Flavor structure of quark and lepton in modular symmetry

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Outline of my talk

1 Introduction

2 Modular symmetry

3 Modular invariant flavor model

4 Spontaneous CP violation

5 Prospect

1 Introduction

There are a lot of works challenging
Flavor Problems of quarks and leptons
by using Modular Symmetries.

Flavor mixing

CP violation

Mass hierarchy

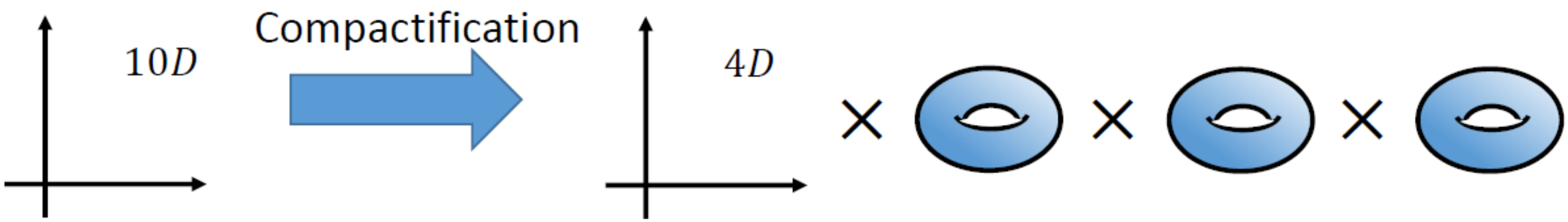
of quarks
and leptons

2 Modular symmetry

Superstring theory 10D
Our universe is 4D

The extra 6D
should be compactified.

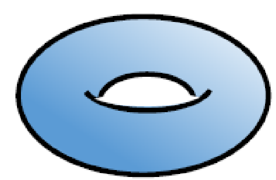
Torus compactification



We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\text{eff}}$$

$\mathcal{L}_{\text{eff}}$ depends on the structure of

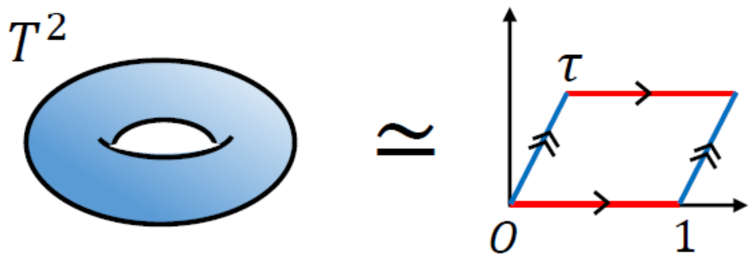


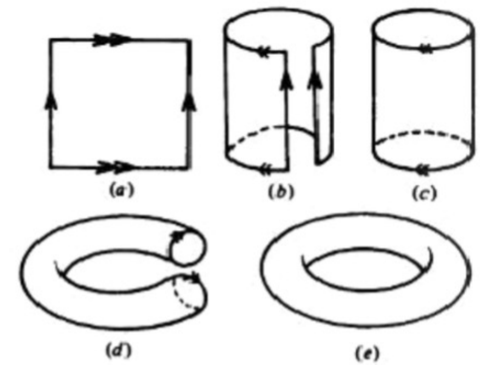
- $4D$ effective theory depends on internal space

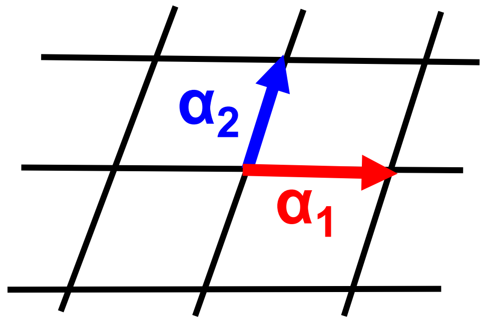
from T.H.Tatsuishi's slides

$2D$ torus (T^2) is equivalent to parallelogram with identification of confronted sides.

by Feruglio







$(x,y) \sim (x,y) + n_1\alpha_1 + n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice, which is spanned by two lattice vectors

$\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$

$\tau = \alpha_2 / \alpha_1$ is a modulus parameter (complex).

