important feature: when $z \to z_j$ one has $\beta \to f_j$ and $\gamma \to 0$, i.e. the DM band collapses to f_j for $z = z_j$

for any given set of input data the DM approach reproduces exactly the known data and it allows to extrapolate the form factor in the whole kinematical range in a parameterization-independent way providing a band of values representing the results of all possible BGL fits satisfying unitarity and passing through the known points

* the DM band represents a uniform distribution which is combined with the multivariate distribution of the input data $\{f_j\}$ to generate the final band for the FF $f(z)$

* kinematical constraint(s) can be rigorously implemented in the DM approach [2105.02497, 2105.08674, 2109.15248]

nonperturbative determination of the susceptibilities

* lattice QCD simulations of 2-point functions can provide a first-principle determination of the unitarity bounds

$$\chi_{1^-}(Q^2) = \frac{1}{4} \int dt\, t^4 \, \frac{j_1(Qt)}{Qt} \, C_{1^-}(t) \qquad C_{1^-}(t) = \frac{1}{3} \sum_{i=1}^{3} \int d^3x \langle 0 | T \left[\bar{b}(x) \gamma_i c(x) \bar{c}(0) \gamma_i b \right] | 0 \rangle$$

t, Q = Euclidean time distance and momentum

$b \to c$ transition (arXiv:2105.07851)

channel	nonPT	with GS subtr.	NNLO PT	with GS subtr.
0^+ [10^{-3}]	7.58 (59)	—	6.204 (81)	—
1^- [10^{-4} GeV^{-2}]	6.72 (41)	5.88 (44)	6.486 (48)	5.131 (48)
0^- [10^{-2}]	2.58 (17)	2.19 (19)	2.41	1.94
1^+ [10^{-4} GeV^{-2}]	4.69 (30)	—	3.894	—

GS = ground state

perturbative
Bigi et al. PRD ’16, PLB ’17, JHEP ’17

differences with NNLO PT ~ 4% for 1^-, ~7% for 0^-, ~20 % for 0^+ and 1^+

$c \to s$ transition (arXiv:2105.02497)

channel	nonPT	with GS subtr.
0^+ [10^{-2}]	0.929 (64)	0.433 (133)
1^- [10^{-3} GeV^{-2}]	7.88 (41)	4.19 (36)
0^- [10^{-2}]	2.48 (15)	0.942 (91)
1^+ [10^{-3} GeV^{-2}]	4.89 (29)	3.74 (56)

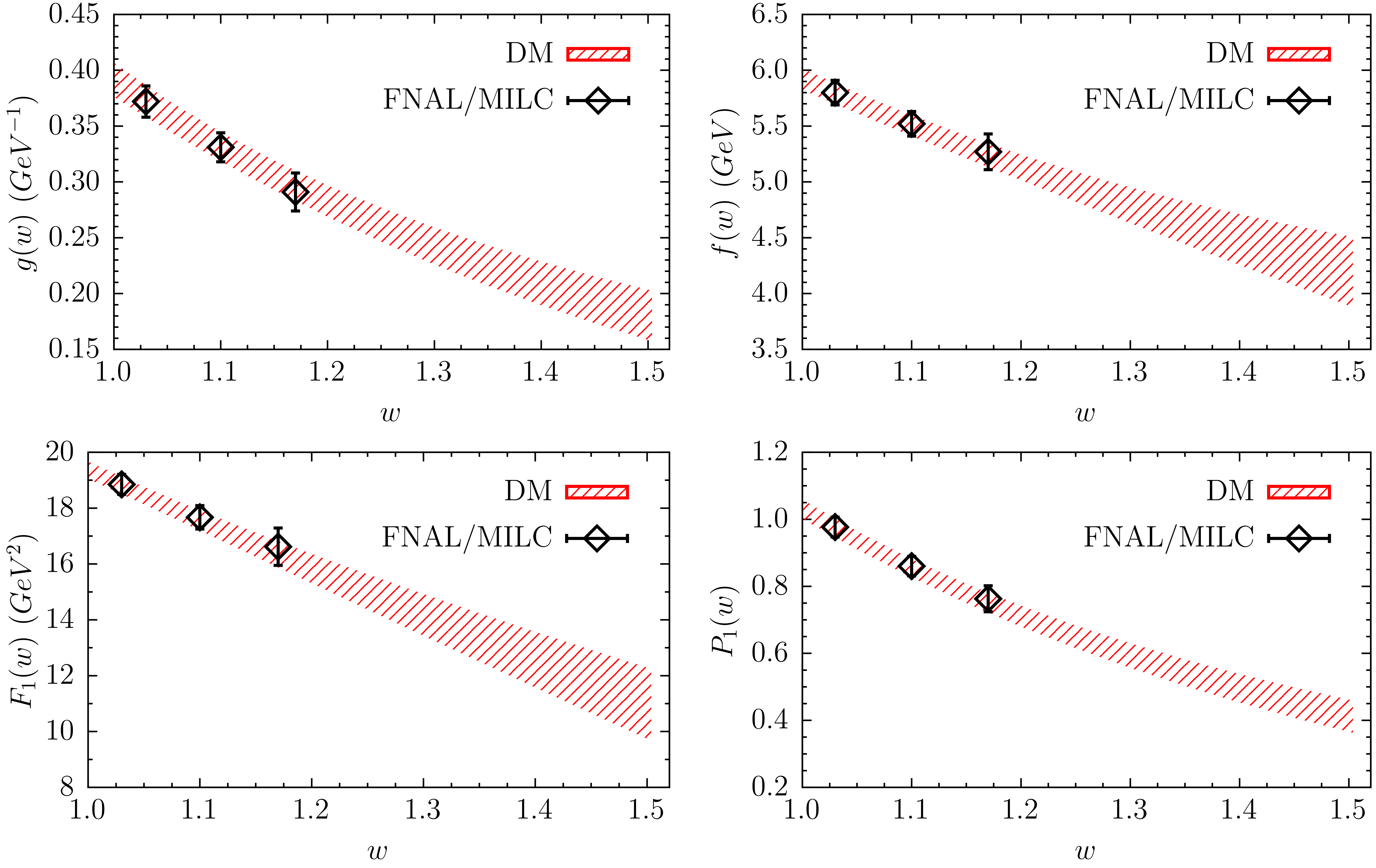
$b \to d$ transition (arXiv:2202.10285)

channel	nonPT	with GS subtr.
0^+ [10^{-2}]	2.04 (20)	—
1^- [10^{-4} GeV^{-2}]	4.88 (1.16)	4.45 (1.16)
0^- [10^{-2}]	2.34 (13)	—
1^+ [10^{-4} GeV^{-2}]	4.65 (1.02)	—

form factors for $B \to D^* \ell \nu_\ell$ decays

[arXiv:2109.15248]

* lattice QCD form factors from **FNAL/MILC arXiv:2105.14019**: synthetic data points at 3 (small) values of the recoil w
* nonperturbative susceptibilities from arXiv:2105.07851 (resonances from Bigi et al., arXiv:1707.09509)



three unitarity bounds

χ_{1^-} for g, χ_{1^+} for f and $\mathscr{F}_1$, χ_{0^-} for P_1

two kinematical constraints

$w = 1$: $\mathscr{F}_1(1) = m_B(1-r)f(1)$

$w = w_{max}$: $P_1(w_{max}) = \dfrac{\mathscr{F}_1(w_{max})}{m_B^2(1+w_{max})(1-r)\sqrt{r}}$

$R^{DM}(D^*) = 0.275 \pm 0.008$

this is a pure SM prediction !

HFLAV '21

$R^{exp.}(D^*) = 0.295 \pm 0.014$

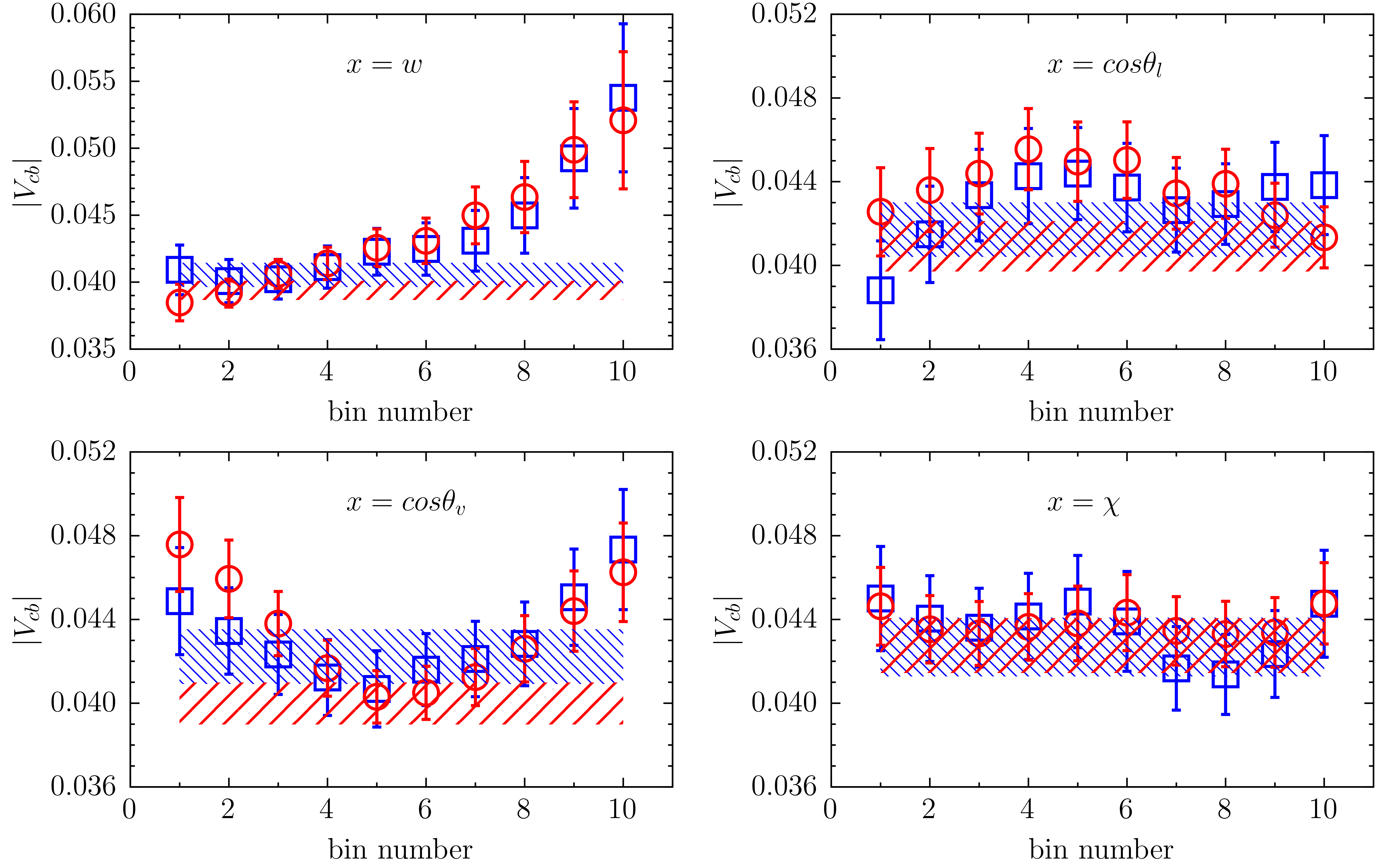
difference of $\simeq 1.2\,\sigma$

extraction of $|V_{cb}|$ from $B \to D^* \ell \nu_\ell$ decays

[arXiv:2109.15248]

*** we do not mix theoretical calculations with experimental data to describe the shape of the FFs ***

$$|V_{cb}|_i \equiv \sqrt{\frac{(d\Gamma/dx)_i^{exp}}{(d\Gamma/dx)_i^{th}}} \qquad i = 1, \ldots, N_{bins}$$



four different differential decay rates $d\Gamma/dx$ where $x = \{w, \cos\theta_v, \cos\theta_\ell, \chi\}$:

- 10 bins for each variable
- total of 80 data points

blue data: Belle 1702.01521

red data: Belle 1809.03290

bands are (correlated) weighted averages

$$|V_{cb}| = \frac{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij} |V_{cb}|_j}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} ,$$

$$\sigma^2_{|V_{cb}|} = \frac{1}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} ,$$

	experiment	$\lvert V_{cb}\rvert(x = w)$	$\lvert V_{cb}\rvert(x = \cos\theta_l)$	$\lvert V_{cb}\rvert(x = \cos\theta_v)$	$\lvert V_{cb}\rvert(x = \chi)$
Belle 1702.01521	Ref. [15]	0.0405 (9)	0.0417 (13)	0.0422 (13)	0.0427 (14)
	$\chi^2/(\text{d.o.f.})$	1.01	0.89	0.66	0.72
Belle 1809.03290	Ref. [16]	0.0394 (7)	0.0409 (12)	0.0400 (10)	0.0427 (13)
	$\chi^2/(\text{d.o.f.})$	1.21	1.36	1.99	0.38

averaging procedure

$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k \ ,$$

$$\sigma_x^2 = \frac{1}{N}\sum_{k=1}^{N}\sigma_k^2 + \frac{1}{N}\sum_{k=1}^{N}(x_k-\mu_x)^2,$$

$$|V_{cb}|_{excl.} \cdot 10^3 = 41.3 \pm 1.7$$

$|V_{cb}|_{incl.} \cdot 10^3 = 42.16 \pm 0.50$ (Bordone et al: arXiv:2107.00604)

$|V_{cb}|_{incl.} \cdot 10^3 = 41.69 \pm 0.63$ (Bernlochner et al: arXiv:2205.10274)

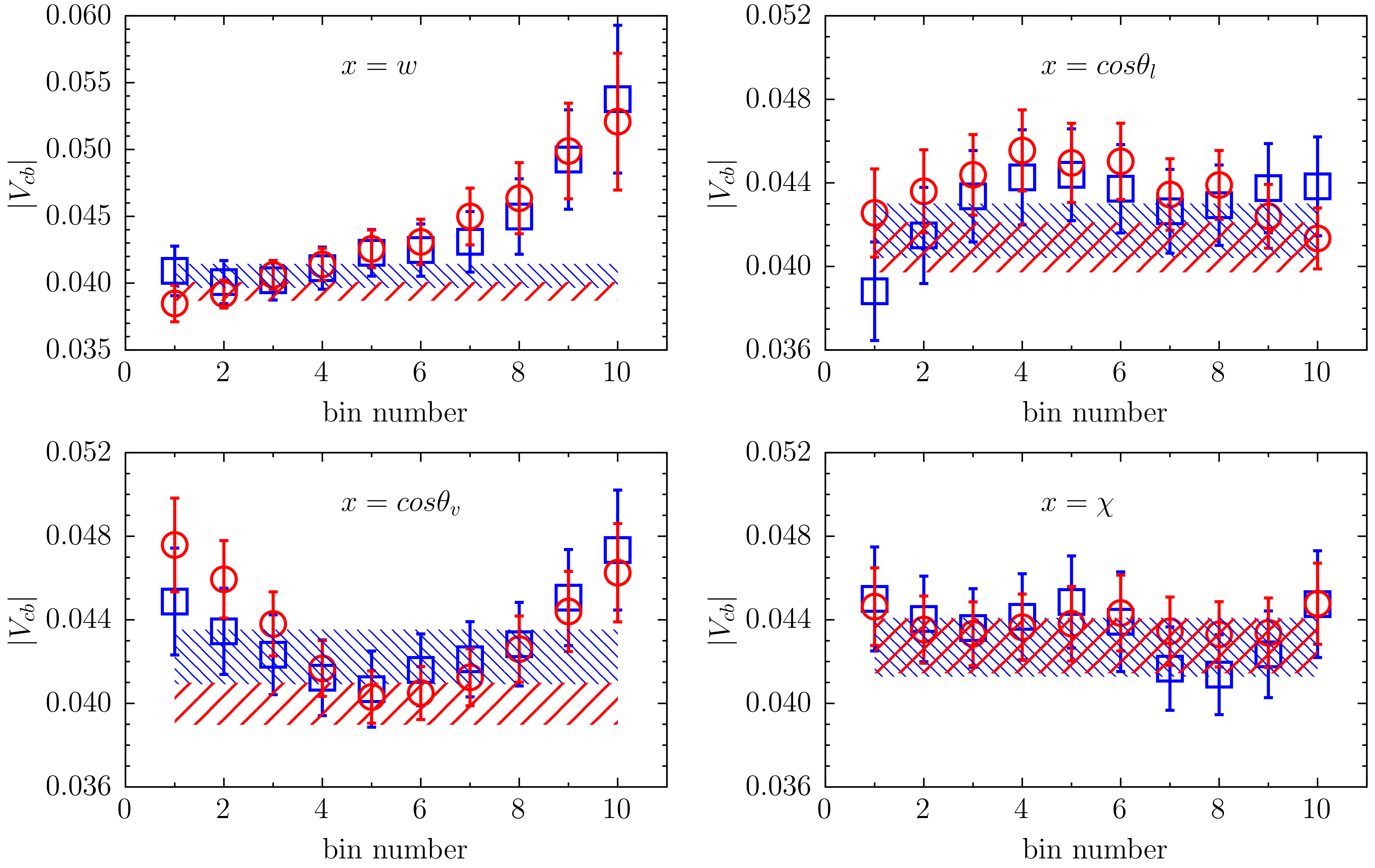
exclusive/inclusive tension reduced to less than 1σ

the use of exp. data to constrain the shape of the FFs leads to smaller errors, but it produces a bias on the extracted value of $|V_{cb}|$ since the experimental and theoretical (FNAL/MILC) slopes differ

$|V_{cb}|_{excl.} \cdot 10^3 = 39.6^{+1.1}_{-1.0}$ Gambino et al., arXiv:1905.08209

$|V_{cb}|_{excl.} \cdot 10^3 = 39.56^{+1.04}_{-1.06}$ Jaiswal et al., arXiv:2002.05726

$|V_{cb}|_{excl.} \cdot 10^3 = 38.86 \pm 0.88$ FLAG '21, arXiv:2111.09849



Remark 1

The value of $|V_{cb}|$ exhibits some dependence on the specific w-bin. The value obtained adopting a constant fit is dominated by the bins at small values of the recoil w, where direct lattice data are available and the lenght of the momentum extrapolation is limited.

