Electromagnetic conductivity of quark-gluon plasma at finite baryon chemical potential

QCD@Work - International Workshop on QCD - Theory and Experiment

UNIVERSITÀ DI PISA

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Outline

- Why to study conductivity?

- Lattice setup and strategy

- Inverse problem

- Results

- Conclusions and Outlook

Why to study the electromagnetic conductivity of quark-gluon matter?

The e.m. conductivity is a transport coefficient which parametrizes the charge transport phenomena.

Its computation is challenging from a phenomenological point of view since in heavy-ion collision experiments large electric and magnetic fields ($\sim m_\pi^2$) are generated and this influences the dynamics of the QGP.

Furthermore it's known that the QGP generated during these collisions has nonzero baryon density (small for LHC and RHIC experiments while large for FAIR and NICA experiments).

Thus, it could be important to study how nonzero baryon density influences the e.m. conductivity of QGP

In an incoming paper, we made its first lattice QCD study at finite baryon density.

Lattice Setup

We used $N_f = 2 + 1$ dynamical staggered quarks at physical quark masses with chemical potentials

$$\mu_u = \mu_d = \mu_B/3, \quad \mu_s = 0.$$

The introduction of a nonzero baryon density leads to the sign problem. Thus, we used imaginary baryon chemical potential for different values.

a, fm	L_s	N_t	T, MeV	$m_l a$	$m_s a$	$\mu_I/3\pi T$	$c(T)$
0.0988	48	10	200	0.0014	0.0394	0.0, 0.14, 0.20, 0.245, 0.285	0.0060(14)
0.0788	48	10	250	0.001119	0.031508	0.0, 0.14, 0.20, 0.245, 0.285	0.0086(14)
0.0820	48	12	200	0.001168	0.032872	0.0, 0.14, 0.20, 0.245, 0.285	0.0076(14)
0.0657	48	12	250	0.000917	0.025810	0.0, 0.14, 0.20, 0.245, 0.285	0.0084(10)

We considered two values of the temperature $T = 200,\ 250MeV$ and most of the simulations are carried out on a 12×48^3 lattice with lattice spacings $a = 0.0820, 0.0657\ fm$ while to check the lattice spacing dependence we also considered a 10×48^3 lattice with $a = 0.0988\ fm$ and $a = 0.0788\ fm$.

To measure the conductivity we went trough two principal steps

- Step 1: the measure of the correlation functions
- Step 2: spectral function extraction and conductivity computation via Kubo formulas

The first step consists in the computation on the lattice of the correlation function

$$C_{ij}(\tau) = \frac{1}{L_s^3}\langle J_i(\tau) J_j(0)\rangle$$

where τ is the Euclidean time and $J_i(\tau)$ is the conserved current

chemical potential

$$J_i(\tau) = \frac{1}{4} e \sum_f q_f \sum_{\vec{x}} \eta_i(x) \left(\bar{\chi}^f_x e^{a\mu_f \delta_{\nu,4}} U^{(2)}_{x,i} \chi^f_{x+i} + \bar{\chi}^f_x e^{-a\mu_f \delta_{\nu,4}} U^{(2)\dagger}_{x,i} \chi^f_{x+i} \right)$$

$\eta_i(x) = (-1)^{x_1 + \ldots x_{i-1}}$ where $x = (\tau, \vec{x})$ and i=1,2,3

$\bar{\chi}^f_x, \chi^f_x$ are staggered fermion fields of f=u,d,s flavours.

This correlator correspond to two different operators for the even $\tau = 2n \times a$ and odd $\tau = (2n+1) \times a$ slices. In the continuum limit it reads

$$C^{e,o}_{ij}(\tau) = \sum_{\vec{x}} \left(\langle A_i(x) A_j(0)\rangle - s^{e,o} \langle B_i(x) B_j(0)\rangle \right)$$