The same lattice is spanned by other bases under the transformation

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

ad-bc=1

a,b,c,d are integer SL(2,Z)

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

↓

$\tau = \alpha_2 / \alpha_1$

$$\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}$$

ad-bc=1
a,b,c,d are integer

Modular transformation

Modular transf. does not change the lattice (torus)

↓

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

e.g.) $\mathcal{L}_{\mathrm{eff}} \supset Y(\tau)_{ij}\phi\overline{\psi_i}\psi_j$

The modular transformation is generated by S and T .

$$\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

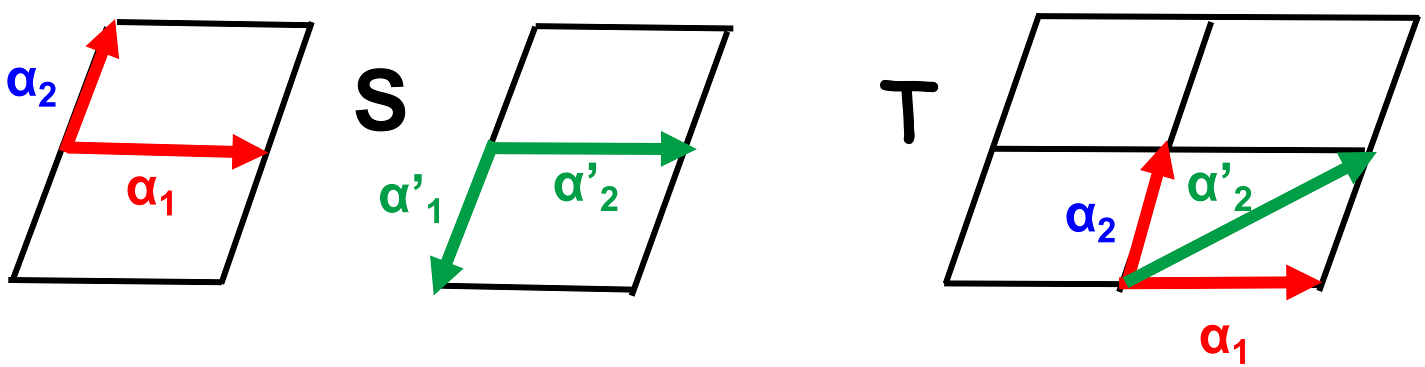
duality

$$T : \tau \longrightarrow \tau + 1$$

Dicrete shift symmetry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



$\tau = \alpha_2 / \alpha_1$

$$S : \tau \longrightarrow -\frac{1}{\tau},$$ Duality

$$T : \tau \longrightarrow \tau + 1.$$ Dicrete shift symmetry

$$S^2 = 1, \qquad (ST)^3 = 1.$$

± 1 is identified

generate infinite discrete group

Modular group

Modular group
$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$

Modular group has subgroups

Impose congruence condition

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\}$$

called principal congruence subgroups (normal subgroup)

$\Gamma_N \equiv \Gamma / \Gamma(N)$ quotient group finite group of level N

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_2 \simeq S_3$ $\Gamma_3 \simeq A_4$ $\Gamma_4 \simeq S_4$ $\Gamma_5 \simeq A_5$

isomorphic

We can consider effective theories with Γ_N symmetry.

$$\mathcal{L}_{\rm eff} \ni f(\tau)\phi^{(1)}\cdots\phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

modular form

In cases of Γ_N (N=2,3,4,5) (S_3, A_4, S_4, A_5) explicit forms of f(τ) have been obtained.

γ= S, T

$$\tau \longrightarrow \tau' = \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Modular transformation

$$f_i(\tau) \longrightarrow f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

modular forms of weight k

Automorphy factor

representation matrix

K is modular weight

Chiral superfields

$$(\phi^{(I)})_i(x) \longrightarrow (c\tau + d)^{-k_I} \rho(\gamma)_{ij} (\phi^{(I)})_j(x)$$

Modular forms are explicitly given if weight k is fixed.
On the other hand, chiral superfields are not modular forms and we have no restriction on the possible value of weight k_I, a priori.