Remark 2

The value of $|V_{cb}|$ deviates from a constant fit for $x = \cos(\theta_v)$. If we try a quadratic fit of the form

$$|V_{cb}| \left[1 + \delta B \cos^2(\theta_v) \right]$$

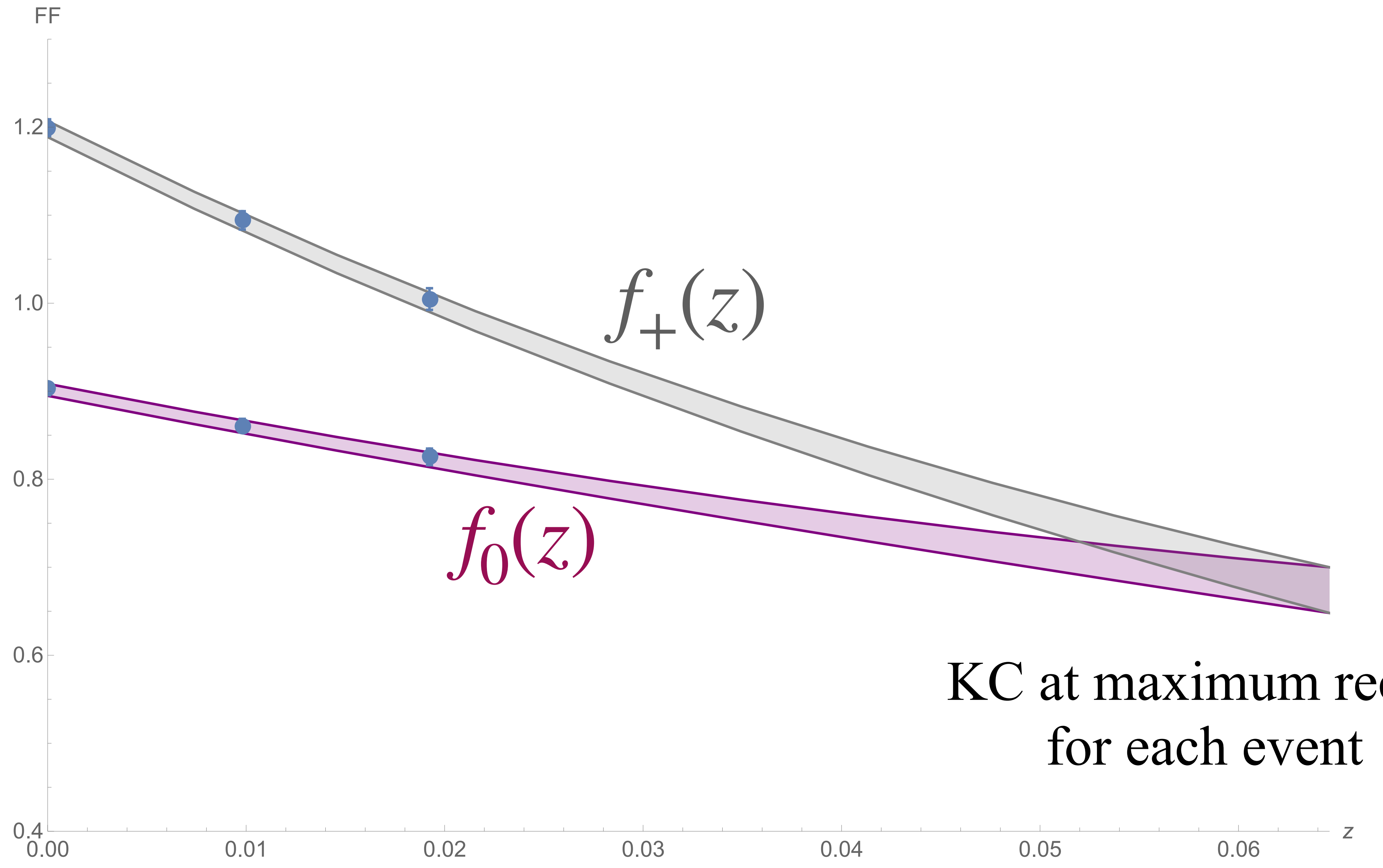
we get $\delta B \neq 0$ (2-3σ level) and $|V_{cb}|$ more consistent between the two sets of Belle data, but still in agreement with the value of $|V_{cb}|$ obtained with a constant fit

Both remarks appear to be related to a different w-slope of the theoretical FFs based on the lattice results from FNAL/MILC with respect to the Belle experimental data. This crucial issue (a kind of ***slope puzzle***) needs to be further investigated by forthcoming calculations of the FFs at non-zero recoil expected from the JLQCD Collaboration as well as by future improvements of the precision of the experimental data.

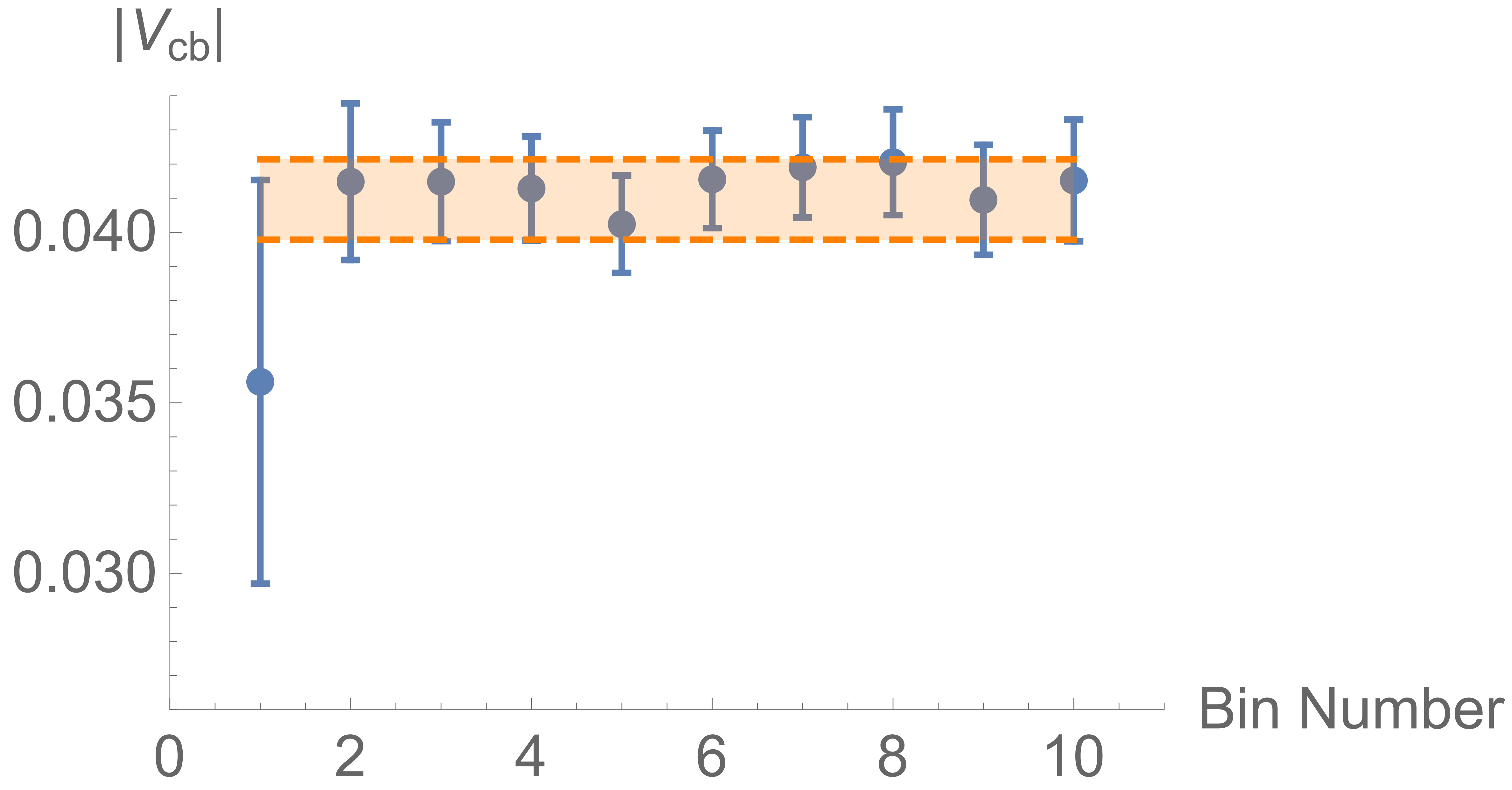
extraction of $|V_{cb}|$ from $B \to D\ell\nu_\ell$ decays

PRD '22 (2105.08674)

* lattice QCD form factors from **FNAL/MILC (arXiv:1503.07237)**: synthetic data points at 3 (small) values of the recoil
* experimental data from Belle collaboration in 10 bins (arXiv:1510.03657)



KC at maximum recoil
for each event



$$R^{DM}(D) = 0.296 \pm 0.008$$

again a pure SM prediction !

$$|V_{cb}|_{excl.} \cdot 10^3 = 41.0 \pm 1.2$$

nice consistency with $|V_{cb}|$ from $B \to D^*$

HFLAV '21 $\quad R^{exp.}(D) = 0.339 \pm 0.030$

difference of $\simeq 1.4\,\sigma$

$|V_{cb}|_{excl.} \cdot 10^3 = 40.49 \pm 0.97$	Gambino et al., arXiv:1606.08030
$|V_{cb}|_{excl.} \cdot 10^3 = 41.0 \pm 1.1$	Jaiswal et al., arXiv:1707.09977
$|V_{cb}|_{excl.} \cdot 10^3 = 40.0 \pm 1.0$	FLAG '21, arXiv:2111.09849

extraction of |V$_{cb}$| from $B_s \to D_s^{(*)} \ell \nu_\ell$ decays

* LQCD form factors from **HPQCD** arXiv:1906.00701($B_s \to D_s$) and arXiv:2105.11433 ($B_s \to D_s^*$)
* two sets of experimental data from **LHCb** collaboration: arXiv:2001.03225 and arXiv:2003.08453

summary of $|V_{cb}|^{\rm DM}$ from $B_{(s)} \to D_{(s)}^{(*)} \ell \nu_\ell$

details in the backup slides

decay	$\|V_{cb}\|^{\rm DM} \cdot 10^3$	incl. [2107.00604]	excl. [FLAG 21]
$B \to D$	41.0 ± 1.2		
$B \to D^*$	41.3 ± 1.7		
$B_s \to D_s$	41.7 ± 1.9		
$B_s \to D_s^*$	40.7 ± 2.4		
average	41.2 ± 0.8	42.16 ± 0.50 ($\simeq 1.0\sigma$)	39.36 ± 0.68 ($\simeq 1.8\sigma$)

nice consistency of $|V_{cb}|^{DM}$ among the four channels

summary of $R(D_{(s)}^{(*)})$ and polarization observables

observable	DM
$R(D_s)$	0.298 (5)
$R(D_s^*)$	0.250 (6)
$P_\tau(D_s^*)$	−0.520 (12)
$F_L(D_s^*)$	0.440 (16)

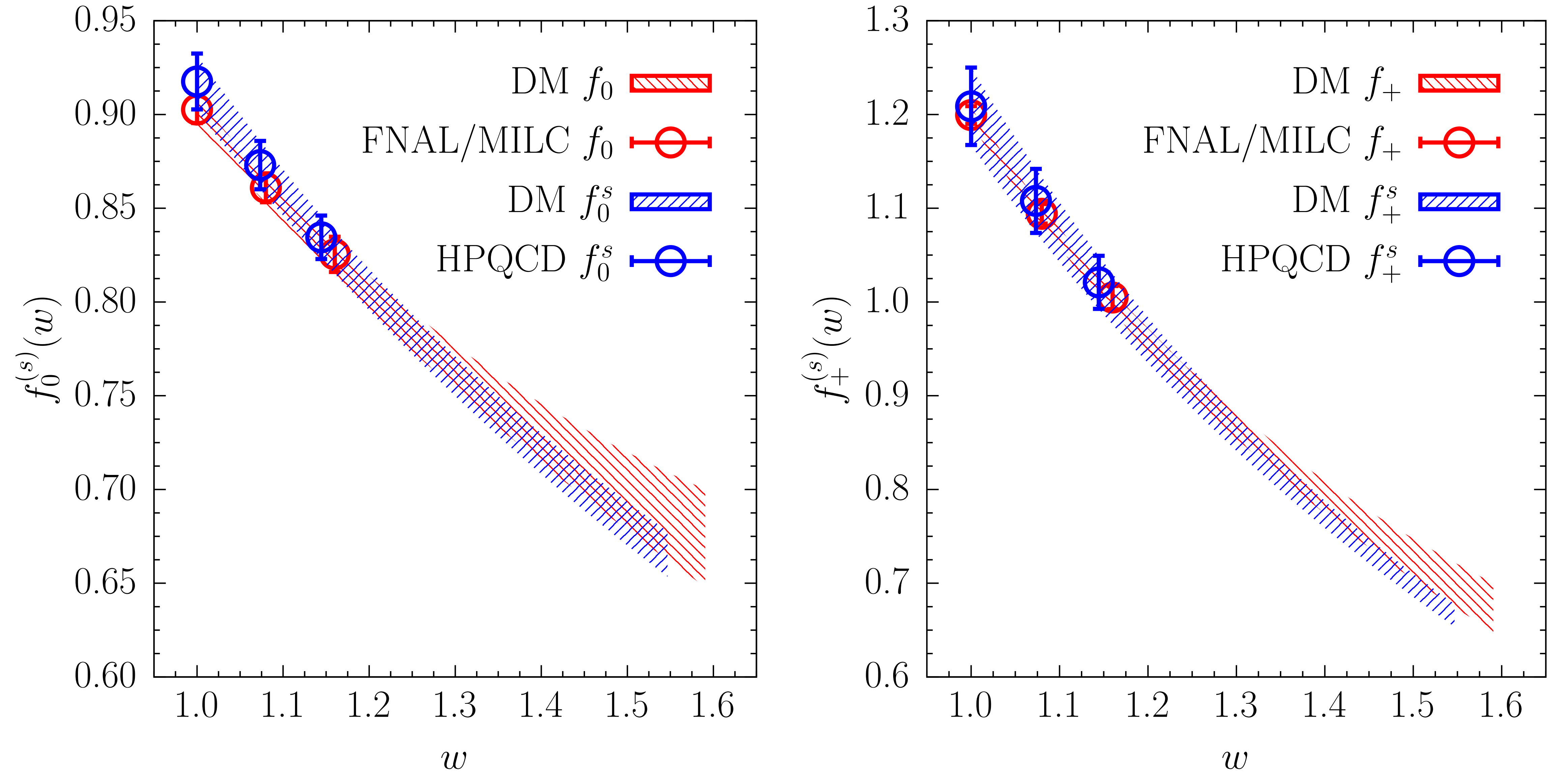
⟷ SU(3)$_F$ breaking ?

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.339 (27) (14)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (10) (10)	$\simeq 1.2\,\sigma$
$P_\tau(D^*)$	−0.52 (1)	$-0.38\,(51)\,(^{+21}_{-16})$	
$F_L(D^*)$	0.42 (1)	0.60 (8) (4)	$\simeq 2.0\,\sigma$

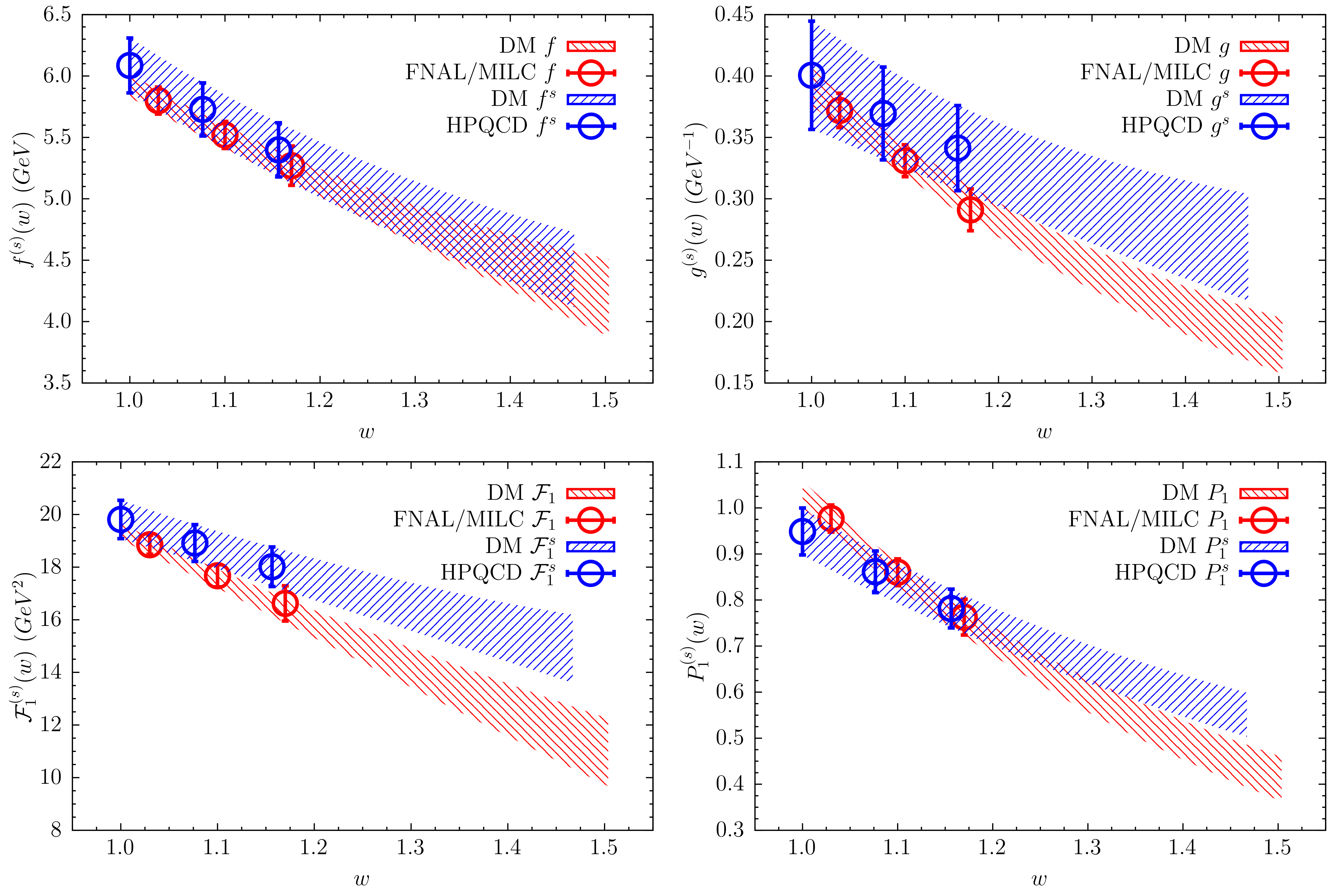
SU(3)$_F$ breaking effects

[arXiv:2204.05925]

$B_{(s)} \to D_{(s)} \ell \nu_\ell$



$B_{(s)} \to D^*_{(s)} \ell \nu_\ell$



red: u/d spectator quark

blue: strange spectator quark

<u>ratios of branching ratios</u>

$$\left.\frac{\mathcal{B}(B_s \to D_s \mu \nu_\mu)}{\mathcal{B}(B \to D \mu \nu_\mu)}\right|_{\text{LHCb}} = 1.09 \pm 0.09 \qquad \left.\frac{\mathcal{B}(B_s \to D_s \mu \nu_\mu)}{\mathcal{B}(B \to D \mu \nu_\mu)}\right|_{\text{DM}} = 1.02 \pm 0.06$$

$$\left.\frac{\mathcal{B}(B_s \to D_s^* \mu \nu_\mu)}{\mathcal{B}(B \to D^* \mu \nu_\mu)}\right|_{\text{LHCb}} = 1.06 \pm 0.10 \qquad \left.\frac{\mathcal{B}(B_s \to D_s^* \mu \nu_\mu)}{\mathcal{B}(B \to D^* \mu \nu_\mu)}\right|_{\text{DM}} = 1.19 \pm 0.11$$