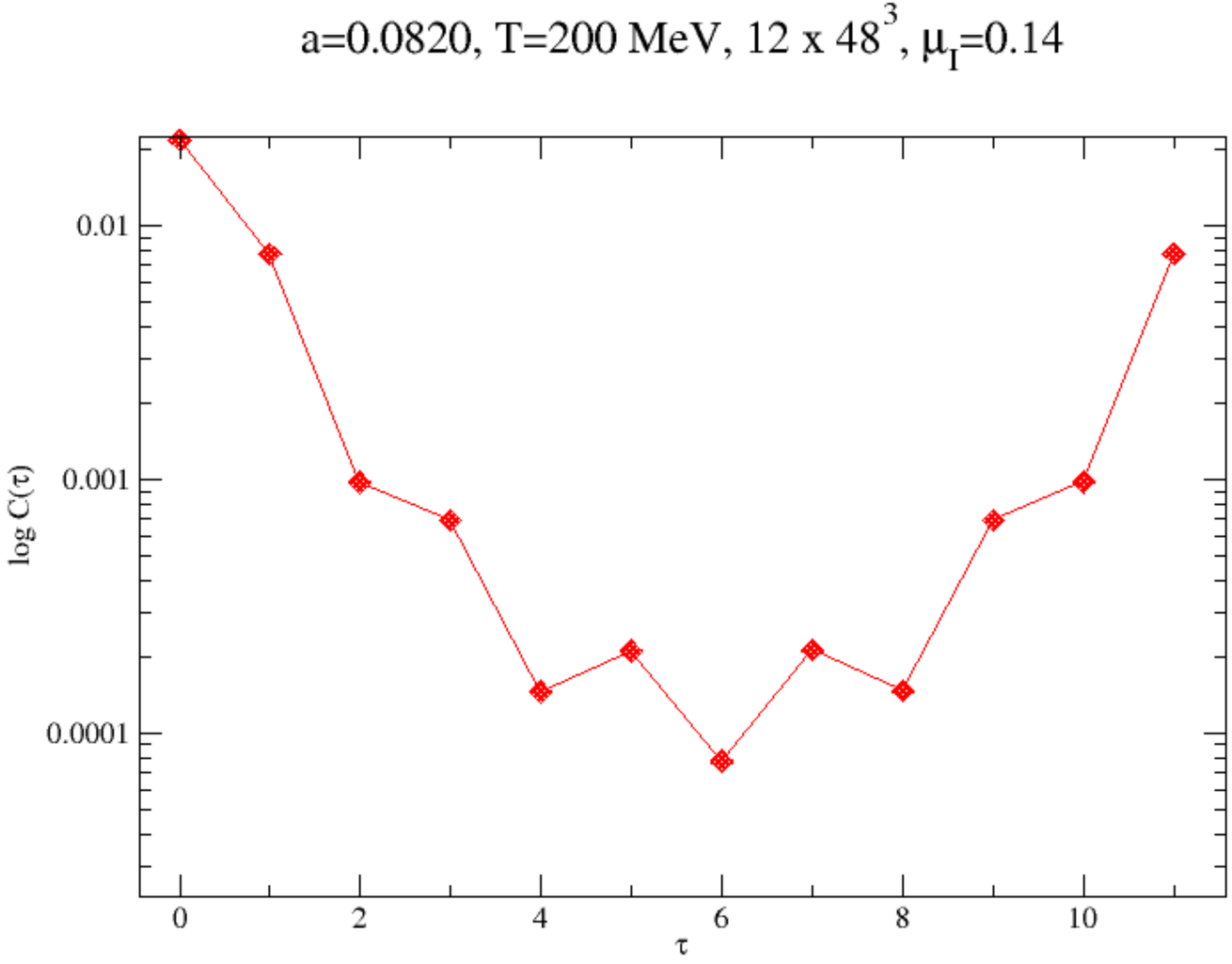
$A_i = e \sum_f q_f \bar{\psi}^f \gamma_i \psi^f$ corresponding to the electromagnetic current in the continuum limit

$s^{e,o} = (-1)^\tau$

$B_i = e \sum_f q_f \bar{\psi}^f \gamma_5 \gamma_4 \gamma_i \psi^f$

The figure shows an example of the result we obtain for a correlation function at nonzero imaginary chemical potential.

The correlator is symmetric and then we'll use only one independent side averaging the symmetric partners. Furthermore, the analysis is made independently for the even and odd values of the Euclidean time.



Having computed the correlators for all the lattice spacing for the even and odd slices, we can study the conductivity.

How?

The current-current Euclidean correlators both for even and odd slices $C_{ij}^{e,o}$ are related to their spectral functions $\rho_{ij}^{e,o}(\omega)$ as

$$C_{ij}^{e,o}(\tau) = \int_0^\infty \frac{d\omega}{\pi} K(\tau,\omega)\rho_{ij}^{e,o}(\omega)$$

where $K(\tau,\omega) = \dfrac{\cosh \omega(\tau - \beta/2)}{\sinh \omega\beta/2}$

If we are able to invert this relation and then extract $\rho_{ij}^{e,o}(\omega)$ from the correlators, we can compute the electromagnetic conductivity σ_{ij} by using the Kubo formulas

$$\frac{\sigma_{ij}}{T} = \frac{1}{2T} \lim_{\omega\to 0} \frac{1}{\omega}\left(\rho_{ij}^{e}(\omega) + \rho_{ij}^{o}(\omega)\right)$$

The problem of the inversion of the correlation function to extract the spectral function is well known in literature.

Then, how can we invert this relation?

The idea is to search for a smearing function that lives in the space spanned by the basis functions $K(\tau, \omega)$

$$\delta(\bar{\omega}, \omega) = \sum_i q_i(\bar{\omega}) K(\tau_i, \omega)$$

in such a way that once the coefficients $q_i(\bar{\omega})$ are known we can extract the smeared spectral density as

$$\hat{\rho}(\bar{\omega}) = \int_0^\infty d\omega \rho(\omega) \delta(\bar{\omega}, \omega) = \sum_i q_i(\bar{\omega}) C(\tau_i) \, .$$

But how do we fix the coefficients $q_i(\bar{\omega})$?

In our work we mainly used the Backus-Gilbert method ***[G. Backus and F. Gilbert, Geophysical Journal International 16, 169 (1968)]*** and a modified version of it recently proposed in [***M. Hansen, A. Lupo, and N. Tantalo, Physical Review D 99 (2019)].***

We also used the so called Tikhonov regularization approach ***[A. N. Tikhonov, Soviet Math. Dokl. 4, 1035 (1963)]*** and the results are very similar to the ones obtained using the BG approach.

Thus, in our discussion we'll focus on the BG method!

In the case of the Backus Gilbert method, to fix the coefficients the idea is to introduce the functional

$$W[\lambda, q] = (1-\lambda)A[q] + \lambda B[q]$$

$A[q] = \int_0^\infty d\omega(\omega - \bar{\omega})^2\{\delta(\bar{\omega}, \omega)\}^2$ that measures the width of the smearing function.

$B[q] = q^T Cov q$, being Cov the covariance matrix of the correlators. It takes into account the fact that the correlators have uncertainties.

$\lambda \in [0,1]$ is a tradeoff between the systematic error (related to the width of the smearing) and the statistic one.