Consider $f_i(\tau)\ \phi^{(I)}\phi^{(J)}H$

Automorphy factor vanishes if $k = k_I + k_J$ $\quad (c\tau+d)^k(c\tau+d)^{-k_I}(c\tau+d)^{-k_J} = (c\tau+d)^{k-k_I-k_J}$

$\mathcal{L}_{\text{eff}}$ is modular invariant if sum of weights satisfy $\sum k_I = k$.

Modular invariant kinetic terms of matters are obtained:

Kähler potential $\quad K^{\text{matter}} = \frac{1}{[i(\bar{\tau}-\tau)]^{k_I}}|\phi^{(I)}|^2$

A_4 Modular symmetry

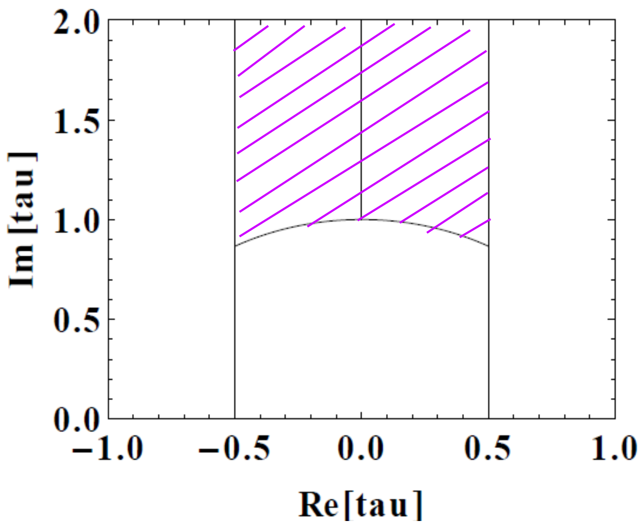
$$\Gamma_N \simeq \{S,T|S^2=\mathbb{I},(ST)^3=\mathbb{I},T^N=\mathbb{I}\}$$

Taking $T^3=1$, we get A_4 modular group (Γ_3). N=3 $\sim A_4$

# of modular forms is k+1 (for N=3) k: weight

There are 3 linealy independent modular forms for weight 2 , which forms A_4 triplet.

Fundamental domain of τ on SL(2,Z)



$$S=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}$$

$$T=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix}$$

A_4 triplet of modular forms with weight 2

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

F. Feruglio, arXiv:1706.08749

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),
\end{aligned}$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Dedekind eta-function

$$Y_2^2 + 2Y_1Y_3 = 0$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$q = e^{2\pi i\tau}$$

Modular forms with higher weights k=4, 6 … are constructed by them.

3 Modular invariant flavor model

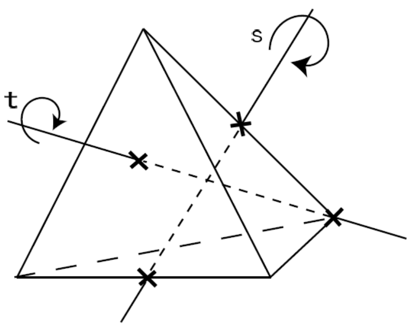
We can construct quark / lepton mass matrices
in the framework of modular symmetry.

Non-Abelian Discrete Symmetry

A_4 group

Irreducible representations: 1, 1′, 1″, 3
The minimum group containing triplet

It could be adjusted to Family Symmetry.



Symmetry of tetrahedron

3: (u_L, c_L, t_L) , 1: u_R , 1″: c_R , 1′: t_R

Flavor symmetry should be broken !
We should know how to break the flavor symmetry.

Key : Modulus τ and Modular forms

Quarks

Mass matrix in SM : $y_q\, q^c \langle H \rangle Q$

Let us introduce Modulus (in the case of A_4).