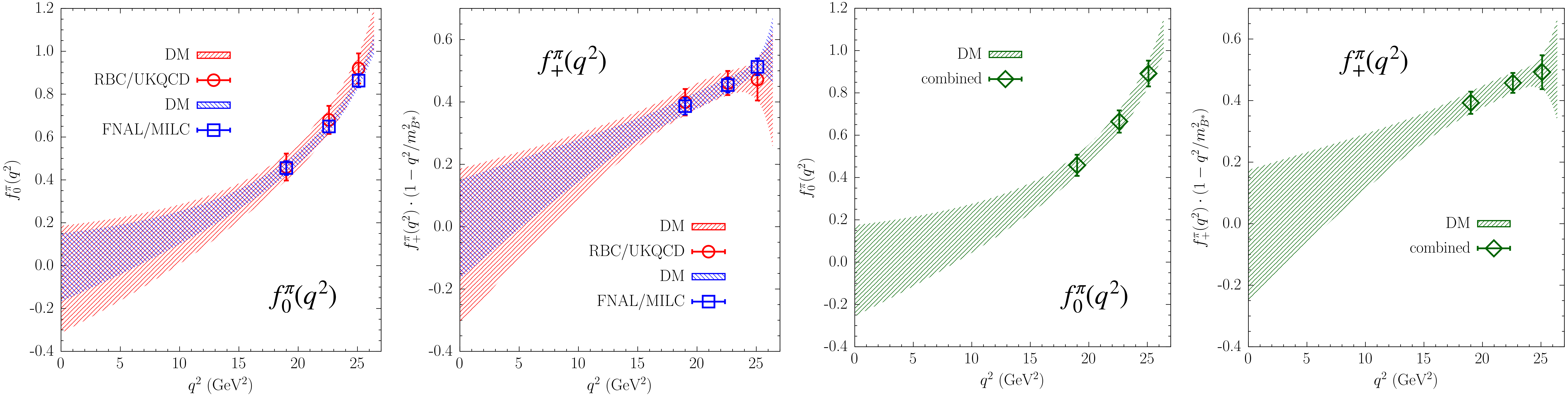
- no SU(3)$_F$ breaking effects in $B_{(s)} \to PS$
- some SU(3)$_F$ breaking effects in $B_{(s)} \to V$

need of more precise exp. and theo. data

DM bands for the form factors of the $B \to \pi \ell \nu_\ell$ decays

[arXiv:2202.10285
to appear in JHEP]

* lattice QCD form factors from **RBC/UKQCD** (arXiv:1501.05363) and **FNAL/MILC** (arXiv:1503.07839): synthetic data points at 3 (large) values of q^2 (19.0, 22.6, 25.1 GeV^2) and their combination
* nonperturbative susceptibilities



red dots: RBC/UKQCD
blue squares: FNAL/MILC

	$f(q^2 = 0)$
RBC/UKQCD	-0.06 ±0.25
FNAL/MILC	-0.01 ±0.16
combined	-0.04±0.22
LCSR	0.28±0.03

arXiv:2102.07233

combined RBC/UKQCD + FNAL/MILC

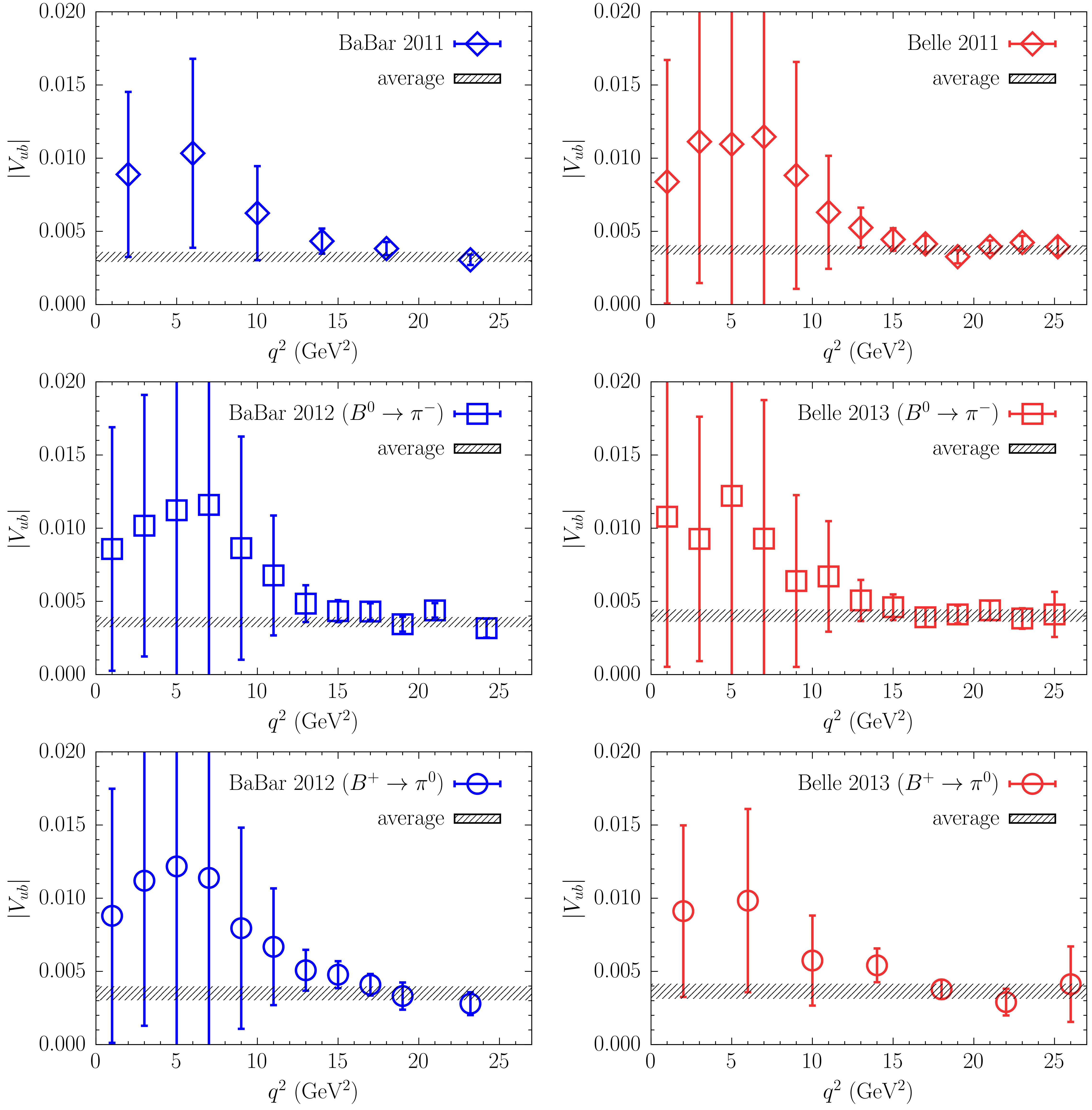
$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k,$$

$$\sigma_x^2 = \frac{1}{N}\sum_{k=1}^{N}\sigma_k^2 + \frac{1}{N}\sum_{k=1}^{N}(x_k-\mu_x)^2.$$

extraction of |V$_{ub}$| from $B \to \pi \ell \nu_\ell$ decays

* six sets of data from **Belle** and **BaBar** collaborations:

BaBar 2011, Belle 2011, BaBar 2012 ($B^0 \to \pi^-$), BaBar 2012 ($B^+ \to \pi^0$), Belle 2013 ($B^0 \to \pi^-$), Belle 2013 ($B^+ \to \pi^0$)



$$|V_{ub}|_j \equiv \sqrt{\frac{(d\Gamma/dq^2)_j^{exp}}{(d\Gamma/dq^2)_j^{th}}} \qquad j = 1, \ldots, N_{bins}$$

bands are (correlated) weighted averages

$$|V_{ub}|_n = \frac{\sum_{i,j} (\mathbf{C}^{-1})_{ij} |V_{ub}|_j}{\sum_{i,j} (\mathbf{C}^{-1})_{ij}}, \qquad \sigma^2_{|V_{ub}|_n} = \frac{1}{\sum_{i,j} (\mathbf{C}^{-1})_{ij}},$$

after averaging over the six exp.'s

input LQCD data	$\|V_{ub}\|^{DM}$ x10^3
RBC/UKQCD	3.52 (49)
FNAL/MILC	3.76 (41)
combined	3.62 (47)
exclusive (FLAG '21)	3.74 (17)
inclusive (PDG '22)	4.13 (26)

extraction of $|V_{ub}|$ from $B_s \to K \ell \nu_\ell$ decays

* LQCD form factors from **HPQCD** (1406.2279), **RBC/UKQCD** (1501.05373) and **FNAL/MILC** (1901.02561)
* two q^2-bins of experimental data from **LHCb** collaboration (2012.05143): $q^2 \leq 7$ GeV^2 and $q^2 \geq 7$ GeV^2

after averaging over the two q^2-bins

input LQCD data	$\lvert V_{ub}\rvert^{DM} \times 10^3$
RBC/UKQCD	3.93 (46)
FNAL/MILC	3.93 (35)
HPQCD	3.54 (35)
combined	3.77 (48)
inclusive (PDG '22)	4.13 (26)

* **improved** extraction of $|V_{ub}|$ from $B \to \pi \ell \nu_\ell$ (unitarization of exp. data): $|V_{ub}|^{DM} \cdot 10^3 = 3.88\ (32)$ [arXiv:2203.16213] CKM '21

***** **average of $|V_{ub}|$ from $B_{(s)} \to \pi(K) \ell \nu_\ell$: $|V_{ub}|^{DM} \cdot 10^3 = 3.85\ (27)$** ***** [paper in preparation]

difference of $\sim 0.7\sigma$ with the incl. PDG value

Conclusions

* the Dispersion Matrix approach is an attractive tool to implement unitarity and lattice QCD calculations in the analysis of exclusive semileptonic decays of hadrons. The main features are:
  - it does not rely on any assumption about the momentum dependence of the hadronic form factors
  - it can be based entirely on first principles using lattice determinations both of the relevant form factors and of the dispersive bounds (the susceptibilities) from appropriate 2-point and 3-point (Euclidean) correlation functions
  - it allows to implement unitarity and kinematical constraints in a rigorous and parameterization-independent way
  - it predicts band of values that are equivalent to the infinite number of BGL fits satisfying unitarity and KCs and reproducing exactly a given set of data points
  - it can be applied to any exclusive semileptonic decay of hadrons

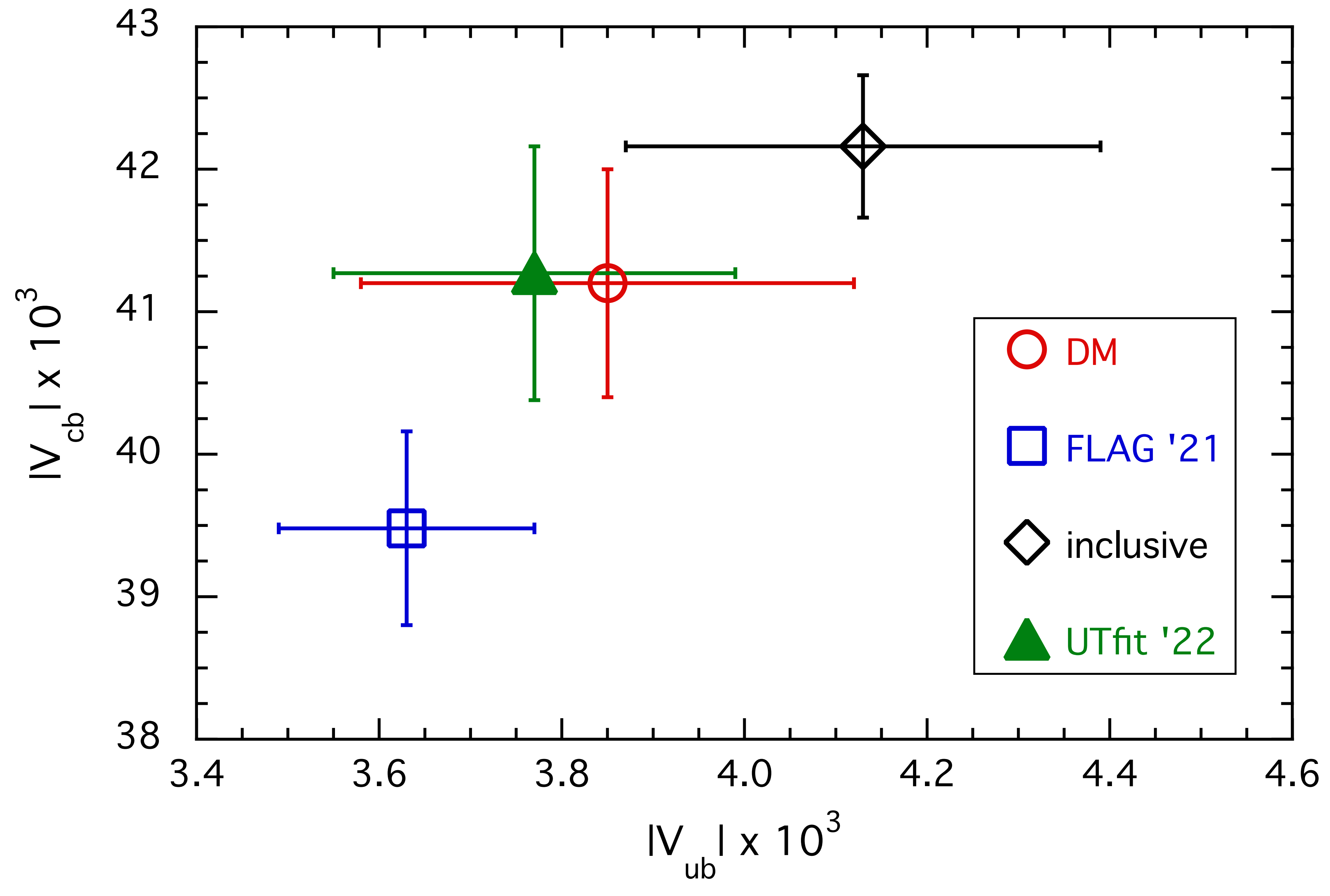
* results for $B_{(s)} \to D^{(*)}_{(s)} \ell \nu_\ell$ decays: extraction of $|V_{cb}|$ and theoretical determination of $R(D^{(*)}_{(s)})$

decay	$\lvert V_{cb}\rvert^{DM} \cdot 10^3$	incl. [2107.00604]	excl. [FLAG 21]
$B \to D$	41.0 ± 1.2		
$B \to D^*$	41.3 ± 1.7		
$B_s \to D_s$	41.7 ± 1.9		
$B_s \to D_s^*$	40.7 ± 2.4		
average	41.2 ± 0.8	42.16 ± 0.50 ($\simeq 1.0\sigma$)	39.36 ± 0.68 ($\simeq 1.8\sigma$)

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.340 (27) (13)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (11) (8)	$\simeq 1.3\,\sigma$
$R(D_s)$	0.298 (5)		
$R(D_s^*)$	0.250 (6)		

* extraction of $|V_{ub}|$ from $B_{(s)} \to \pi(K) \ell \nu_\ell$ decays:

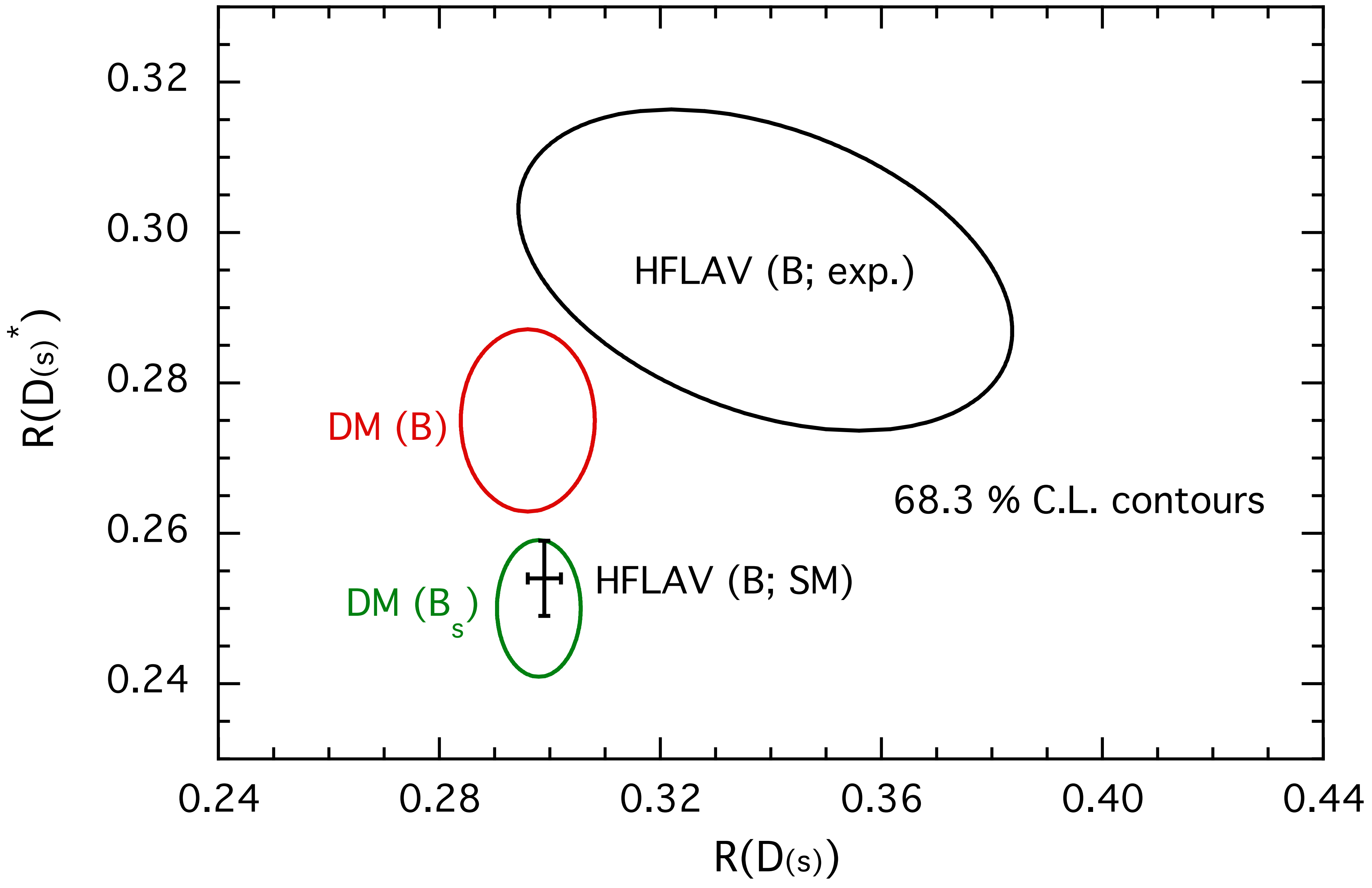
decay	$\lvert V_{ub}\rvert^{DM} \cdot 10^3$	incl. [PDG 22]	excl. [FLAG 21]
$B \to \pi$	3.88 ± 0.32		
$B_s \to K$	3.77 ± 0.48		
average	3.85 ± 0.27	4.13 ± 0.26 ($\sim 0.7\sigma$)	3.63 ± 0.14 ($\sim 0.7\sigma$)



	decays	DM	FLAG '21	inclusive	UTfit '22
$\|V_{cb}\| \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.2 (8)	39.48 (68)	42.16 (50)	41.27 (89)
$\|V_{ub}\| \cdot 10^3$	$B_{(s)} \to \pi(K)$	3.85 (27)	3.63 (14)	4.13 (26)	3.77 (22)