By minimizing it and imposing the unit area constraint, $\int_0^\infty d\omega\delta(\bar{\omega}, \omega) = 1$, we can extract an expression for the coefficients

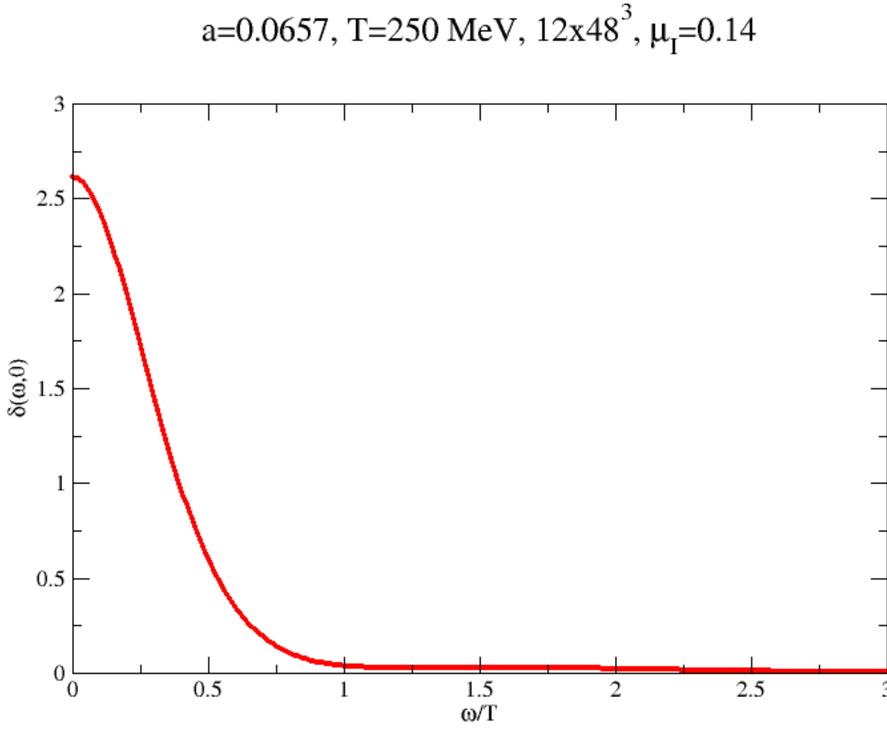
$$q(\bar{\omega}, \lambda) = \frac{W^{-1}(\lambda, \bar{\omega})R}{R^T W^{-1}(\lambda, \bar{\omega})R}$$

$W_{ij}(\lambda, \bar{\omega}) = (1-\lambda)A_{ij} + \lambda Cov_{ij}$, being

$A_{ij}(\bar{\omega}) = \int_0^\infty d\omega(\omega - \bar{\omega})^2 K(\tau_i, \omega)K(\tau_j, \omega)$

$R_i = \int_0^\infty d\omega K(\tau_i, \omega)$

through which we can extract the smearing function.



Thus, fixed λ, this allows us to extract the smearing function and then the value of the spectral function in the generic point $\bar{\omega}$:

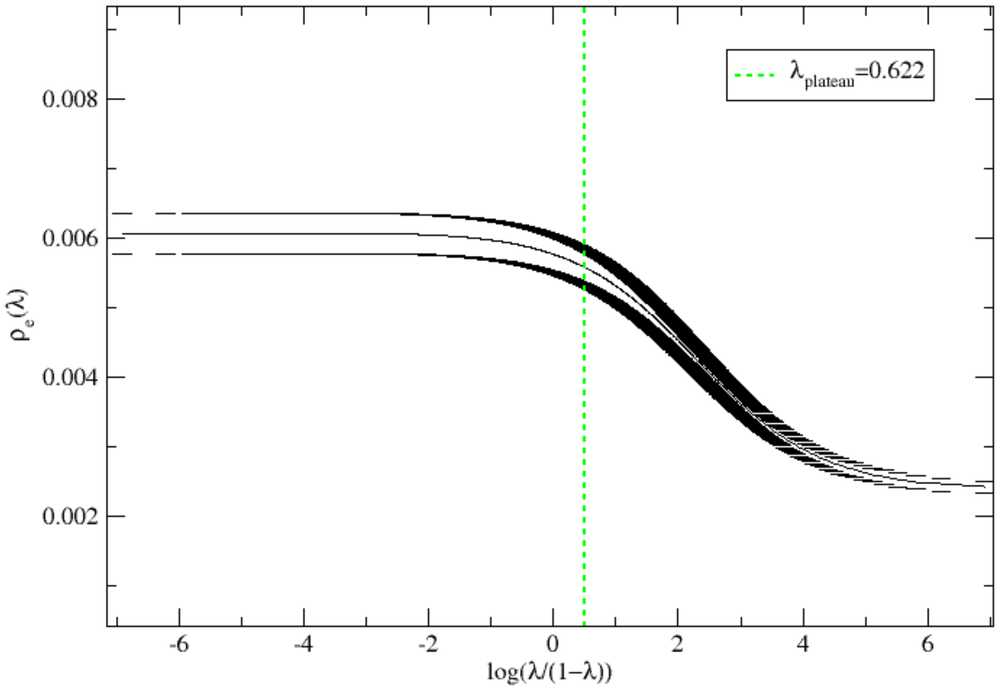
$$\hat{\rho}_\lambda(\bar{\omega}) = \sum_i q_i(\bar{\omega}, \lambda) C(\tau_i)$$

But how do we fix λ?

We look at spectral function for different values of the λ parameter and then choose the value in correspondence of the first point (starting from larger values of λ) in which a plateau, namely the region where the value stabilizes inside the errors, begins.