The modulus τ plays the role of a spurion

Up-type quark sector

$$q_1 = u^c\,, \quad q_{1"} = c^c\,, \quad q_{1'} = t^c\,, \quad Q_3 = \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L\,, \quad Y_3^{(k)}(\tau) = \begin{pmatrix} Y_1^{(k)}(\tau) \\ Y_2^{(k)}(\tau) \\ Y_3^{(k)}(\tau) \end{pmatrix}$$

A_4 representation

1 x1 x 3 x 3 ⇒ 1

$$\alpha\, q_1\, H\, Y_3^{(k)}(\tau)\, Q_3 = \alpha u^c [\, Y_1^{(k)}(\tau) u + Y_3^{(k)}(\tau) c + Y_2^{(k)}(\tau) t\,] H$$

VEV of H and τ give quark mass terms

$$\alpha \langle H \rangle Y_1^{(k)}(\langle \tau \rangle)\; u^c u + \alpha \langle H \rangle Y_3^{(k)}(\langle \tau \rangle)\; u^c c + \alpha \langle H \rangle Y_2^{(k)}(\langle \tau \rangle)\; u^c t$$

up-type quark masses are given as

$$\alpha_u\, u_1^c\, H\, Y_3^{(k)}(\tau)\, Q_3 + \beta_u\, c_{1"}^c\, H\, Y_3^{(k)}(\tau)\, Q_3 + \gamma_u\, t_{1'}^c\, H\, Y_3^{(k)}(\tau)\, Q_3$$

We can construct a simple mass matrix by using weight 2 modular forms

A_4 assignments: left-handed doublet 3 right-handed singlets 1, 1”, 1’

$$M_f = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

Typical mass matrix of fermions by using weight 2 modular forms

for both up- and down-quarks and charged leptons

6 real prarameters α_q, β_q, γ_q and τ are responsible for quark mass matrices.

However, this simple structure for up and down quarks is inconsistent with V_{ub} .

Towards multi-Higgs or higher weight modular forms

An example quark mass matrix

H. Okada, M. Tanimoto, Eur. Phys. J C 81 (2020) 5, 053B05, arXiv:1905.13421

	Q	(d^c, s^c, b^c)	(u^c, c^c, t^c)	$H_{u,d}$	$\mathbf{Y}_{\mathbf{3}}^{(2)}$	$\mathbf{Y}_{\mathbf{3}}^{(6)}, \ \mathbf{Y}_{\mathbf{3'}}^{(6)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1}'',\ \mathbf{1}')$	$(\mathbf{1},\ \mathbf{1}'',\ \mathbf{1}')$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}\ ,\quad \mathbf{3}'$
$-k_I$	-2	$(0,\ 0,\ 0)$	$(-4,\ -4,\ -4)$	0	$k=2$	$k=6$

Sum of weights vanishes due to $\sum k_I = k$

$$M_d = \begin{pmatrix} \alpha_d & 0 & 0 \\ 0 & \beta_d & 0 \\ 0 & 0 & \gamma_d \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL} ,$$

weight 2 modular forms

$$M_u = \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \left[\begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix} + \begin{pmatrix} g_{u1} & 0 & 0 \\ 0 & g_{u2} & 0 \\ 0 & 0 & g_{u3} \end{pmatrix} \begin{pmatrix} Y_1'^{(6)} & Y_3'^{(6)} & Y_2'^{(6)} \\ Y_2'^{(6)} & Y_1'^{(6)} & Y_3'^{(6)} \\ Y_3'^{(6)} & Y_2'^{(6)} & Y_1'^{(6)} \end{pmatrix} \right]_{RL}$$

weight 6 modular forms

We have additional 3 complex parameters.