UTfit → CP violation and rare decays measurements into a global UT fit

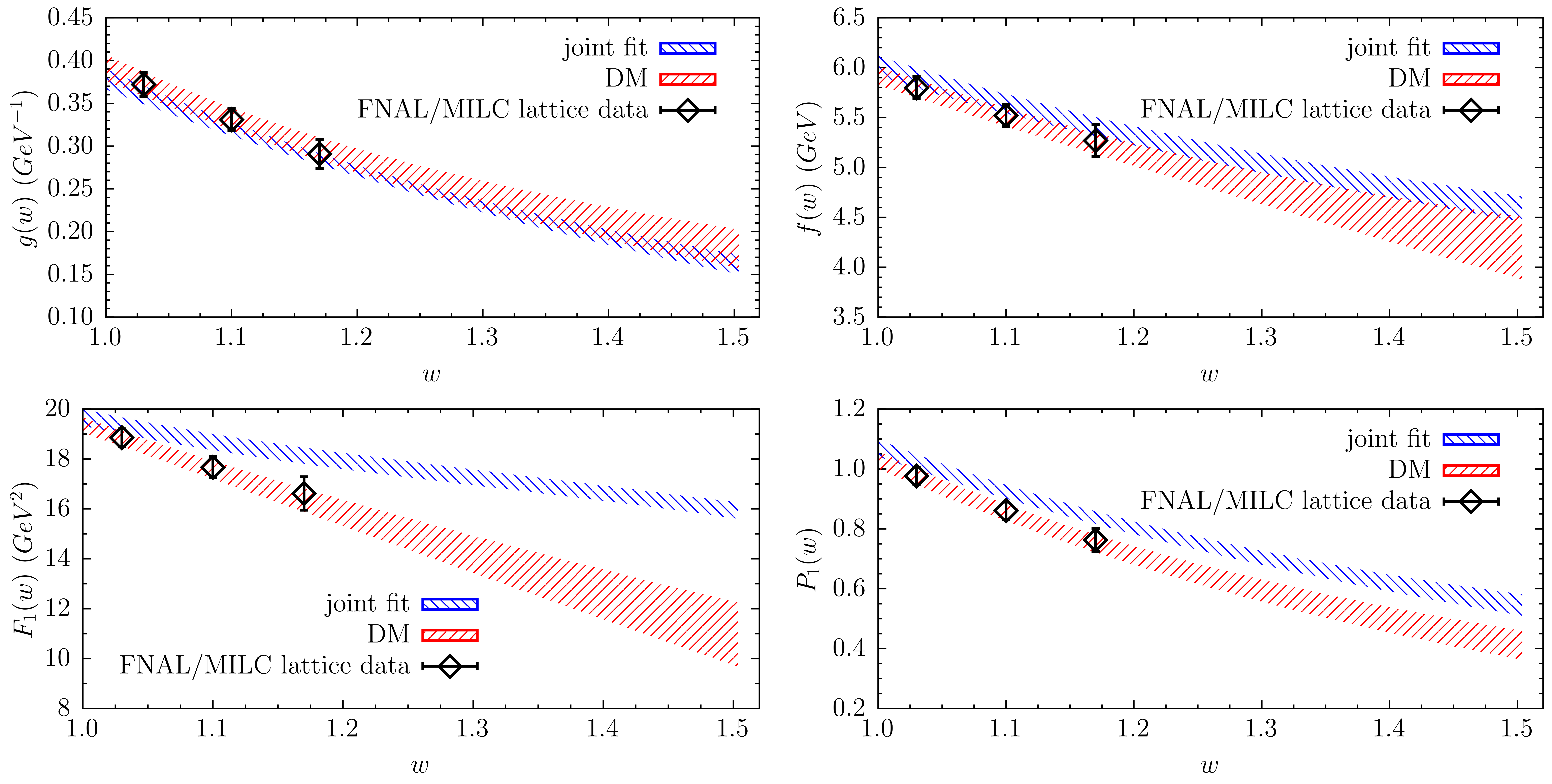
reduced tensions in $|V_{cb}|$, $|V_{ub}|$



	DM	HFLAV '21 (exp.)	HFLAV '21 (SM)
R(D)	0.296 (8)	0.339 (26) (14)	0.299 (3)
R(D*)	0.275 (8)	0.295 (10) (10)	0.254 (5)
$R(D_s)$	0.298 (5)		
$R(D_s^*)$	0.250 (6)		

reduced tension in $R(D^*)$ (using the FNAL ff's)

backup slides



joint fit

$R(D^*) = 0.2483 \pm 0.0013$

DM

$R(D^*) = 0.275 \pm 0.008$

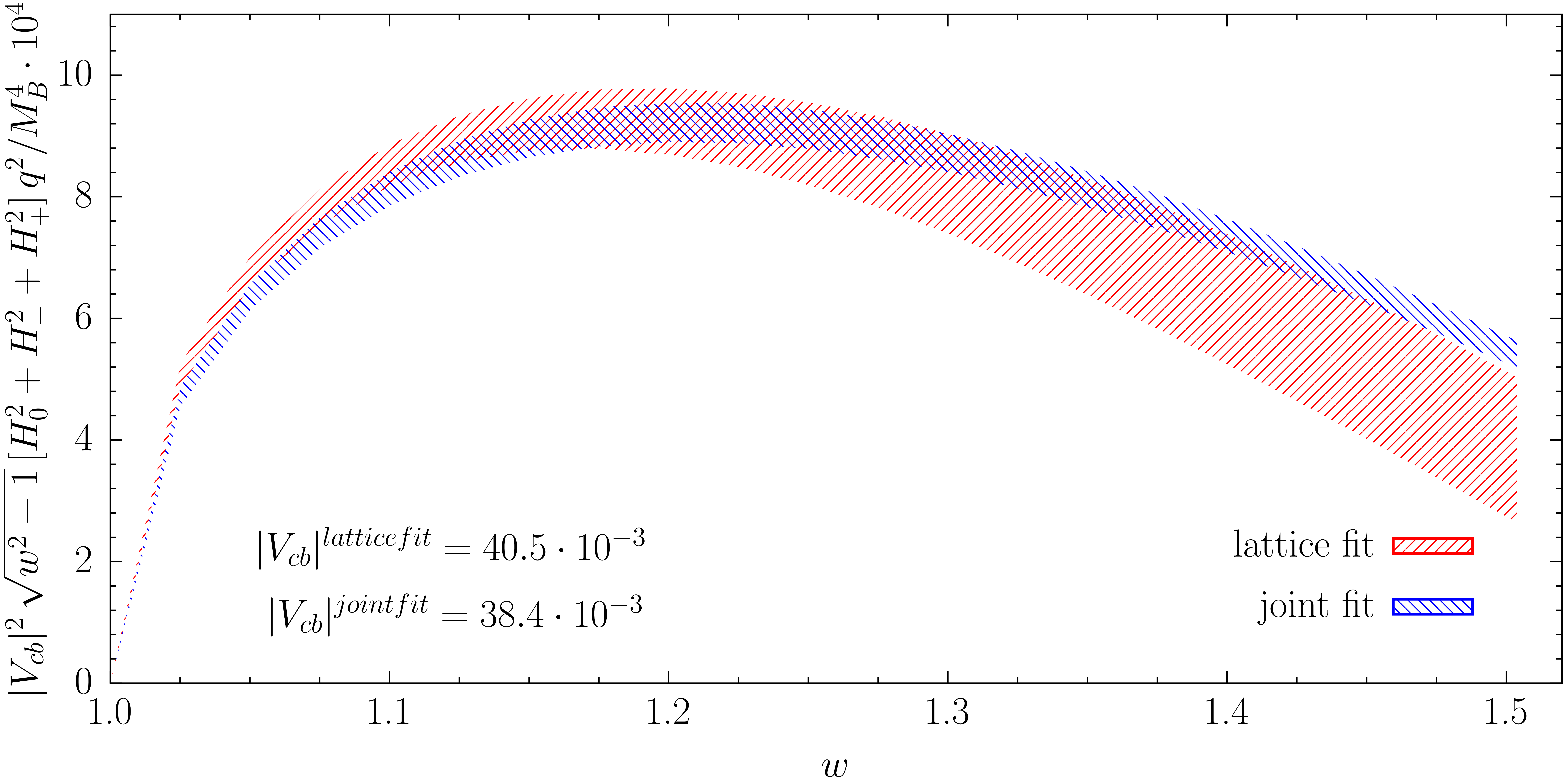
$$R(D^*) = \frac{\int_{m_\tau^2}^{q^2_{max}} dq^2 \left[\left.\frac{d\Gamma}{dq^2}\right|_{m_\ell=0} + \frac{m_\tau^2}{q^2}\frac{d\widetilde{\Gamma}}{dq^2} \right]}{\int_0^{q^2_{max}} dq^2 \left.\frac{d\Gamma}{dq^2}\right|_{m_\ell=0}}$$

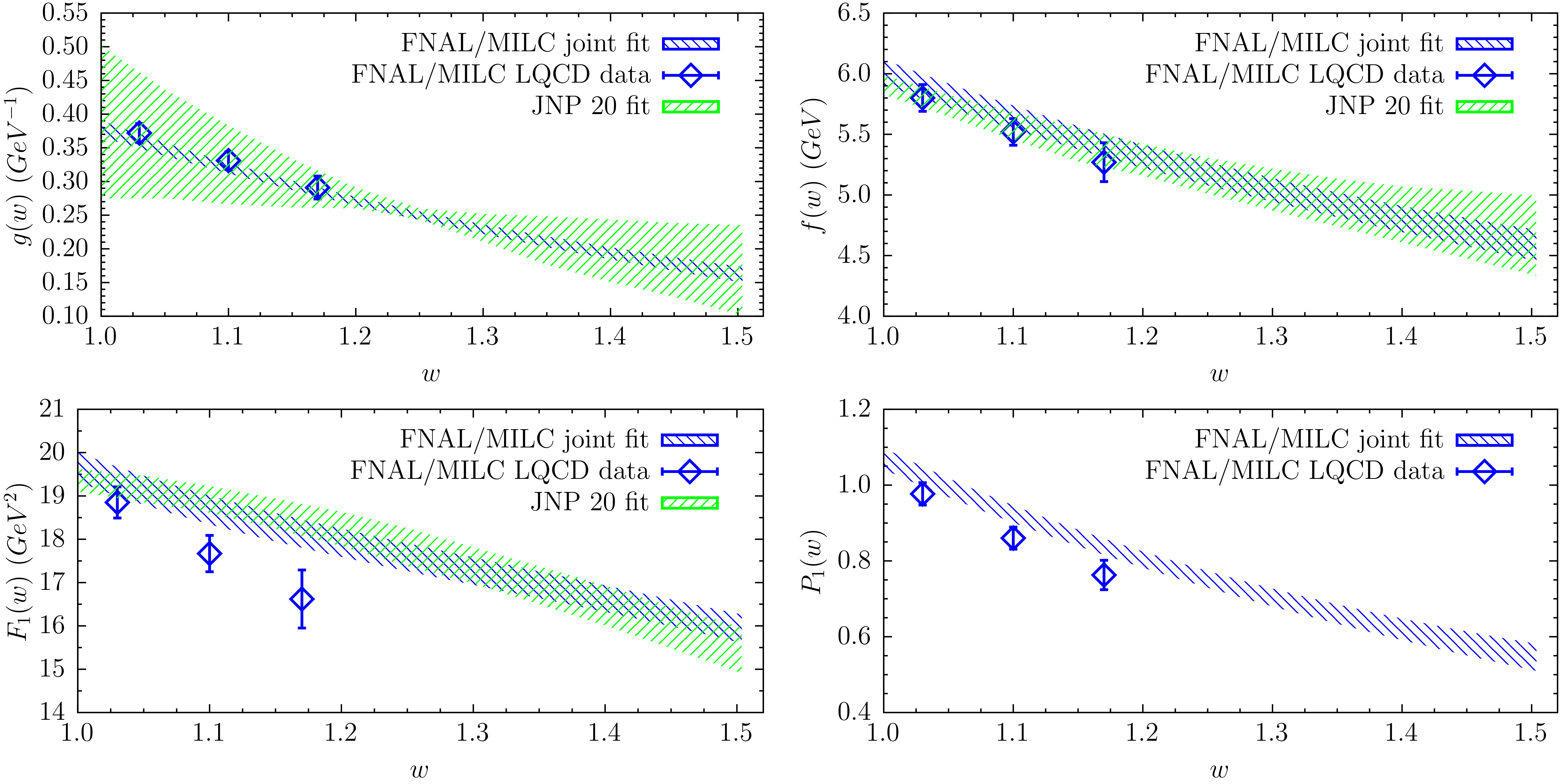
$q^2 = m_\tau^2 \simeq 3.2\ \text{GeV}^2 \to w \simeq 1.32$

at small q^2 (large recoil)
the form factor F_1 dominates

~ 10 % of R(D*)

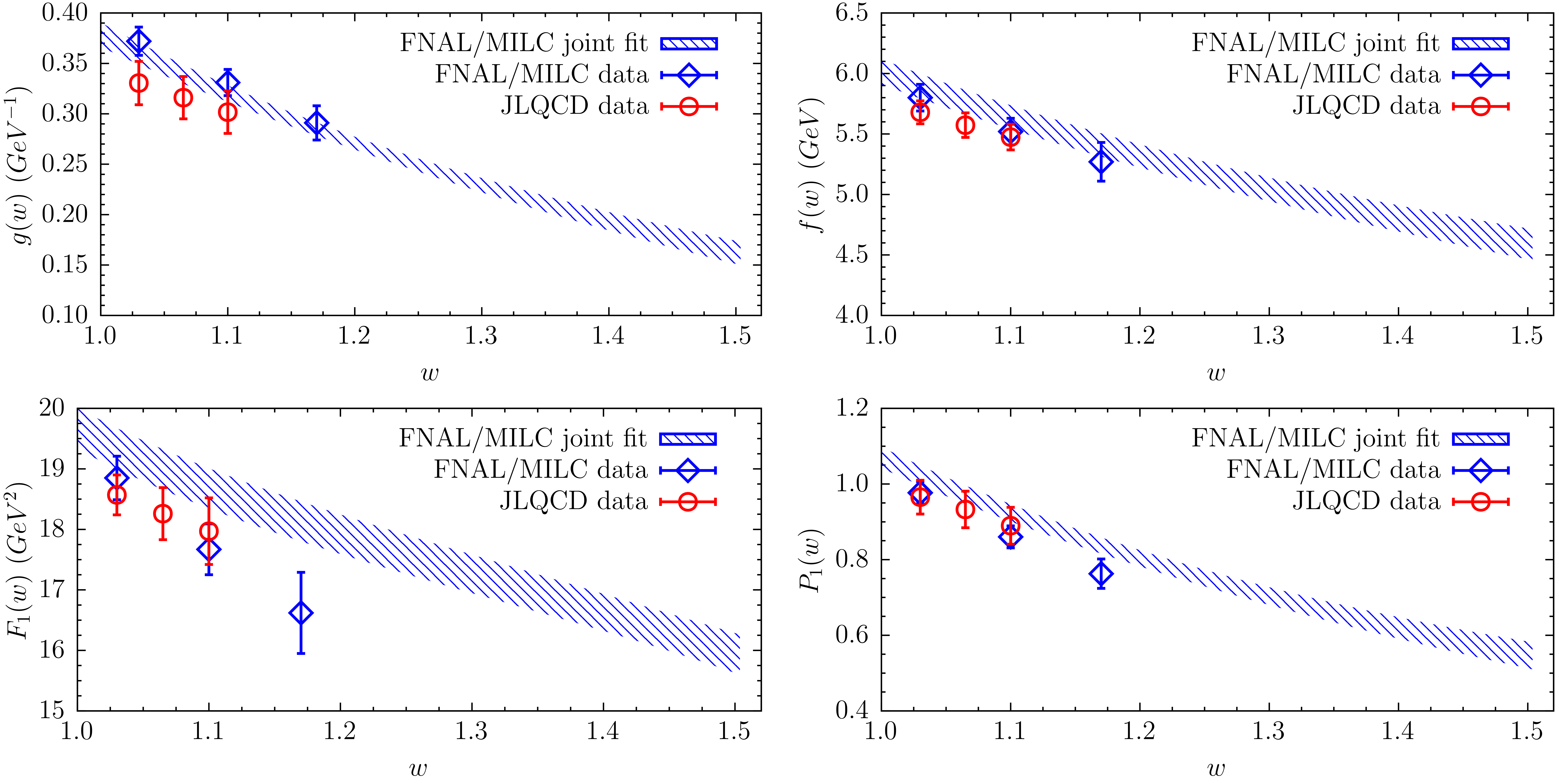
$$\frac{d\Gamma}{dw} \propto |V_{cb}|^2 \sqrt{w^2-1}\,\frac{q^2}{M_B^4}\left[H_0^2(w)+H_-^2(w)+H_+^2(w)\right] = |V_{cb}|^2\sqrt{w^2-1}\left\{\left(\frac{\mathscr{F}_1(w)}{M_B^2}\right)^2 + 2\frac{q^2}{M_B^2}\left[\left(\frac{f(w)}{M_B}\right)^2 + r^2(w^2-1)\,m_B^2\,g^2(w)\right]\right\} \qquad m_\ell = 0$$





FNAL/MILC joint fit (arXiv:2105.14019) uses Belle+BaBar data and new FNAL/MILC LQCD points

JNP 20 fit (Jaiswal et al. JHEP '20) uses Belle data + old FNAL/MILC LQCD point $h_{A_1}(1)$



FNAL/MILC data from arXiv:2105.14019

JLQCD data from slides of T. Kaneko at CKM '21

FNAL/MILC fit to lattice points (arXiv:2105.14019)

TABLE XI. Results of linear, quadratic, and unitarity-constrained cubic z expansions using only lattice-QCD data.