Once we fixed λ we can extract the value of the spectral function in the unknown point $\bar{\omega}$.

a=0.0656, T=250MeV, $12x48^3$, μ_I=0.245



In the case of the modified BG the main difference is the change of the choice of the first term of the functional

$$A[q] = \int_0^\infty d\omega(\omega - \bar{\omega})^2\{\delta(\bar{\omega},\omega)\}^2 \longrightarrow A[q] = \int d\omega\rho(\omega)(\delta(\omega,\bar{\omega}) - \delta_0(\omega,\bar{\omega}))^2$$

The idea is that instead of minimizing the width of the smearing, we minimize the deviation from a reference chosen function $\delta_0(\omega,\bar{\omega})$.

This leads to an analogous expression for the coefficients

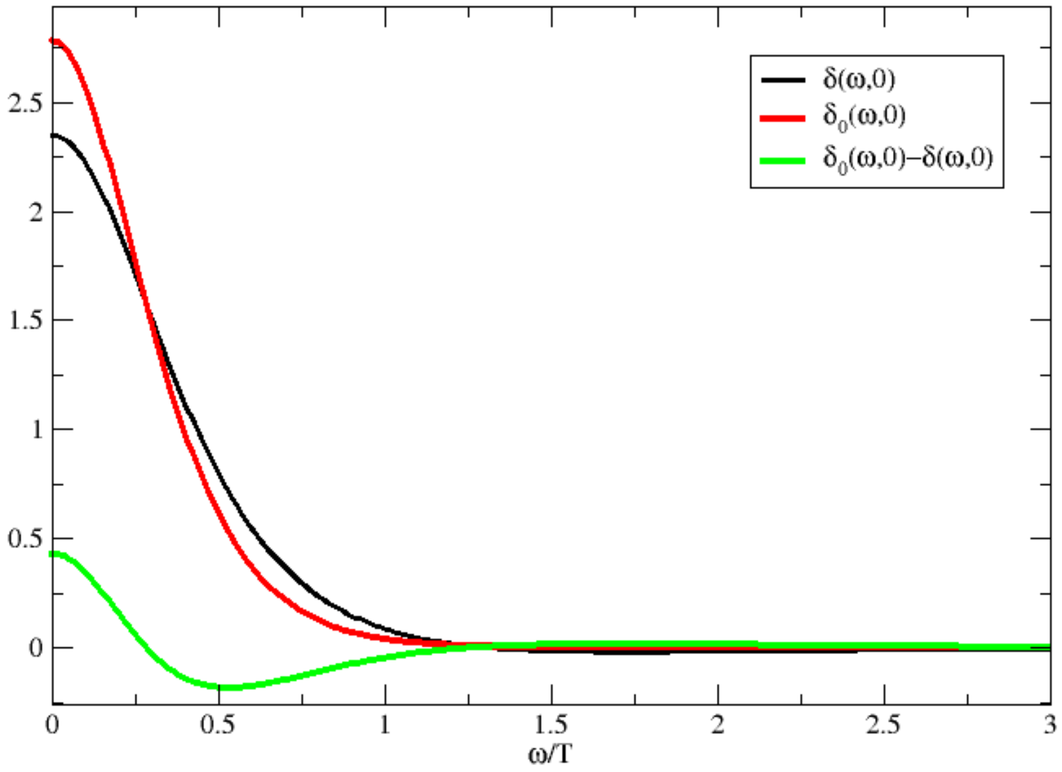
$$\hat{\rho}_\lambda(\bar{\omega}) = \sum_i q_i(\bar{\omega},\lambda)C(\tau_i).$$

where λ will be fixed with the same strategy as before.

Here we'll have a new uncertainty which results from the deviation of the resolution function $\delta(\omega,\bar{\omega})$ from the target one $\delta_0(\omega,\bar{\omega})$ that can be computed as

$$\Delta_{syst} = |r|\bar{\rho}(\bar{\omega})$$

where $r = 1 - \dfrac{\delta(\bar{\omega},\bar{\omega})}{\delta_0(\bar{\omega},\bar{\omega})}$ is the relative deviation at the peak.



Note that the calculation is hindered by large UV contribution of $\rho(\omega)$. One could subtract those contributions from the correlation function. However this leads to large uncertainties.

We considered instead of the correlation function at nonzero chemical potential $C^{e,o}_{\mu_I}$, the difference

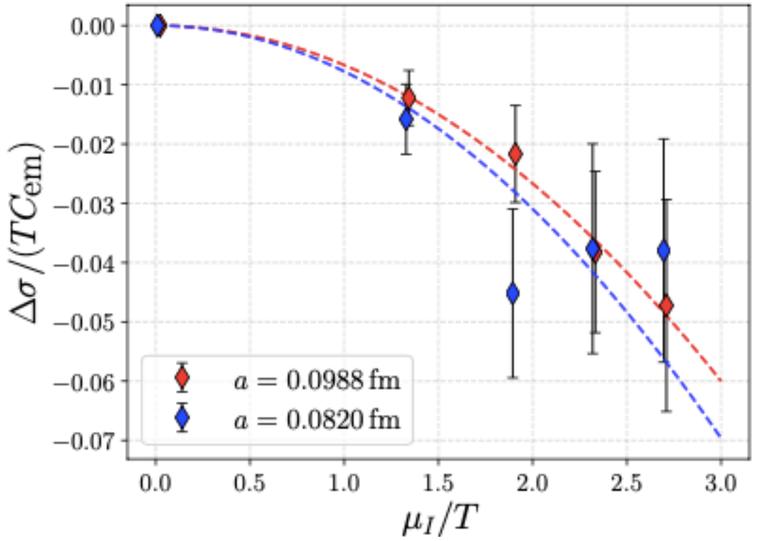
$$\Delta C^{e,o} = C^{e,o}_{\mu_I} - C^{e,o}_{\mu_I=0}$$