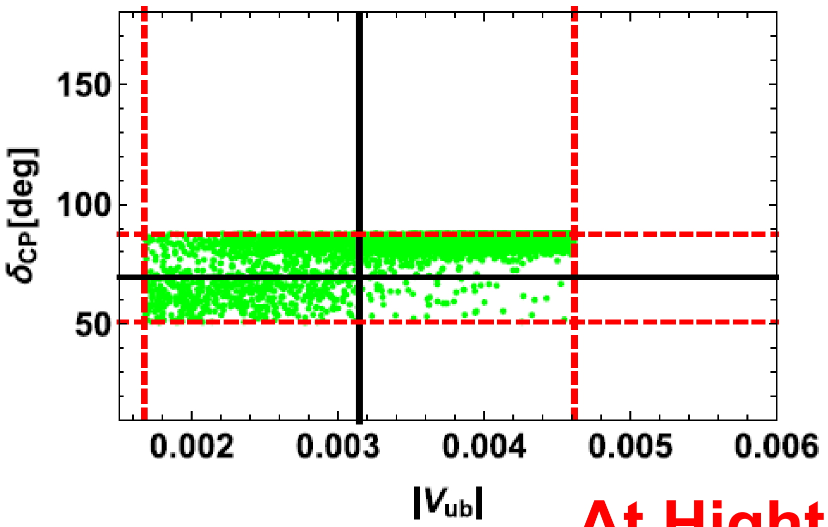
of modular forms is k+1

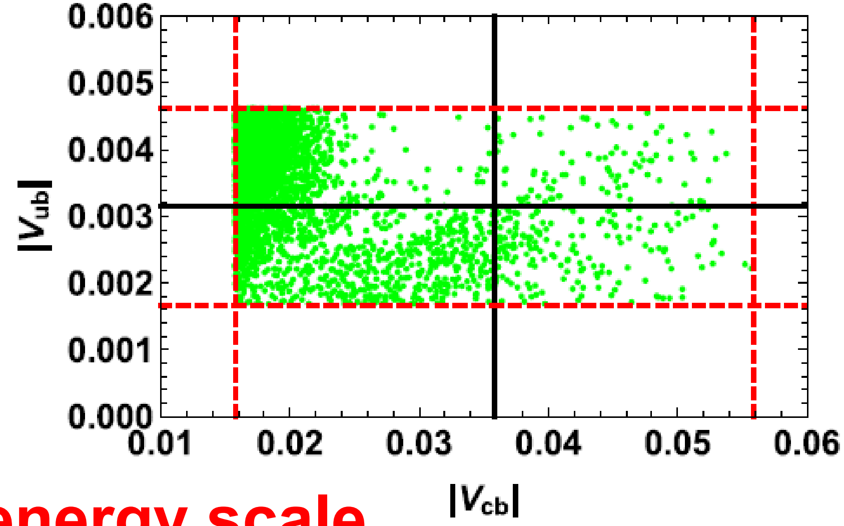
weight 6 k=6

7 modular forms

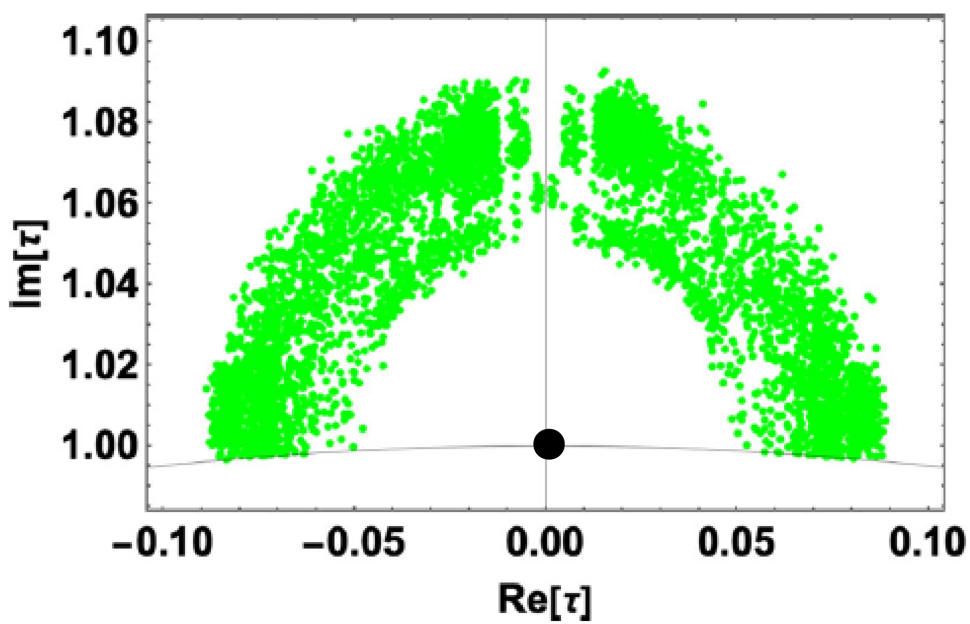
$$Y_{\mathbf{1}}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3 \ ,$$

$$Y_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \ , \qquad Y_{\mathbf{3'}}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}$$