	Linear	Quadratic	Cubic
a_0	0.0330(12)	0.0330(12)	0.0330(12)
a_1	−0.157(52)	−0.155(55)	−0.155(55)
a_2		−0.12(98)	−0.12(98)
a_3			−0.004(1.000)
b_0	0.01229(23)	0.01229(24)	0.01229(23)
b_1	−0.002(10)	−0.003(12)	−0.003(12)
b_2		0.07(53)	0.05(55)
b_3			−0.01(1.00)
c_1	−0.0057(22)	−0.0058(25)	−0.0057(25)
c_2		−0.013(91)	−0.02(10)
c_3			0.10(95)
d_0	0.0508(15)	0.0509(15)	0.0509(15)
d_1	−0.317(59)	−0.327(67)	−0.327(67)
d_2		−0.03(96)	−0.02(96)
d_3			−0.0006(1.0000)
χ^2/dof	0.83/5	0.64/3	0.64/3
$\sum_i^N a_i^2$	0.026(16)	0.04(24)	0.04(24)
$\sum_i^N (b_i^2 + c_i^2)$	0.000193(69)	0.005(70)	0.01(18)
$\sum_i^N d_i^2$	0.103(37)	0.110(61)	0.110(52)

quadratic fit

$$\sum_{i=1}^{2} a_i^2 = 0.04 \pm 0.24 \quad ???$$

indeed: $a_2 = -0.12 \pm 0.98$

with 1σ one has $|a_2| > 1$!!!

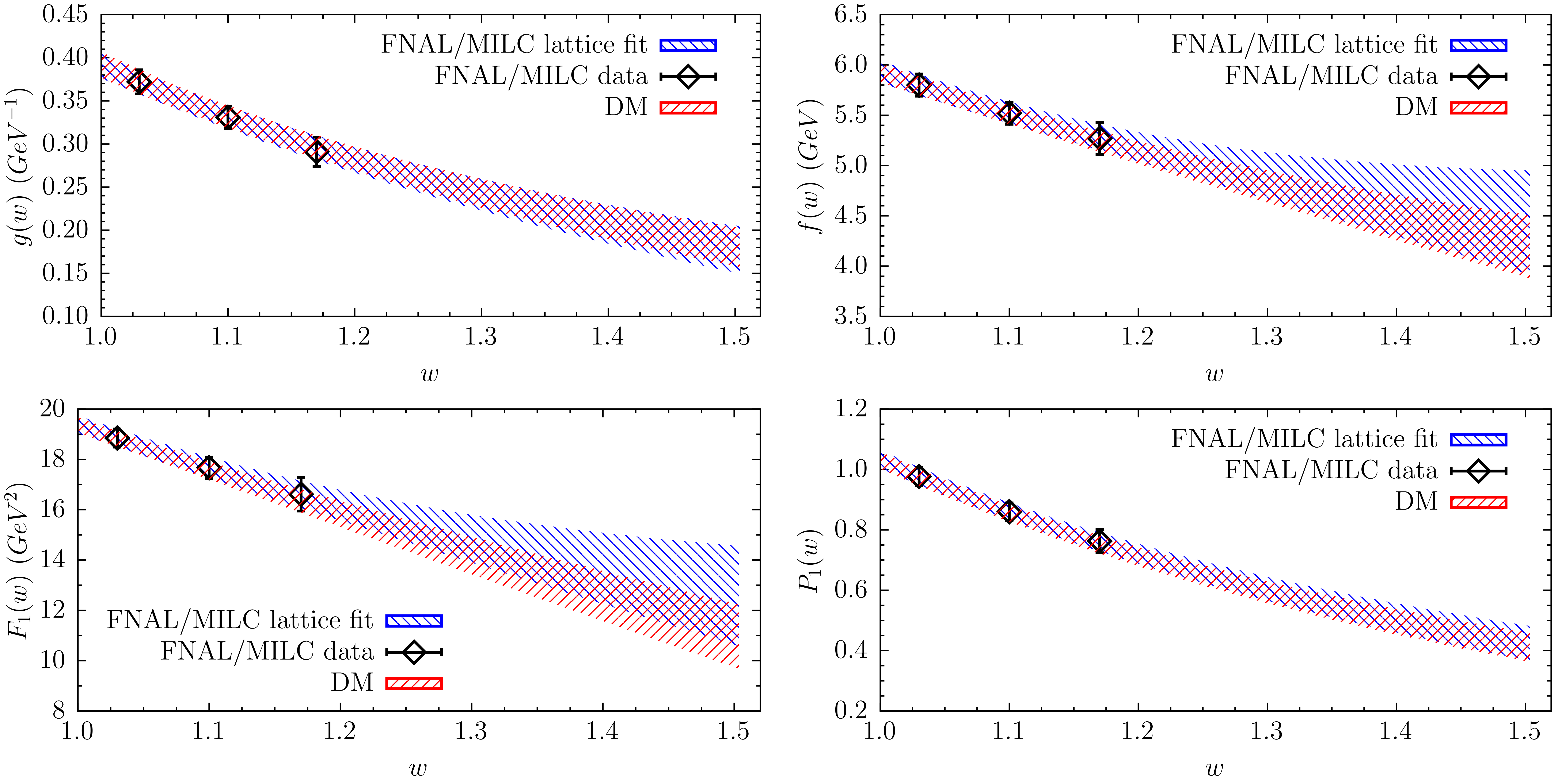
what's going on ?

linearization of the error

$$(a_2 + \delta a_2)^2 - a_2^2 \to 2|a_2|\delta a_2 \approx 0.24$$

wrong when $\delta a_2 >> |a_2|$

* comparison with FNAL/MILC “lattice fit” from arXiv:2105.14019 → blue bands: quadratic BGL fit of LQCD points only



blue bands

$$\sum_i a_i^2 \leq 1 \qquad 68\,\% \quad (g)$$

$$\sum_i (b_i^2 + c_i^2) \leq 1 \qquad 94\,\% \; (f + \mathscr{F}_1)$$

$$\sum_i d_i^2 \leq 1 \qquad 67\,\% \quad (P_1)$$

43 % of events satisfy unitarity

KC at w=1: OK

KC at w=w_{max}: not applied

red bands (DM)

100 % of events satisfying unitarity

KC at w=1: OK

KC at w=w_{max}: OK

(after iterative procedure)

* overall consistency, differences hidden in the correlations among the FFs at different values of w

* some differences for $\mathscr{F}_1(w_{max})$: some impact on $R(D^*)$

$$R(D^*) = 0.265 \pm 0.013$$

$$R(D^*) = 0.275 \pm 0.008$$

nonperturbative determination of the susceptibilities

* lattice QCD simulations can provide a first-principle determination of the unitarity bounds [arXiv:2105.02497]

time-momentum representation (Q = Euclidean 4-momentum)

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^+}(t)\ ,$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4 \frac{j_1(Qt)}{Qt}\ C_{1^-}(t)\ ,$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^-}(t)\ ,$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4 \frac{j_1(Qt)}{Qt}\ C_{1^+}(t)\ ,$$

2-point Euclidean correlation functions

$$C_{0^+}(t) = \widetilde{Z}_V^2 \int d^3x \langle 0|T\left[\bar{q}_1(x)\gamma_0 q_2(x)\ \bar{q}_2(0)\gamma_0 q_1(0)\right]|0\rangle\ ,$$

$$C_{1^-}(t) = \widetilde{Z}_V^2\ \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{q}_1(x)\gamma_j q_2(x)\ \bar{q}_2(0)\gamma_j q_1(0)\right]|0\rangle\ ,$$

$$C_{0^-}(t) = \widetilde{Z}_A^2 \int d^3x \langle 0|T\left[\bar{q}_1(x)\gamma_0\gamma_5 q_2(x)\ \bar{q}_2(0)\gamma_0\gamma_5 q_1(0)\right]|0\rangle\ ,$$

$$C_{1^+}(t) = \widetilde{Z}_A^2\ \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{q}_1(x)\gamma_j\gamma_5 q_2(x)\ \bar{q}_2(0)\gamma_j\gamma_5 q_1(0)\right]|0\rangle\ ,$$

* in arXiv:2105.02497, 2105.07851 and 2202.10285 we have calculated the $\chi's$ for the $c \to s$, $b \to c$ and $b \to u$ transitions at $Q^2 = 0$ using the $N_f = 2+1+1$ gauge ensembles generated by ETMC

$b \to c$

- subtraction of discretization effects evaluated in perturbation theory at order $\mathcal{O}(\alpha_s^0)$
- implementation of WI for the 0^+ and 0^- channels to avoid exactly contact terms
- use of the ETMC ratio method (hep-lat/0909.3187) to reach the physical b-quark point

applicable also at $Q^2 \neq 0$

- start from a set of input data $\{f_i\}$ with a given covariance matrix C_{ij} and a (eventually correlated) susceptibility χ
- generate a multivariate distribution of N_{boot} events
- for each event $k = 1,2,\ldots,N_{boot}$ evaluate the lower $f^k_{lo}(t)$ and upper $f^k_{up}(t)$ values of the form factor at a given t

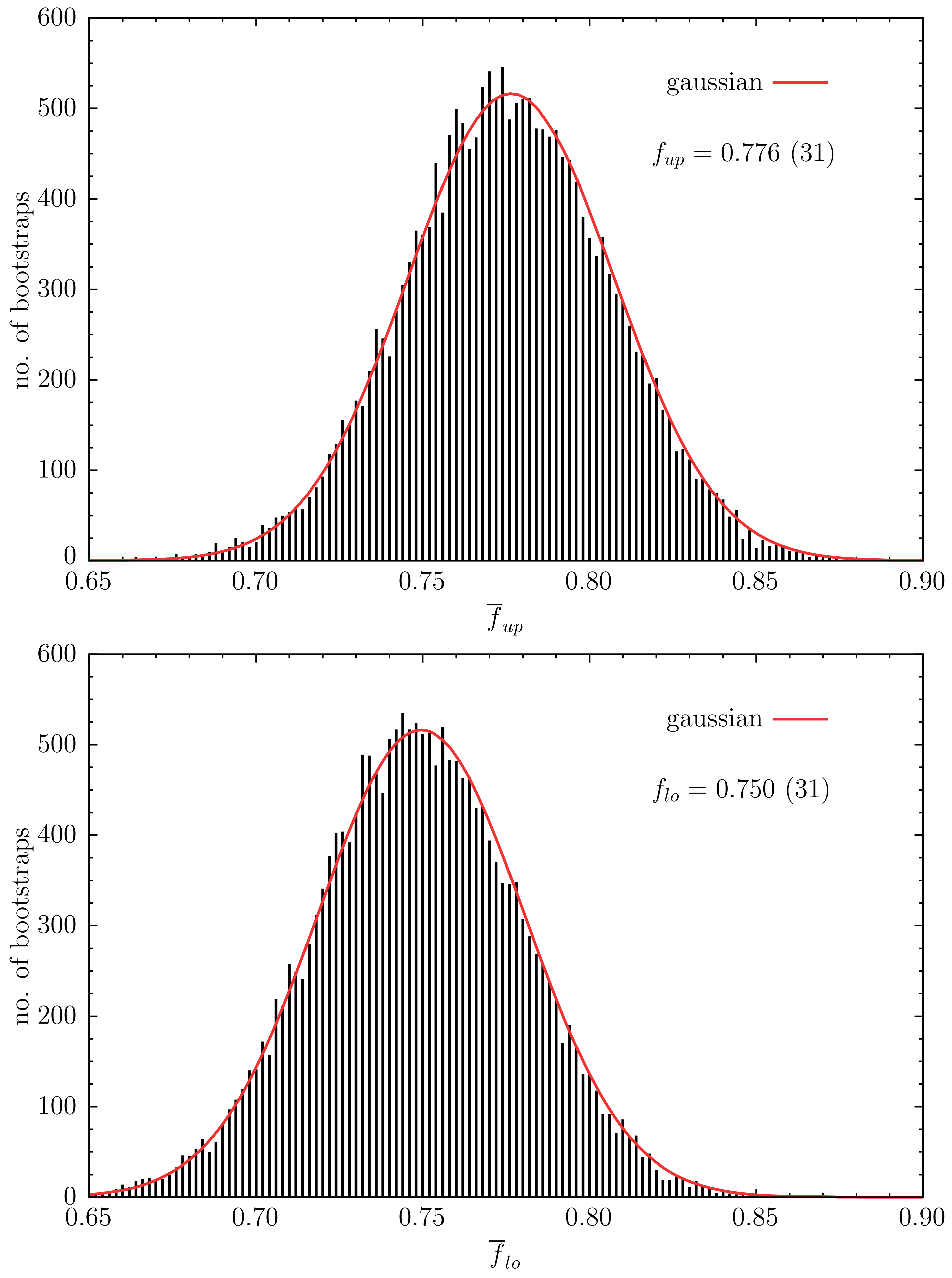


FIG. 1. Histograms of the values of $\bar{f}_{\rm up}$ (upper panel) and $\bar{f}_{\rm lo}$ (lower panel) for the bootstrap events that pass the unitarity filter in the case of the vector form factor $f_+(t = 0\ {\rm GeV}^2)$ of the $D \to K$ transition.

averages: $\bar{f}_{lo(up)}(t) = \dfrac{1}{N_{boot}} \sum_{k=1}^{N_{boot}} f^k_{lo(up)}(t)$

covariance: $C_{L(U),L(U)} \equiv \dfrac{1}{N_{boot}-1} \sum_{k=1}^{N_{boot}} \left[f^k_{lo(up)}(t) - \bar{f}_{lo(up)}(t)\right] \left[f^k_{lo(up)}(t) - \bar{f}_{lo(up)}(t)\right]$

correlated bivariate: $P_{LU}(f_L, f_U) = \dfrac{\sqrt{\det(C^{-1})}}{2\pi} e^{-\frac{1}{2}\left[C^{-1}_{LL}(f_L-\bar{f}_{lo})^2 + 2C^{-1}_{LU}(f_L-\bar{f}_{lo})(f_U-\bar{f}_{up}) + C^{-1}_{UU}(f_U-\bar{f}_{up})^2\right]}$

uniform distribution: $P(f) = \dfrac{1}{f_U - f_L}\theta(f - f_L)\theta(f_U - f)$

final average: $f(t) \equiv \dfrac{\bar{f}_{lo}(t) + \bar{f}_{up}(t)}{2}$

final variance: $\sigma^2_f(t) \equiv \dfrac{1}{12}\left[\bar{f}_{lo}(t) - \bar{f}_{up}(t)\right]^2 + \dfrac{1}{3}\left[C_{LL}(t) + C_{UU}(t) + C_{LU}(t)\right]$

* kinematical constraint: $f_+(0) = f_0(0)$

for each event $k = 1,2,\ldots,N_{boot}$: $f(0)|_{lo} \leq f(0) \leq f(0)|_{up}$

$$f_0(0)|_{lo} = \max(f_0(0)|_{lo}, f_+(0)|_{lo})$$
$$f_0(0)|_{up} = \min(f_0(0)|_{up}, f_+(0)|_{up})$$

addition of one (common) point at $q^2 = 0$ in the dispersion matrices of f_0 and f_+ uniformely distributed in $\left[f(0)|_{lo}, f(0)|_{up}\right]$

* when the percentage of events satisfying the unitarity and/or kinematical constraints is too low, the reliability of the DM bands may become questionable and we apply a procedure to recover a larger percentage of events passing the filters

<u>skeptical procedure</u> (from D'Agostini, arXiv: 2001.03466)

1. modify the standard deviations σ_i of the input data by a factor r_i while keeping fixed the averages f_i (a common value r is typically enough)
2. enlarge the number of bootstraps by extracting Nr values of r distributed according to a exponential distribution
3. select the the events passing the filters and compute their average value r*
4. select the event with r closest to r*

<u>iterative procedure</u> [arXiv:2109.15248]

1. recalculate the mean values and the covariance matrix of the subset of inpout data passing the filters
2. generate a new multivariate distribution
3. check unitarity and kinematical constraints
4. repeat steps 1-3 until convergence of the percentage of events passing the filters is reached

⇓

simpler and more effective procedure

experimental data for $B \to D^* \ell \nu_\ell$ decays

- two sets of data from Belle collaboration arXiv:1702.01521 and arXiv:1809.03290
- four different differential decay rates $d\Gamma/dx$ where $x = \{w, \cos\theta_v, \cos\theta_\ell, \chi\}$: 10 bins for each variable

total of 80 data points

*** we do not mix theoretical calculations with experimental data to describe the shape of the FFs ***

$$|V_{cb}|_i \equiv \sqrt{\frac{(d\Gamma/dx)_i^{exp}}{(d\Gamma/dx)_i^{th}}} \qquad i = 1, \ldots, N_{bins}$$

* issue with the covariance matrix C_{ij}^{exp} of the Belle data: $\Gamma^{exp} \equiv \sum_{i=1}^{10} \left(\frac{d\Gamma}{dx}\right)_i^{exp}$ should be the same for all the variables x

- we recover the above property by evaluating the correlation matrix of the experimental ratios