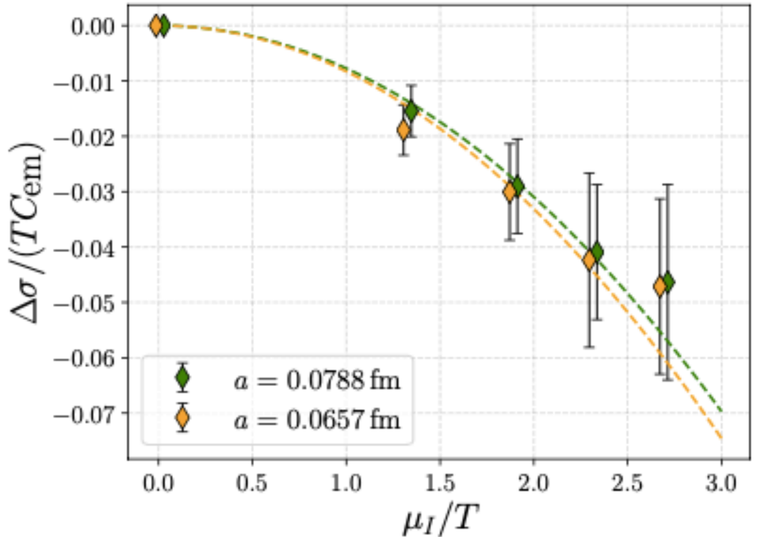
In fact, for the chosen lattice spacings the UV regime starts around $\omega \sim 2GeV$ and $\mu_I \ll \omega$ for frequencies in the UV regime and baryon chemical potential. Thus, one can consider the UV spectral function independent on the imaginary chemical potential and assume that $\Delta C^{e,o}$ does not contain UV contributions.

Thus, by applying the inversion methods on the difference of the correlators separately for the even and odd slices, we can extract the spectral functions $\Delta\rho^{e,o}$ independent from UV contributions. By summing them, we can finally extract the conductivity $\Delta\sigma$ using the Kubo formula

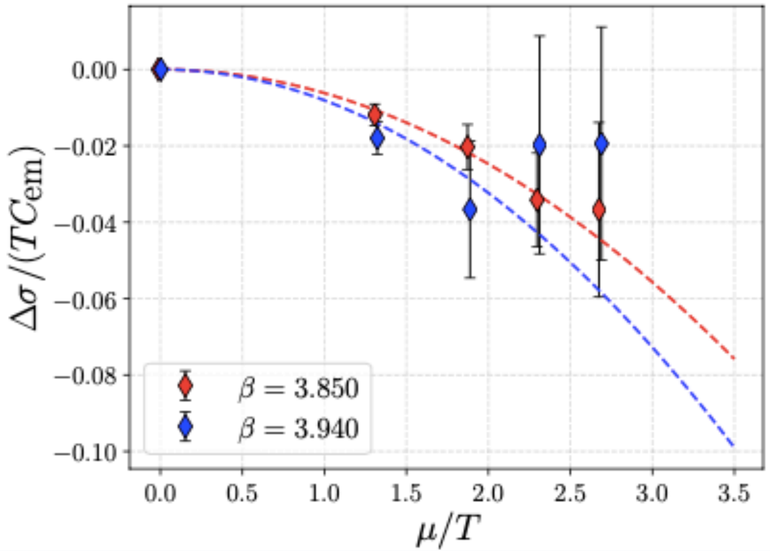
$$\frac{\Delta\sigma_{ij}}{T} = \frac{1}{2T}\lim_{\omega\to 0}\frac{1}{\omega}\left(\Delta\rho^{e}_{ij}(\omega) + \Delta\rho^{o}_{ij}(\omega)\right).$$

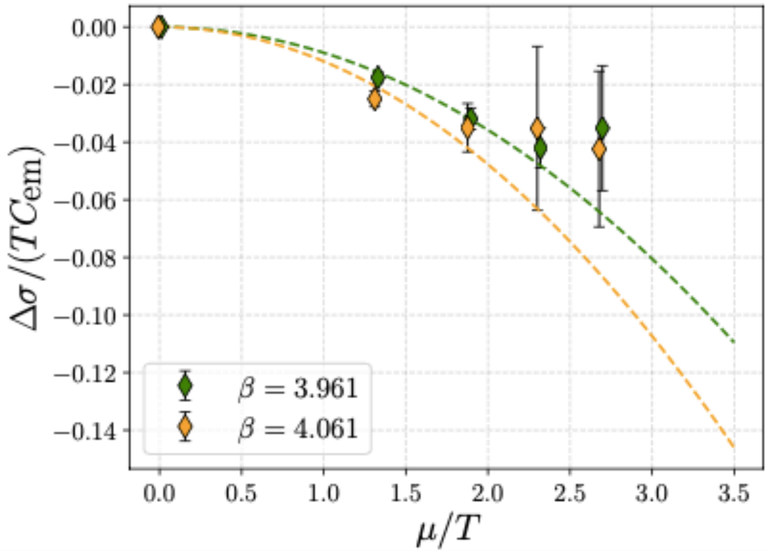
In Figure we show the change of the e.m. conductivity $\Delta\sigma = \sigma_{\mu_I} - \sigma_{\mu_I=0}$ as a function of imaginary chemical potential.