At Hight energy scale



Charged Leptons

Left-handed **3 of A_4: (ℓe, $\ell \mu$, $\ell \tau$) ,** Right-handed **1: e_R , 1”: μ_R , 1’: τ_R**

$$w_e = \alpha e_R H_d(LY) + \beta \mu_R H_d(LY) + \gamma \tau_R H_d(LY)$$

$$1_R^{(')('')} \times 3_L \times 3_Y \Rightarrow 1$$

α, β, γ are fixed by the charged lepton masses

$$M_E = \mathrm{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}$$

If τ (complex) is fixed, modular symmetry is broken, and flavor structure of mass matrices is determined including CP violation !

Modulus τ controls flavor mixing and CP phase ???

4 Spontaneous CP violation

At present,

CP violation is observed in quark sector (CKM matrix),

where CP is explicitly broken.

In the near future, CP violation will be confirmed

by the neutrino oscillation experiments.

In addition,

T violation is expected to be observed in EDM of q/l.

Is there other souces of CP (T) violation ?

What is the origin of CP (T) violation ?

High energy theory is often CP invariant,

then CP is violated spontaneously.

VEV of modulus τ is possibly the source of CP violation !

CP symmetry

P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 07 (2019) 165

CP transformation

$$\tau \xrightarrow{\text{CP}} -\tau^* , \qquad \psi(x) \xrightarrow{\text{CP}} X_r \overline{\psi}(x_P) , \qquad \mathbf{Y}_\mathbf{r}^{(\mathrm{k})}(\tau) \xrightarrow{\text{CP}} \mathbf{Y}_\mathbf{r}^{(\mathrm{k})}(-\tau^*) = \mathbf{X}_\mathbf{r} \mathbf{Y}_\mathbf{r}^{(\mathrm{k})*}(\tau)$$

$$\mathbf{X}_\mathbf{r} \rho_\mathbf{r}^*(g) \mathbf{X}_\mathbf{r}^{-1} = \rho_\mathbf{r}(g') , \qquad g, g' \in G$$

$\mathbf{X}_\mathbf{r} = \mathbb{1}_\mathbf{r}$ can be taken in the base of symmetric S and T.

After fixing τ, real part of τ provides imaginary part of the mass matrices.

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots , \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$

$$q = e^{2\pi i \tau}$$

$|q| \ll 1$

$$Y_2^2 + 2Y_1Y_3 = 0$$

CP conserved modular invariant theory

$$\tau \xrightarrow{\mathrm{CP}} -\tau^*\,, \qquad \psi(x) \xrightarrow{\mathrm{CP}} X_r\overline{\psi}(x_P)\,, \qquad \mathbf{Y}^{(\mathrm{k})}_{\mathbf{r}}(\tau) \xrightarrow{\mathrm{CP}} \mathbf{Y}^{(\mathrm{k})}_{\mathbf{r}}(-\tau^*) = \mathbf{X}_{\mathbf{r}}\mathbf{Y}^{(\mathrm{k})*}_{\mathbf{r}}(\tau)$$

We can construct CP invariant mass matrices.

$$M_E(\tau) = M_E^*(\tau)\,, \qquad M_\nu(\tau) = M_\nu^*(\tau) \qquad \mathbf{X}_{\mathbf{r}} = \mathbb{1}_{\mathbf{r}}$$

Is the CP violation realized by τ, which is consistent with the observed lepton mixing angles and neutrino masses ?