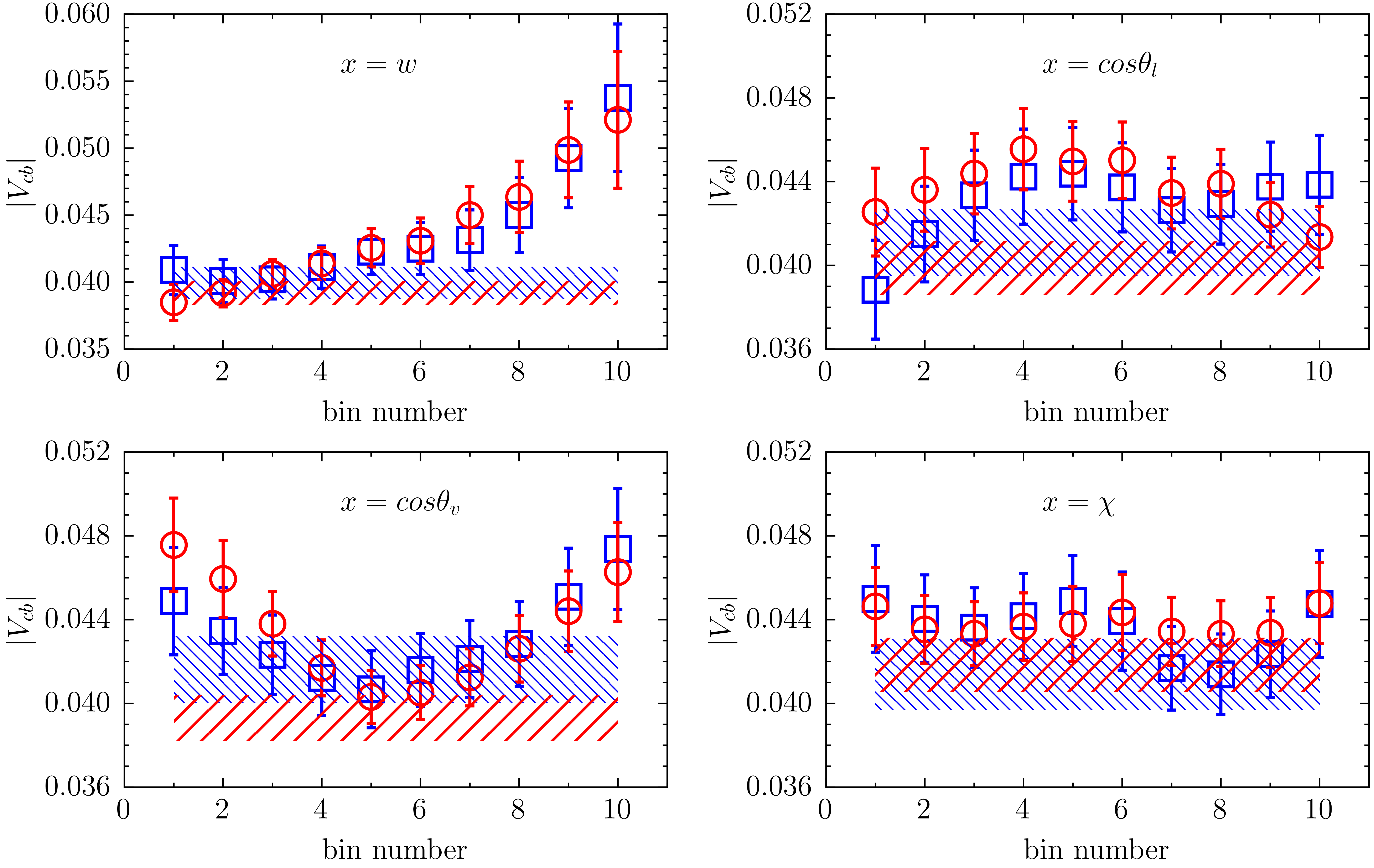
$$\frac{1}{\Gamma^{exp}} \left(\frac{d\Gamma}{dx}\right)_i^{exp}$$

and by considering the new covariance matrix of the experimental data given by (see arXiv:2105.08674)

$$\widetilde{C}_{ij}^{exp} \to \rho_{ij}^{ratios} \sqrt{C_{ii}^{exp.} C_{jj}^{exp}}$$

extraction of $|V_{cb}|$ from $B \to D^* \ell \nu_\ell$ decays

original covariance matrix of Belle data



blue data: Belle 1702.01521

red data: Belle 1809.03290

bands are (correlated) weighted averages

$$|V_{cb}| = \frac{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij} |V_{cb}|_j}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} \, ,$$

$$\sigma^2_{|V_{cb}|} = \frac{1}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} \, ,$$

$$|V_{cb}| \cdot 10^3 = 40.5 \pm 1.7$$

$R(D)$, $R(D^*)$ and polarization observables

* pure theoretical and parameterization-independent determinations within the DM approach

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.339 (26) (14)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (10) (10)	$\simeq 1.2\,\sigma$
$P_\tau(D^*)$	-0.52 (1)	-0.38 (51) $\left(^{+21}_{-16}\right)$	
$F_L(D^*)$	0.42 (1)	0.60 (8) (4)	$\simeq 2.0\,\sigma$

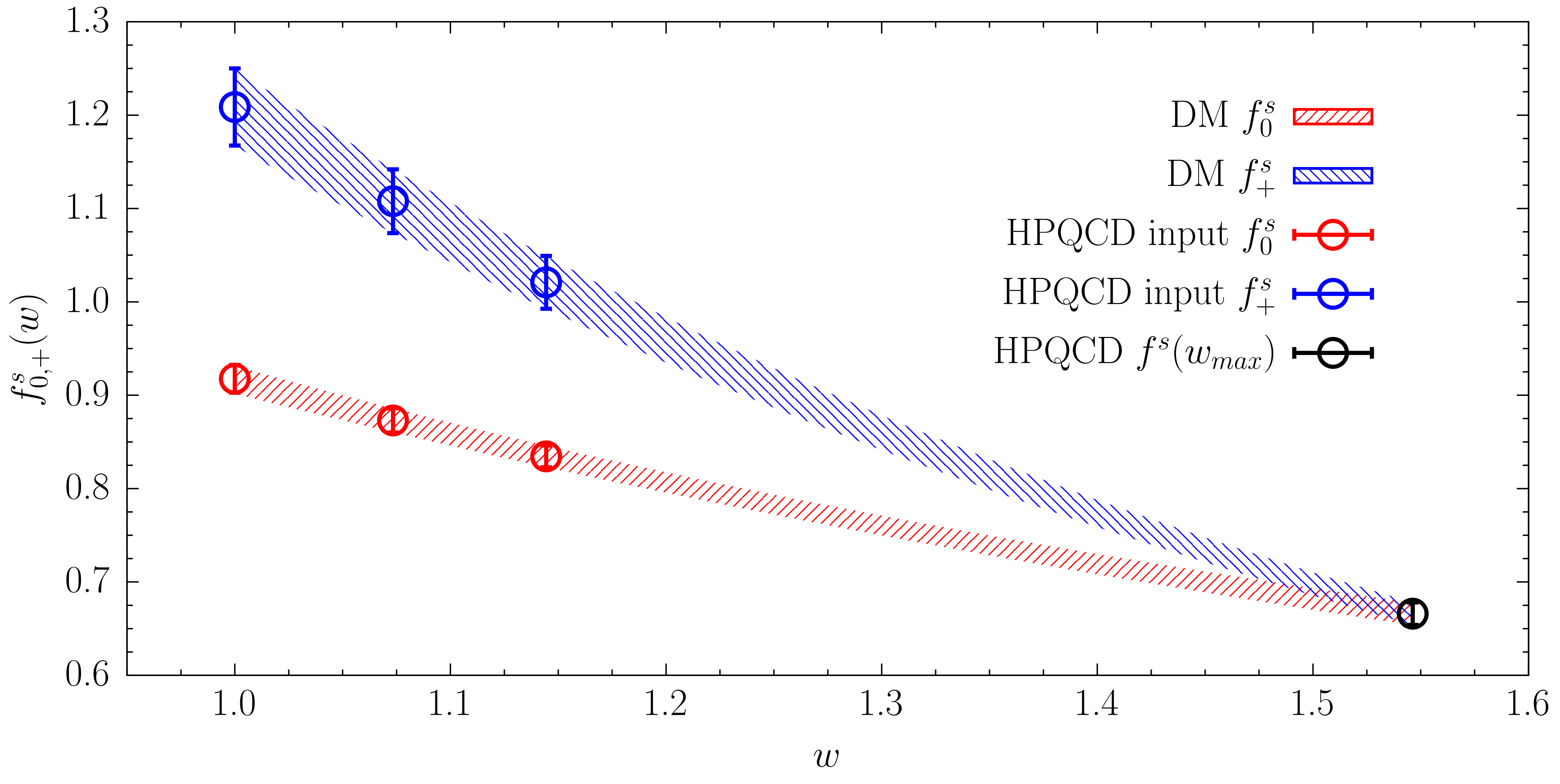
***** exp/SM tension significantly reduced for *R*(*D**) *****

form factors for $B_s \to D_s^{(*)} \ell \nu_\ell$ decays

[arXiv:2204.05925]

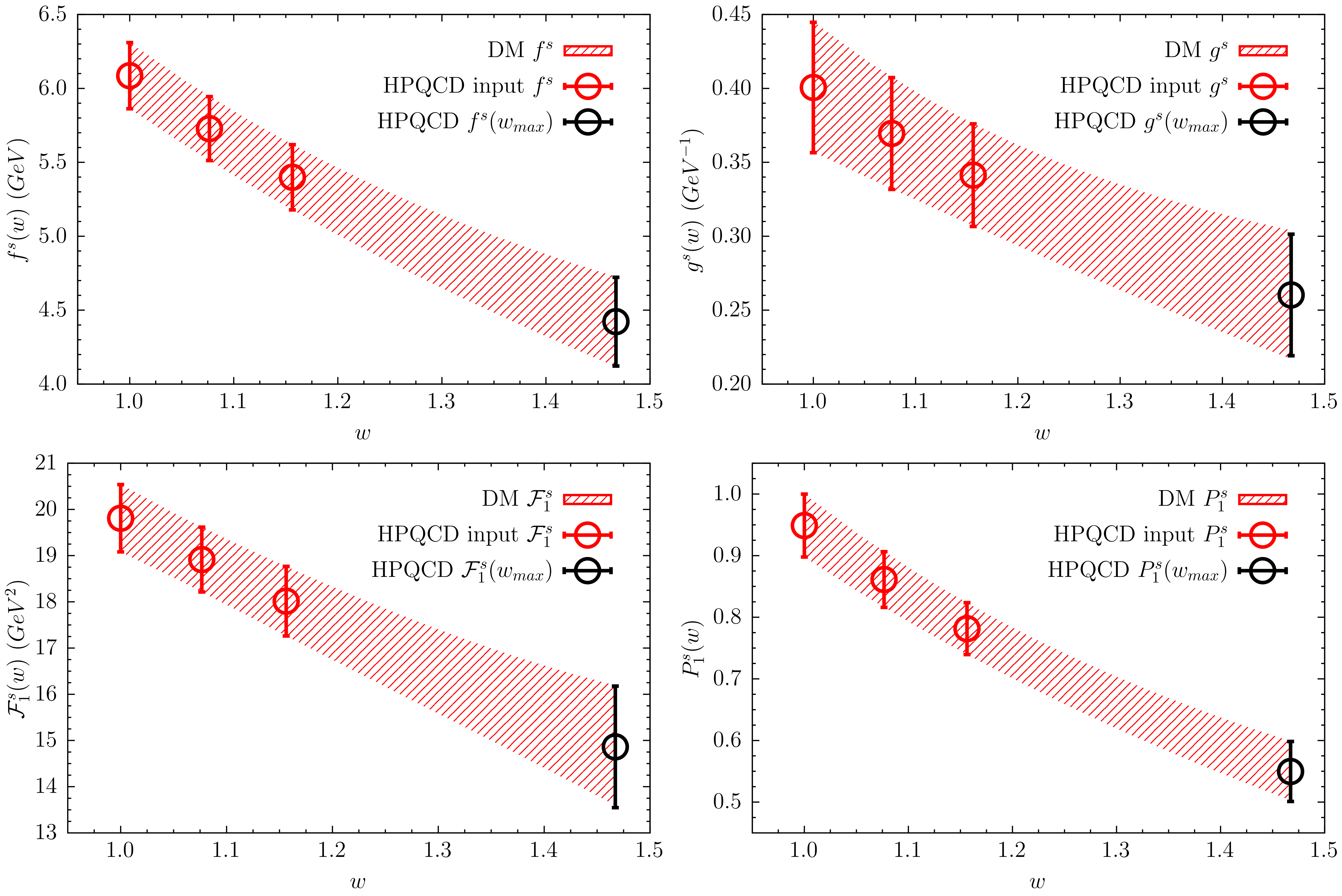
* lattice QCD form factors from **HPQCD arXiv:1906.00701**($B_s \to D_s$) and **arXiv:2105.11433** ($B_s \to D_s^*$) in the form of BCL fits in the whole kinematical range
* we extract 3 data points for the FFs at small values of the recoil, to which we apply the DM approach

$B_s \to D_s \ell \nu_\ell$



* nice agreement in the whole kinematical range

$B_s \to D_s^* \ell \nu_\ell$



extraction of $|V_{cb}|$ from $B_s \to D_s^{(*)} \ell \nu_\ell$ decays

two different runs at LHC

* two sets of experimental data from LHCb collaboration: arXiv:2001.03225, 2003.08453, 2103.06810

* first analysis: ratios of branching ratios [2103.06810]

$$\frac{\mathscr{B}(B_s \to D_s \mu \nu_\mu)}{\mathscr{B}(B \to D \mu \nu_\mu)} = 1.09 \pm 0.05_{stat} \pm 0.06_{syst} \pm 0.05_{inputs} = 1.09 \pm 0.09$$

$$\frac{\mathscr{B}(B_s \to D_s^* \mu \nu_\mu)}{\mathscr{B}(B \to D^* \mu \nu_\mu)} = 1.06 \pm 0.05_{stat} \pm 0.07_{syst} \pm 0.05_{inputs} = 1.06 \pm 0.10$$

- using the PDG values for $\mathscr{B}(B \to D^{(*)} \mu \nu_\mu)$ and the B_s-meson lifetime one gets

$$\Gamma^{\rm LHCb}(B_s \to D_s \mu \nu_\mu) = (1.04 \pm 0.10) \cdot 10^{-14} \ \text{GeV}$$
$$\Gamma^{\rm LHCb}(B_s \to D_s^* \mu \nu_\mu) = (2.26 \pm 0.24) \cdot 10^{-14} \ \text{GeV}$$

to be compared with

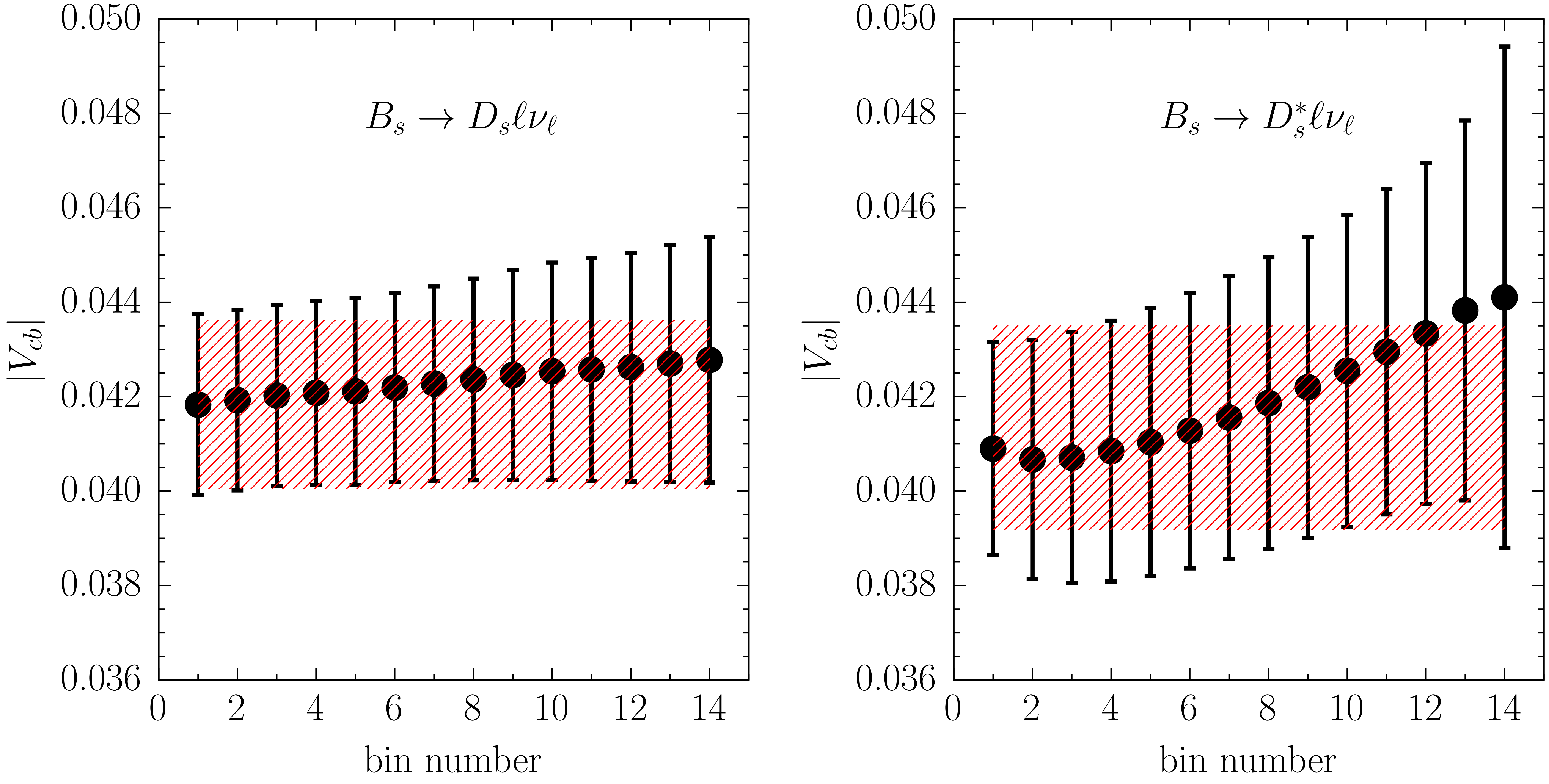
$$\Gamma^{\rm DM}(B_s \to D_s \mu \nu_\mu)/|V_{cb}|^2 = (6.04 \pm 0.23) \cdot 10^{-12} \ \text{GeV}$$
$$\Gamma^{\rm DM}(B_s \to D_s^* \mu \nu_\mu)/|V_{cb}|^2 = (1.39 \pm 0.11) \cdot 10^{-11} \ \text{GeV}$$

⇩

decays	$\|V_{cb}\|^{\rm DM} \cdot 10^3$
$B_s \to D_s \ell \nu_\ell$	41.5 ± 2.1
$B_s \to D_s^* \ell \nu_\ell$	40.3 ± 2.7

* second analysis: differential decay rates reconstructed from the LHCb fits of $p_\perp$ distributions (BGL/CLN parameterizations for the FFs) carried out in arXiv:2001.03225 (see also arXiv:2103.06810)

bin-per-bin analysis: $|V_{cb}|_j \equiv \sqrt{\dfrac{d\Gamma^{\text{LHCb}}/dw_j}{d\Gamma^{\text{DM}}/dw_j}}$ $j = 1, \ldots, N_{bins}$ we adopted $N_{bins} = 14$ w-bins



correlated weighted averages

$$|V_{cb}| = \frac{\sum_{i,j=1}^{N_{\text{bins}}} (\mathbf{C}^{-1})_{ij} |V_{cb}|_j}{\sum_{i,j=1}^{N_{\text{bins}}} (\mathbf{C}^{-1})_{ij}}$$

$$\sigma^2_{|V_{cb}|} = \frac{1}{\sum_{i,j=1}^{N_{\text{bins}}} (\mathbf{C}^{-1})_{ij}}$$