Backus-Gilbert (Tikhonov) approach results





Modified Backus-Gilbert approach results

The results obtained using the different methods are compatible.

The results are well described by the quadratic polynomial the we analytically continue to the real chemical potential

$$\frac{\Delta\sigma}{TC_{em}} = -\,c(T)\left(\frac{\mu_I}{T}\right)^2 \longrightarrow \frac{\Delta\sigma}{TC_{em}} = c(T)\left(\frac{\mu_B}{T}\right)^2$$

a, fm	T, MeV	c_{standard}	c_{improved}
0.0988	200	0.0066(12)	0.0061(9)
0.0788	250	0.0077(12)	0.0089(6)
0.0820	200	0.0077(16)	0.0080(18)
0.0657	250	0.0082(12)	0.011(1)

By looking at the results of the fit showed in the table we can make the following observations:

- We don’t see a noticeable dependence on the lattice spacing;

- The values of the coefficients obtained for the different methods are compatible. This is probably because the smearing functions used are all very similar (close widths);

- The temperature dependence of the $c(T)$ is within the uncertainties. Thus, we don’t see a dependence of the coefficients on the temperature;

-The coefficients for all the lattice parameters are positive and so the conclusion is that real ***baryon density enhances the e.m. conductivity.***

Conclusions and Outlook

We studied the electromagnetic conductivity in dense quark-gluon plasma obtained within lattice simulations with $N_f = 2 + 1$ dynamical quarks.

The simulations were performed at imaginary chemical potential and to reconstruct the e.m. conductivity we employed the Backus-Gilbert method, both in the normal and modified versions, and the Tikhonov regularization method.

Our results were analytically continued to real values of baryon chemical potential.

The study indicates that e.m. conductivity of QGP raises with real baryon density and this dependence is quite strong.

We are concluding the study of the e.m. conductivity also in presence of strong magnetic fields ($eB \sim 4,\ 9\ GeV^2$) for different values of the temperature.

See Lorenzo Maio's talk

We plan to show and discuss these results very early in the future.

Thanks for your attention!

Back-up slides

Chemical potentials μ_f $(f = u, d, s)$ are coupled to quark number operators, $\mathcal{Z}(T, \mu_u, \mu_d, \mu_s)$, in a setup for which

$$\mu_u = \mu_d = \mu_B/3,\ \mu_s = 0$$

The path integral formulation $\mathcal{Z}(T, \mu_B)$, discretized via improved rooted staggered fermions and using exponentiated implementation of the chemical potentials, reads

$$\mathcal{Z}(T, \mu_B) = \int \mathcal{D}U e^{-S_{YM}} \prod_{f=u,d,s} \det \left[M^f_{st}(U\,.\,u_f) \right]^{1/4}$$

where

$$S_{YM} = -\frac{\beta}{3} \sum_{i,\mu\neq\nu} \left(\frac{5}{6} W^{1\times1}_{i;\mu,\nu} - \frac{1}{12} W^{1\times2}_{i;\mu\nu} \right)$$

is the Symanzik improved action and the staggered fermion matrix is defined as

$$M^f_{st}(U, \mu_f) = a m_f \delta_{i,j} + \sum_{\nu=1}^{4} \frac{\eta_{i;\nu}}{2} [e^{a\mu_f \delta_{\nu,4}} U^{(2)}_{i;\nu} \delta_{i,j-\hat{\nu}} - e^{-a\mu_f \delta_{\nu,4}} U^{\dagger}_{i-\hat{\nu};\nu} \delta_{i,j+\hat{\nu}}]$$

The naivest way we can think is to fix them such that the smearing function is as more peaked as possible. Then, we could fix the coefficients by minimizing the deterministic functional

$$A[q] = \int_0^\infty d\omega(\omega - \bar{\omega})^2 \{\delta(\bar{\omega}, \omega)\}^2$$

that can be interpreted as a measure of the width of the smearing function.

Minimizing the functional by imposing the unit area constraint, namely $\int_0^\infty d\omega \delta(\bar{\omega}, \omega) = 1$, we can extract an expression for the coefficients

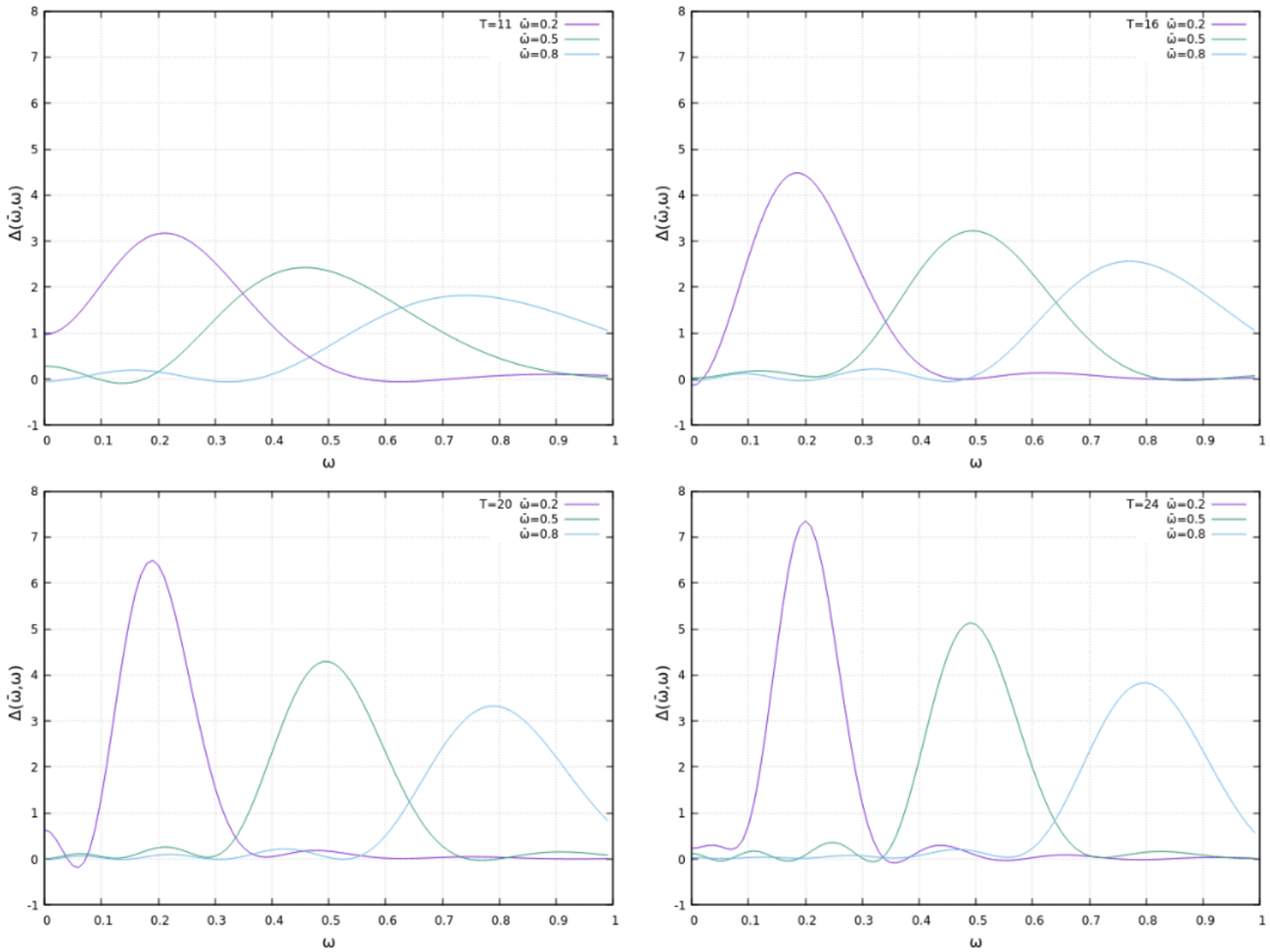
$$A_{ij}(\bar{\omega}) = \int_0^\infty d\omega(\omega - \bar{\omega})^2 K(\tau_i, \omega) K(\tau_j, \omega)$$

$$q(\bar{\omega}) = \frac{A^{-1}(\bar{\omega})R}{R^T A^{-1}(\bar{\omega})R}$$

$$R_i = \int_0^\infty d\omega K(\tau_i, \omega)$$

This would allow us to compute the smeared spectral function in the point $\bar{\omega}$ by using the relation $\hat{\rho}(\bar{\omega}) = \sum_i q_i(\bar{\omega}) C(\tau_i)$.

Note that the coefficients only depend on the point $\bar{\omega}$ where we want extract information and on the Euclidean time. Obviously the more point we have the tighter will be the smearing function $\delta(\bar{\omega}, \omega)$.



We show the smearing functions for different values of the point $\bar{\omega}$ in correspondence of which we want to extrapolate and for different number of points N_t.

At fixed $\bar{\omega}$, the smearing becomes more similar to a Dirac-δ function increasing N_t. This is due to the fact that, in absence of the second term, the functional tends to reduce the width of the smearing function as much as possible:

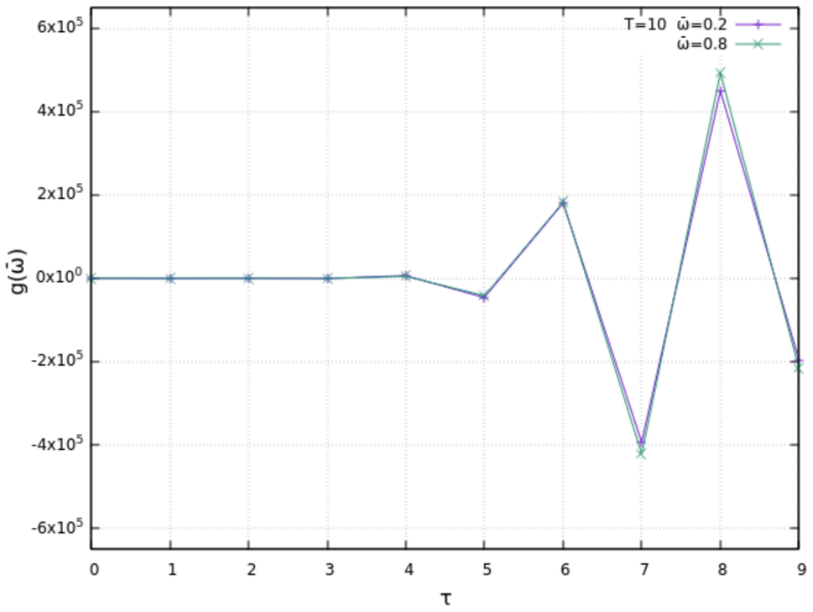
$$\lim_{N_t \to \infty} \Delta(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega}) \,.$$