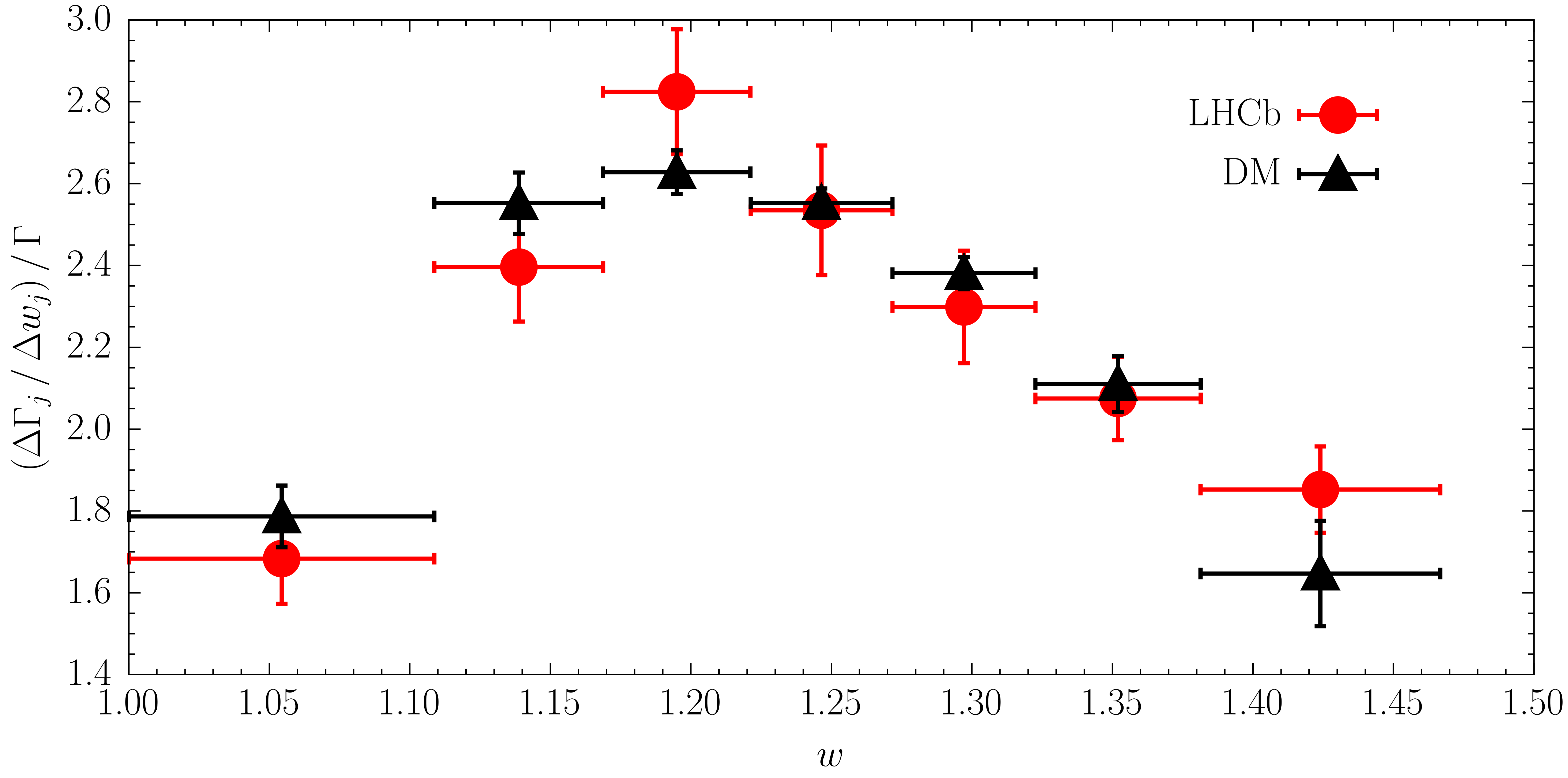
decays	$\lvert V_{cb}\rvert^{\text{DM}} \cdot 10^3$
$B_s \to D_s \ell \nu_\ell$	41.8 ± 1.8
$B_s \to D_s^* \ell \nu_\ell$	41.3 ± 2.2

$|V_{cb}|^{\text{LHCb}} \cdot 10^3 = 41.7 \pm 1.6$

* third analysis: LHCb ratios from arXiv:2003.08453

$$\Delta r_j = \frac{\Delta\Gamma_j(B_s \to D_s^* \mu \nu_\mu)}{\Gamma(B_s \to D_s^* \mu \nu_\mu)} \qquad j = 1,\ldots,7$$

j	1	2	3	4	5	6	7
w-bin	1.000 - 1.1087	1.1087 - 1.1688	1.1688 - 1.2212	1.2212 - 1.2717	1.2717 - 1.3226	1.3226 - 1.3814	1.3814 - 1.4667
Δw_j	0.1087	0.0601	0.0524	0.0505	0.0509	0.0588	0.0853
$\Delta r_j^{\mathrm{LHCb}}$	0.183(12)	0.144(8)	0.148(8)	0.128(8)	0.117(7)	0.122(6)	0.158(9)
Δr_j^{DM}	0.1942(82)	0.1534(45)	0.1377(28)	0.1289(18)	0.1212(20)	0.1241(40)	0.1405(110)



consistency within $\sim 1\sigma$

⇩

shape of theoretical FFs is consistent with the one of the experimental data

* to determine $|V_{cb}|$ we evaluate the integrated differential decay rates for each bin

$$\Delta\Gamma_j^{\rm exp} = \Delta r_j^{\rm LHCb} \cdot \Gamma^{\rm LHCb}(B_s \to D_s^* \mu \nu_\mu) \qquad j = 1, \ldots, 7$$

and the covariance matrix: $\Gamma_{ij}^{\rm exp} = R_{ij}^{\rm LHCb}\left[\overline{\Gamma}^2 + \sigma_{\overline{\Gamma}}^2\right] + \Delta r_i^{\rm LHCb} \Delta r_j^{\rm LHCb} \sigma_{\overline{\Gamma}}^2$

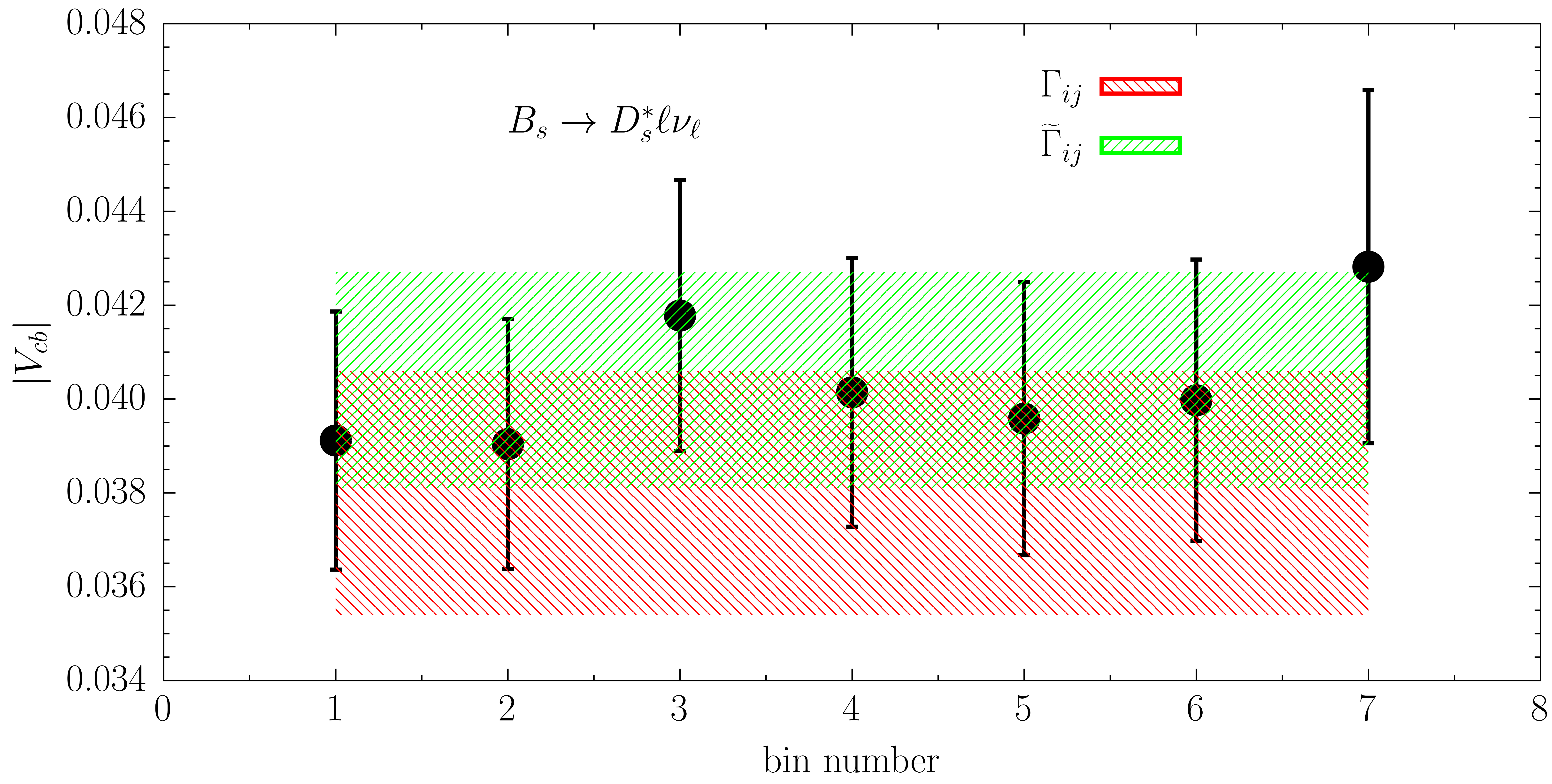
$\Gamma^{\rm LHCb}(B_s \to D_s^* \mu \nu_\mu)$ from arXiv:2103.06810

$$\overline{\Gamma} \pm \sigma_{\overline{\Gamma}} = (2.26 \pm 0.24) \cdot 10^{-14} \text{ GeV}$$

*** uncorrelated with $\Delta r_j^{\rm LHCb}$ ***

general property: $\sum_{i,j=1}^{N_{bins}} \Gamma_{ij}^{\rm exp} = \sigma_{\overline{\Gamma}}^2 \iff \sum_{i=1}^{N_{bins}} \Delta r_i^{\rm LHCb} = 1$ and $\sum_{i,j=1}^{N_{bins}} R_{ij}^{\rm LHCb} = 0$

- D'Agostini effect (NIMA '94): negative bias on constant fits to data affected by an overall normalization uncertainty; it depends upon $\sigma_{\overline{\Gamma}}$ and $\Delta r_i^{\rm LHCb} \neq \Delta r_j^{\rm LHCb}$



modified covariance matrix

$$\widetilde{\Gamma}_{ij}^{\rm exp} = R_{ij}^{\rm LHCb}\left[\overline{\Gamma}^2 + \sigma_{\overline{\Gamma}}^2\right] + \sigma_{\overline{\Gamma}}^2 / N_{bins}^2$$

$$\sum_{i,j=1}^{N_{bins}} \widetilde{\Gamma}_{ij}^{\rm exp} = \sum_{i,j=1}^{N_{bins}} \Gamma_{ij}^{\rm exp} = \sigma_{\overline{\Gamma}}^2$$

correlated weighted averages

$$|V_{cb}| \cdot 10^3 = 38.0 \pm 2.6$$

$$|V_{cb}| \cdot 10^3 = 40.4 \pm 2.3$$

$|V_{cb}|^{DM} \cdot 10^3$ from $B_s \to D_s^{(*)} \ell \nu_\ell$

analysis	$B_s \to D_s$	$B_s \to D_s^*$
first	41.5 ± 2.1	40.3 ± 2.7
second	41.8 ± 1.8	41.3 ± 2.2
third		40.4 ± 2.3
average	41.7 ± 1.9	40.7 ± 2.4

$$x = \sum_{k=1}^{N} \omega_k x_k$$

$$\sigma^2 = \sum_{k=1}^{N} \omega_k \left[\sigma_k^2 + (x_k - x)^2\right]$$

$$\omega_k = (1/\sigma_k^2) / \sum_{j=1}^{N} (1/\sigma_j^2)$$

summary of $|V_{cb}|^{DM}$ from $B_{(s)} \to D_{(s)}^{(*)} \ell \nu_\ell$

decay	$\lvert V_{cb}\rvert^{DM} \cdot 10^3$	inclusive [2107.00604]	exclusive [FLAG 21]
$B \to D$	41.0 ± 1.2		
$B \to D^*$	41.3 ± 1.7		
$B_s \to D_s$	41.7 ± 1.9		
$B_s \to D_s^*$	40.7 ± 2.4		
average	41.2 ± 0.8	42.16 ± 0.50	39.36 ± 0.68
difference		$\simeq 1.0\,\sigma$	$\simeq 1.8\,\sigma$

summary of $R(D_{(s)})$, $R(D_{(s)}^*)$ and polarization observables

observable	DM
$R(D_s)$	0.298 (5)
$R(D_s^*)$	0.250 (6)
$P_\tau(D_s^*)$	−0.520 (12)
$F_L(D_s^*)$	0.440 (16)

⟷ SU(3)F breaking ?

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.339 (27) (14)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (10) (10)	$\simeq 1.2\,\sigma$
$P_\tau(D^*)$	−0.52 (1)	$-0.38\,(51)\,(^{+21}_{-16})$	
$F_L(D^*)$	0.42 (1)	0.60 (8) (4)	$\simeq 2.0\,\sigma$

Γ_i : mean values $\overline{\Gamma}_i$ and covariance matrix C_{ij} $\qquad i, j = 1, \ldots, N$

$$\Gamma = \sum_{i=1}^{N} \Gamma_i$$

$$\text{mean value } \overline{\Gamma} = \sum_{i=1}^{N} \overline{\Gamma}_i \text{ and variance } \sigma_\Gamma^2 = \sum_{i,j=1}^{N} C_{ij}$$

$$\sum_{i,j=1}^{N} C_{ij} = \sum_{i,j=1}^{N} \langle (\Gamma_i - \overline{\Gamma}_i)(\Gamma_j - \overline{\Gamma}_j) \rangle = \left\langle \left[\sum_{i=1}^{N} (\Gamma_i - \overline{\Gamma}_i) \right]^2 \right\rangle = \langle (\Gamma - \overline{\Gamma})^2 \rangle \equiv \sigma_\Gamma^2$$

$$\Gamma_i = r_i \cdot \Gamma \qquad i = 1,\ldots,N$$

r_i : mean values $\bar{r}_i$ and covariance matrix R_{ij}

Γ : mean value $\overline{\Gamma}$ and variance σ_Γ^2 uncorrelated with all the $r_i's$

$$\Gamma_i : \text{mean values } \overline{\Gamma}_i = \bar{r}_i \cdot \overline{\Gamma} \text{ and covariance matrix } C_{ij} = R_{ij} \cdot \left[\overline{\Gamma}^2 + \sigma_\Gamma^2\right] + \bar{r}_i \bar{r}_j \sigma_\Gamma^2$$

$$r_i = \bar{r}_i + \sqrt{R_{ii}} \sum_{k=1}^{N} U_{ik}^T \sqrt{\lambda_k} \cdot \xi_k \qquad R_{ij} = \sqrt{R_{ii} R_{jj}} \sum_{k=1}^{N} U_{ik}^T \lambda_k U_{kj}$$

ξ_k : uncorrelated variables

$< \xi_k > = 0$ and $< \xi_k \xi_{k'} > = \delta_{kk'}$

$$\Gamma = \overline{\Gamma} + \sigma_\Gamma \cdot \xi_\Gamma$$

ξ_Γ : uncorrelated variable with all the ξ_k variables

$< \xi_\Gamma > = 0$ and $< \xi_\Gamma^2 > = 1, < \xi_\Gamma \xi_k > = 0$

$$C_{ij} = < (r_i \cdot \Gamma - \bar{r}_i \cdot \overline{\Gamma})(r_j \cdot \Gamma - \bar{r}_j \cdot \overline{\Gamma} > = \sqrt{R_{ii} R_{jj}} \sum_{k=1}^{N} U_{ik}^T \lambda_k U_{jk}^T \left[\overline{\Gamma}^2 + \sigma_\Gamma^2\right] + \bar{r}_i \bar{r}_j \sigma_\Gamma^2 = R_{ij} \left[\overline{\Gamma}^2 + \sigma_\Gamma^2\right] + \bar{r}_i \bar{r}_j \sigma_\Gamma^2$$

* LHCb ratios from arXiv:2003.08453

$$\Delta r_j = \frac{\Delta\Gamma_j(B_s \to D_s^* \mu\nu_\mu)}{\Gamma(B_s \to D_s^* \mu\nu_\mu)} \qquad j = 1,\ldots,7$$

- constrain $\sum_{j=1}^{7} \Delta r_j = 1$ ⇨
  1) one null eigenvalue of the covariance matrix R_{ij}^{LHCb} (six independent ratios)
  2) $\sum_{i,j} R_{ij}^{\mathrm{LHCb}} = 0$ (null variance for the sum)

- experimental covariance matrix R_{ij}^{LHCb} :

  eigenvalues $\lambda_j = \{0.072,\ 0.21,\ 0.33,\ 0.53,\ 0.73,\ 1.03,\ 2.33\} \cdot 10^{-4}$ and $\sum_{i,j} R_{ij}^{\mathrm{LHCb}} = 1.45 \cdot 10^{-3}$

  → ~ 3.8% to be added (in quadrature) to the error of the total decay rate

- modified covariance matrix $\widetilde{R}_{ij}^{\mathrm{LHCb}}$: $\widetilde{\Delta r}_j = \Delta r_j / \sum_{k=1}^{7} \Delta r_k$

  eigenvalues $\widetilde{\lambda}_j = \{0.0,\ 0.073,\ 0.22,\ 0.34,\ 0.53,\ 0.74,\ 1.14\} \cdot 10^{-4}$ and $\sum_{i,j} \widetilde{R}_{ij}^{\mathrm{LHCb}} = 0$

LQCD form factors for $B \to \pi \ell \nu_\ell$ decays

	RBC/UKQCD	FNAL/MILC	Combined
$f_+^\pi(19.0\ \mathrm{GeV}^2)$	1.21(10)(9)	1.17(8)	1.19(11)
$f_+^\pi(22.6\ \mathrm{GeV}^2)$	2.27(13)(14)	2.24(12)	2.25(16)
$f_+^\pi(25.1\ \mathrm{GeV}^2)$	4.11(51)(29)	4.46(23)	4.29(48)
$f_0^\pi(19.0\ \mathrm{GeV}^2)$	0.46(3)(5)	0.46(3)	0.46(5)
$f_0^\pi(22.6\ \mathrm{GeV}^2)$	0.68(3)(6)	0.65(3)	0.66(5)
$f_0^\pi(25.1\ \mathrm{GeV}^2)$	0.92(3)(6)	0.86(3)	0.89(6)

RBC/UKQCD (arXiv:1501.05363)

FNAL/MILC (arXiv:1503.07839)

combined:

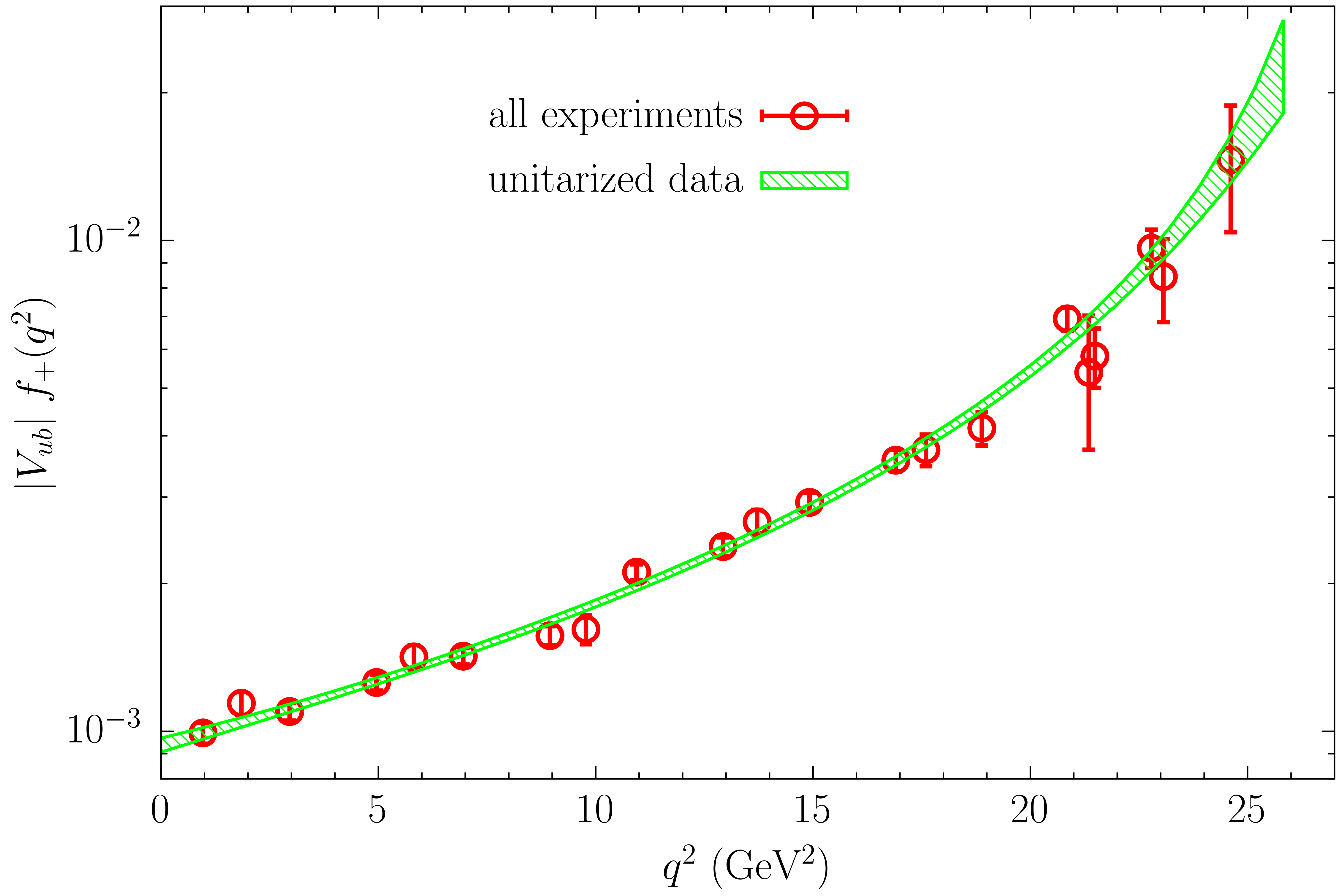
$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k,$$

$$\sigma_x^2 = \frac{1}{N}\sum_{k=1}^{N}\sigma_k^2 + \frac{1}{N}\sum_{k=1}^{N}(x_k - \mu_x)^2.$$

$$C(x_i, x_j) \equiv \frac{1}{N}\sum_{k=1}^{N} C(x_i, x_j)_k + \frac{1}{N}\sum_{k=1}^{N}(x_k^i - \mu_x^i)(x_k^j - \mu_x^j).$$

a new strategy: unitarization of the data

* construct the experimental values of $|V_{ub} f_+(q_i^2)| = \sqrt{\Delta\Gamma_i / z_i}$ (z_i = kinematical coefficient in the i-th bin)
* apply the DM method on the data points $|V_{ub} f_+(q_i^2)|$ using the unitarity bound $|V_{ub}|^2 \chi_{1^-}(0)$ with an initial guess for $|V_{ub}|$
* determine $|V_{ub}|$ using the theoretical DM bands and iterate the procedure until consistency for $|V_{ub}|$ is reached



we still keep separate the theoretical calculations and the experimental data for describing the shape of the FFs

$$|V_{ub}|_{DM} \cdot 10^3 = 3.88 \pm 0.32$$

$$|V_{ub}|_{incl.} \cdot 10^3 = 4.32 \pm 0.29$$

difference of $\approx 1\sigma$

$$|V_{ub}|_{excl.} \cdot 10^3 = 3.74 \pm 0.17 \qquad \text{(FLAG '21)}$$

the FLAG error is much smaller because the exp. data are used to describe the shape of the FFs

In the massless lepton limit ($m_\ell = 0$) the differential rate for the semileptonic $B \to \pi \ell \nu_\ell$ decay is given by

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2}{24\pi^3} |V_{ub}|^2 p_\pi^3 f_+^2(q^2) \ , \tag{1}$$

where

$$p_\pi^3 = \frac{m_B^3}{8} \left[(1 + r^2 - q^2/m_B^2)^2 - 4r^2 \right]^{3/2} \tag{2}$$

with $r \equiv m_\pi / m_B$.

Let us consider a series of bins in q^2, namely from $q_i^2 - \Delta_i/2$ to $q_i^2 + \Delta_i/2$ with $i = 1, 2, ..., N$. Using the experimental data for the integrated rate in the various bins, $\Delta\Gamma_i \equiv C_v \Delta B_i / \tau_B$, we can obtain the values of $|V_{ub}|^2 f_+^2(q^2)$ by choosing a series of values $\overline{q}_i^2$ by requiring the vanishing of the contribution of the slope of the form factor $f_+^2(q^2)$ in the given bin. This leads to

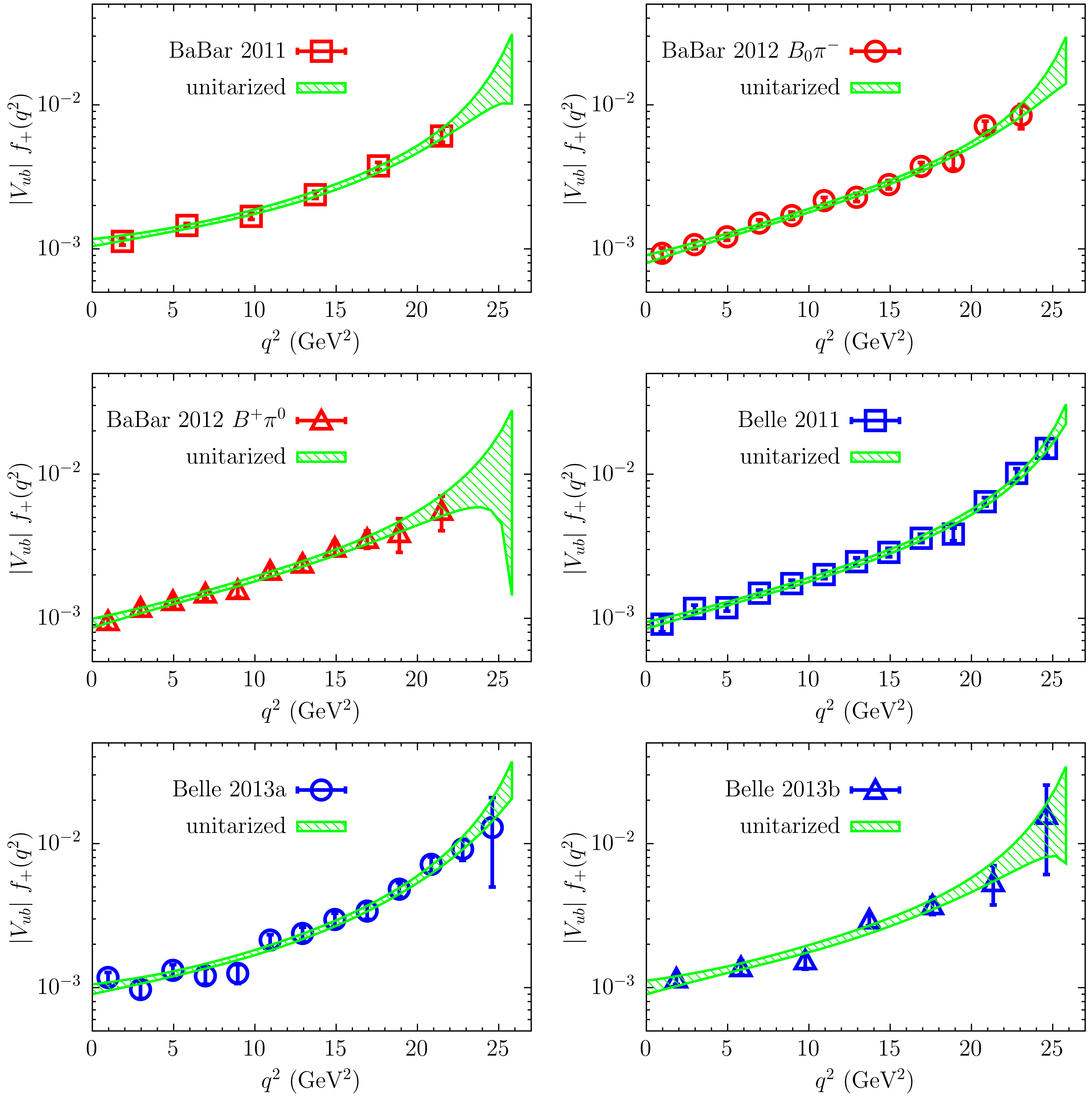
$$|V_{ub} \ f_+(\overline{q}_i^2)| = \sqrt{\frac{\Delta\Gamma_i}{z_i}} \ , \tag{3}$$

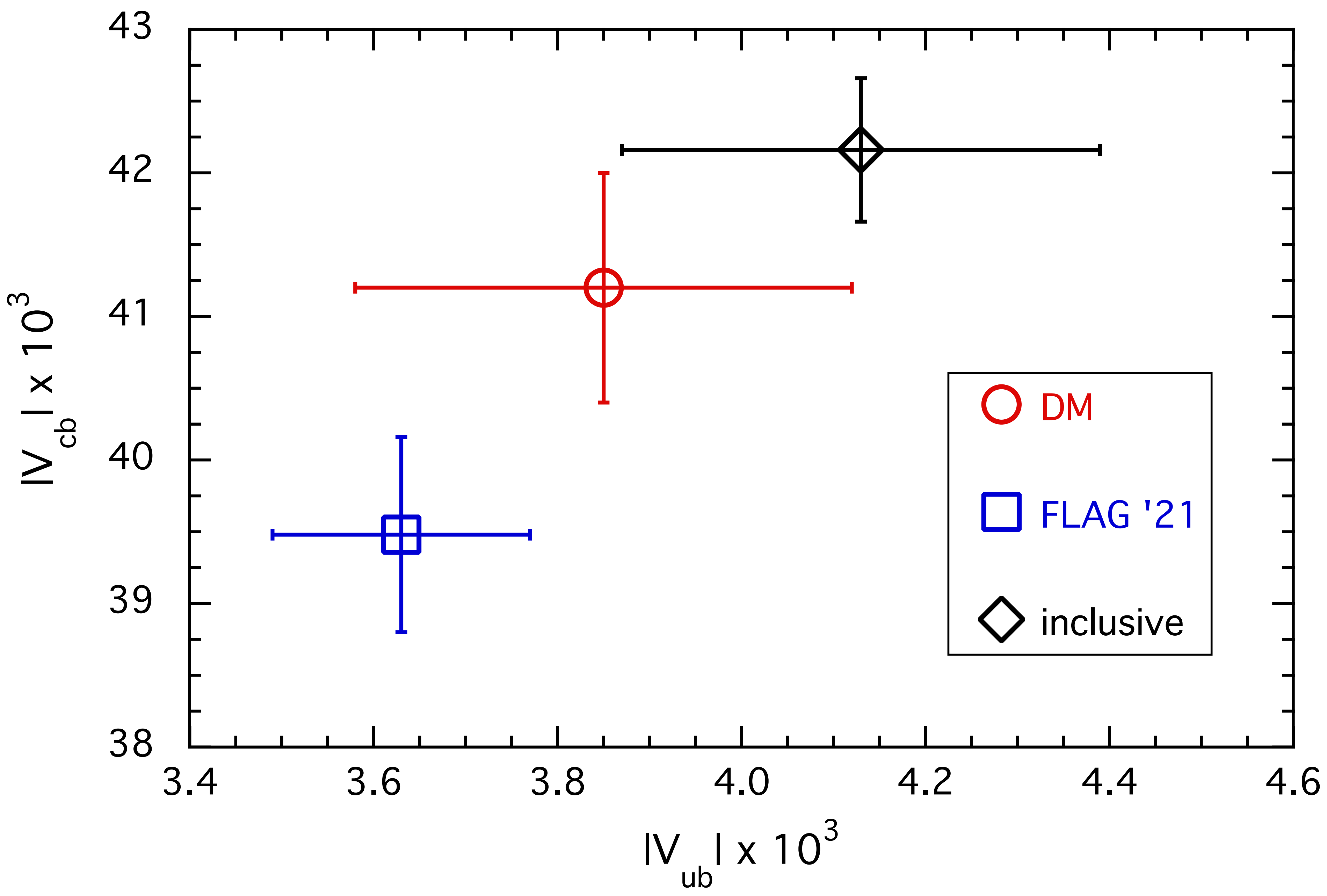
where

$$z_i \equiv \frac{G_F^2}{24\pi^3} \frac{m_B^3}{8} \int_{q_i^2 - \Delta_i/2}^{q_i^2 + \Delta_i/2} dq^2 \left[(1 + r^2 - q^2/m_B^2)^2 - 4r^2 \right]^{3/2} \tag{4}$$

and

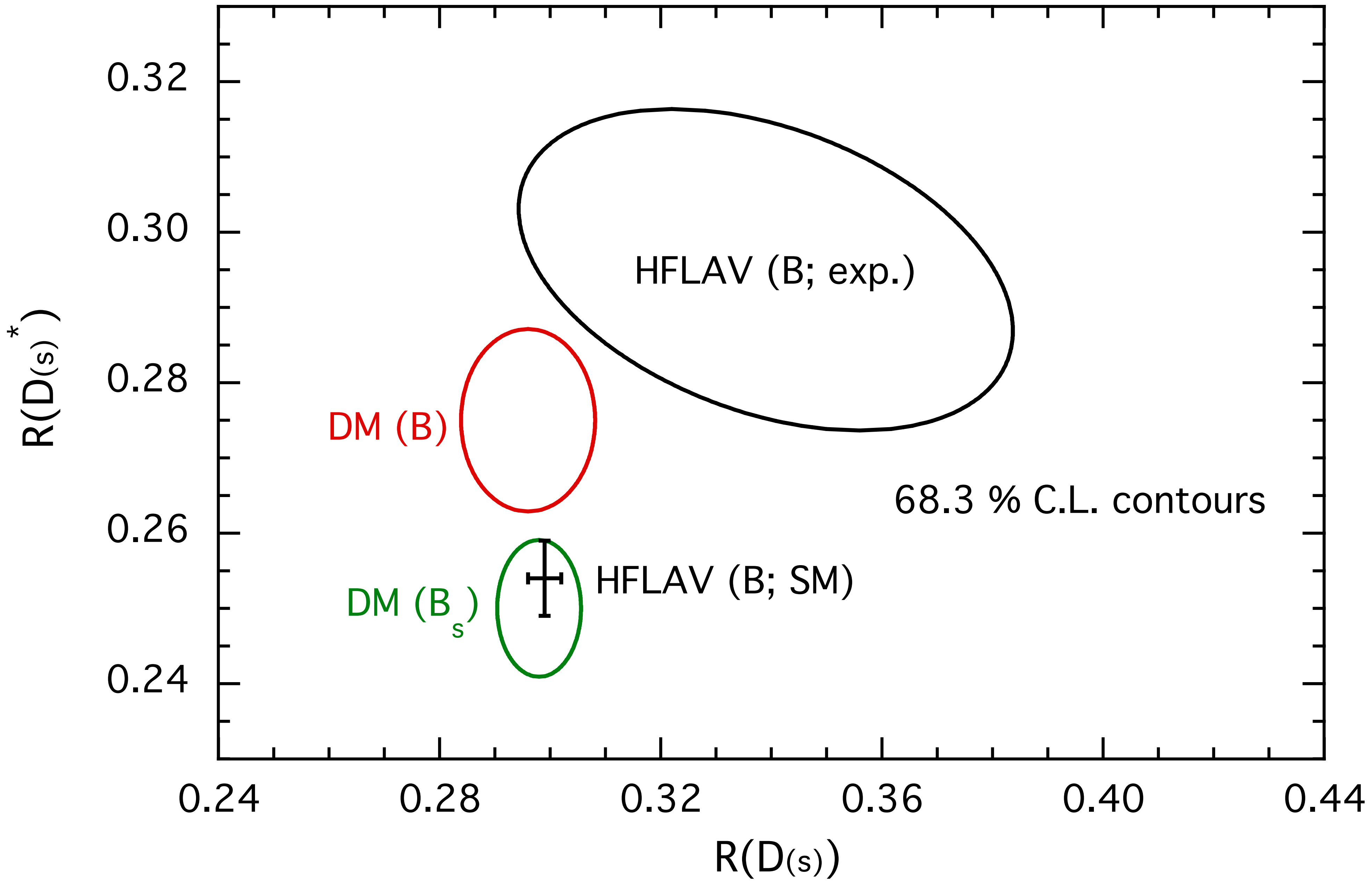
$$\overline{q}_i^2 \equiv \frac{\int_{q_i^2 - \Delta_i/2}^{q_i^2 + \Delta_i/2} dq^2 q^2 \left[(1 + r^2 - q^2/m_B^2)^2 - 4r^2 \right]^{3/2}}{\int_{q_i^2 - \Delta_i/2}^{q_i^2 + \Delta_i/2} dq^2 \left[(1 + r^2 - q^2/m_B^2)^2 - 4r^2 \right]^{3/2}} \ . \tag{5}$$


BaBar 2011
unitarized
BaBar 2012 $B_0\pi^-$
unitarized
BaBar 2012 $B^+\pi^0$
unitarized
Belle 2011
unitarized
Belle 2013a
unitarized
Belle 2013b
unitarized
$|V_{ub}|\ f_+(q^2)$
10^{-2}
10^{-3}
0
5
10
15
20
25
q^2 (GeV^2)



	decays	DM	FLAG '21	inclusive
$\lvert V_{cb}\rvert \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.2 (8)	39.48 (68)	42.16 (50)
$\lvert V_{ub}\rvert \cdot 10^3$	$B_{(s)} \to \pi(K)$	3.85 (27)	3.63 (14)	4.13 (26)

reduced tensions in $|V_{cb}|$, $|V_{ub}|$



	DM	HFLAV '21 (exp.)	HFLAV '21 (SM)
R(D)	0.296 (8)	0.339 (26) (14)	0.299 (3)
R(D*)	0.275 (8)	0.295 (10) (10)	0.254 (5)
$R(D_s)$	0.298 (5)		
$R(D_s^*)$	0.250 (6)		

reduced tension in $R(D^*)$ (using the FNAL ff's)

channel	# LQCD	# exp's
$B \to D$	1	1
$B \to D^*$	1	2/3
$B_s \to D_s$	1	2
$B_s \to D_s^*$	1	2
$B \to \pi$	2	6/7
$B_s \to K$	3	1