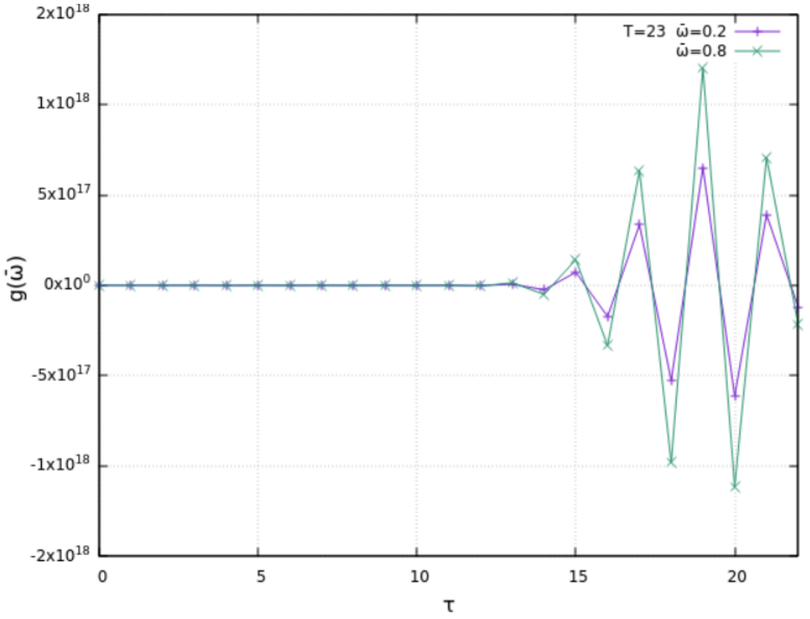
However there is a problem. In fact in the real case, where the correlators are affected by uncertainties, if the width of the smearing function is too small then the method becomes unstable and susceptible to noise in the data.

In fact when N_t becomes large, then also the coefficients become large and oscillating.

Thus, if we consider the errors associated to the correlators $C(t_i)$, we obtain that

$$\sum_i g_i(\bar{\omega})(C(\tau_i) + \delta C(\tau_i)) = \hat{\rho}(\bar{\omega}) + \sum_i g_i(\bar{\omega})\delta C(\tau_i)$$





Being the coefficients g_i large, also this term will be large and then the final error will be unacceptably large.

For this reason, we need to add the second term in the expression of the functional to take into account that we want a balance also with the statistical uncertainties.

Tikhonov regularization method

While in the in the BG method we add the second term that takes into account uncertainties in the TR scheme it's regularized the Singular Value Decomposition (SVD) of

$$A^{-1} = VDU^T$$

The diagonal matrice $D = diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1})$ might have large entries that represent the susceptibility of the data to noise.

Thus, the regularization is done by adding the regularizer γ to all the entries as

$$\tilde{D} = diag((\sigma_1\gamma)^{-1}, (\sigma_2 + \gamma)^{-1}, \ldots, (\sigma_n + \gamma)^{-1})$$

In this way, small σ_i will be smoothly cut-off.

Extrapolation of $\rho(\omega)/\omega$ for $\omega \to 0$

For the Kubo formula

$$\frac{\sigma_{ij}}{T} = \frac{1}{2T} \lim_{\omega \to 0} \frac{1}{\omega} \left(\rho^e_{ij}(\omega) + \rho^o_{ij}(\omega) \right)$$

we want to compute the ratio of the $\rho(\omega)/\omega$ for $\omega \to 0$.

We directly extract it by using the BG method. In fact, we have that

$$C^{e,o}_{ij}(\tau) = \int_0^\infty \frac{d\omega}{\pi} K(\tau,\omega) \rho^{e,o}_{ij}(\omega) = \int_0^\infty \frac{d\omega}{\pi} \tilde{K}(\tau,\omega) \frac{\rho^{e,o}_{ij}(\omega)}{\omega}$$

$$\tilde{K}(\tau,\omega) = \omega \frac{\cosh \omega(\tau - \beta/2)}{\sinh \omega\beta/2}$$

In this way, we can find the smeared spectral function as

$$\hat{\rho}_\lambda(\bar{\omega}) = \bar{\omega} \sum_i q_i(\bar{\omega}, \lambda) C(\tau_i)$$

Fixing $\bar{\omega} = 0$, this allows us to directly extract $\rho(\omega)/\omega$ for $\omega \to 0$.

To treat the statistical uncertainties, we made a binned bootstrap analysis.

We produced a number of new correlators samples by resampling the original sample. For each sample, we carried the analysis arriving at the final determination of the even and odd spectral functions and of the value of the conductivity. Then, we estimated the statistical error on the conductivity by monitoring it fluctuation over the different bootstrap samples.

In the error bar we reported, in the case of the modified BG, also the systematic error for both even and odd spectral function by using the definition

$$\Delta_{syst} = |r|\bar{\rho}(\bar{\omega})$$

where $r = 1 - \dfrac{\delta(\bar{\omega},\bar{\omega})}{\delta_0(\bar{\omega},\bar{\omega})}$ is the relative deviation at the peak.

When we summed the even and odd spectral functions to obtain the conductivity, we also summed in modulus the systematic errors. Then, we summed in modulus the final value of the systematic error to the statistical uncertainty coming from the bootstrap analysis.

Backus Gilbert VS Modified Backus Gilbert

The Backus-Gilbert, such as the Tikhonov regularization procedure, has three main problems:

- The resolution function depends on the lattice data and then it's non strictly correct to compare the spectral functions obtained at different lattice spacings;

- There is a systematic uncertainty related to the choice of the parameter λ which is difficult to estimate;

- The resolution function in the BG approach is not a simple Ansatz, but it's an output of the method. This complicates the comparison of the obtained results with different models.

These problems can be solved within the modified approach that also allow us to fix a common target function for different lattice spacings. This allows us also to perform the continuum limit.