Simple model of CP violation in Lepton sector

H.Okada, M.Tanimoto, JHEP 03(2021),010 [arXiv:2012.01688 [hep-ph]]

	L	(e^c, μ^c, τ^c)	H_u	H_d	$\mathbf{Y_r^{(2)}}$, $\mathbf{Y_r^{(4)}}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1}, \mathbf{1}'', \mathbf{1}')$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$, $\{\mathbf{3}, \mathbf{1}, \mathbf{1}'\}$
$-k_I$	-2	$(0, 0, 0)$	0	0	2, 4

$$M_E = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

3, 1, 1'

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L \mathbf{Y_r^{(k)}})_\mathbf{1}$$

Weinberg operator by using weight 4 modular forms

5 modular forms

$$M_\nu = \frac{v_u^2}{\Lambda}\left[\begin{pmatrix} 2Y_1^{(4)} & -Y_3^{(4)} & -Y_2^{(4)} \\ -Y_3^{(4)} & 2Y_2^{(4)} & -Y_1^{(4)} \\ -Y_2^{(4)} & -Y_1^{(4)} & 2Y_3^{(4)} \end{pmatrix} + g_{\nu 1} \mathbf{Y}_\mathbf{1}^{(4)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_{\nu 2} \mathbf{Y}_{\mathbf{1'}}^{(4)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]_{LL}$$

of modular forms is k+1

weight 4 k=4

5 modular forms

$$Y_{\mathbf{1}}^{(4)}(\tau)=Y_1(\tau)^2+2Y_2(\tau)Y_3(\tau)\,,\qquad Y_{\mathbf{1}'}^{(4)}(\tau)=Y_3(\tau)^2+2Y_1(\tau)Y_2(\tau)\,,$$

$$Y_{\mathbf{1}''}^{(4)}(\tau)=Y_2(\tau)^2+2Y_1(\tau)Y_3(\tau)=0\,,\qquad Y_{\mathbf{3}}^{(4)}(\tau)=\begin{pmatrix}Y_1^{(4)}(\tau)\\ Y_2^{(4)}(\tau)\\ Y_3^{(4)}(\tau)\end{pmatrix}=\begin{pmatrix}Y_1(\tau)^2-Y_2(\tau)Y_3(\tau)\\ Y_3(\tau)^2-Y_1(\tau)Y_2(\tau)\\ Y_2(\tau)^2-Y_1(\tau)Y_3(\tau)\end{pmatrix}$$

$$M_E(\tau) \xrightarrow{CP} M_E(-\tau^*) = M_E^*(\tau) = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1(\tau)^* & Y_3(\tau)^* & Y_2(\tau)^* \\ Y_2(\tau)^* & Y_1(\tau)^* & Y_3(\tau)^* \\ Y_3(\tau)^* & Y_2(\tau)^* & Y_1(\tau)^* \end{pmatrix}_{RL}$$

$$M_\nu(\tau) \xrightarrow{CP} M_\nu(-\tau^*) = M_\nu^*(\tau)$$

$$= \frac{v_u^2}{\Lambda} \left[\begin{pmatrix} 2Y_1^{(4)*}(\tau) & -Y_3^{(4)*}(\tau) & -Y_2^{(4)*}(\tau) \\ -Y_3^{(4)*}(\tau) & 2Y_2^{(4)*}(\tau) & -Y_1^{(4)*}(\tau) \\ -Y_2^{(4)*}(\tau) & -Y_1^{(4)*}(\tau) & 2Y_3^{(4)*}(\tau) \end{pmatrix} + g_1^\nu \mathbf{Y_1^{(4)*}}(\tau) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_2^\nu \mathbf{Y_{1'}^{(4)*}}(\tau) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]$$

$$\mathbf{Y_r^{(k)}}(\tau) \xrightarrow{\mathrm{CP}} \mathbf{Y_r^{(k)}}(-\tau^*) = \mathbf{X_r} \mathbf{Y_r^{(k)*}}(\tau) \qquad \mathbf{X_r} = \mathbb{1}_\mathbf{r}$$

Impose CP invariance

$M_E(\tau) = M_E^*(\tau)$, $\quad M_\nu(\tau) = M_\nu^*(\tau)$ $\quad$ which leads to g_1^ν and g_2^ν being real.

6 parameters + τ = 8 parameters

3 charged lepton masses+ 2 neutrino mass differences+ 3 mixing angles = 8

CP phase and mass absolute values can be predicted !

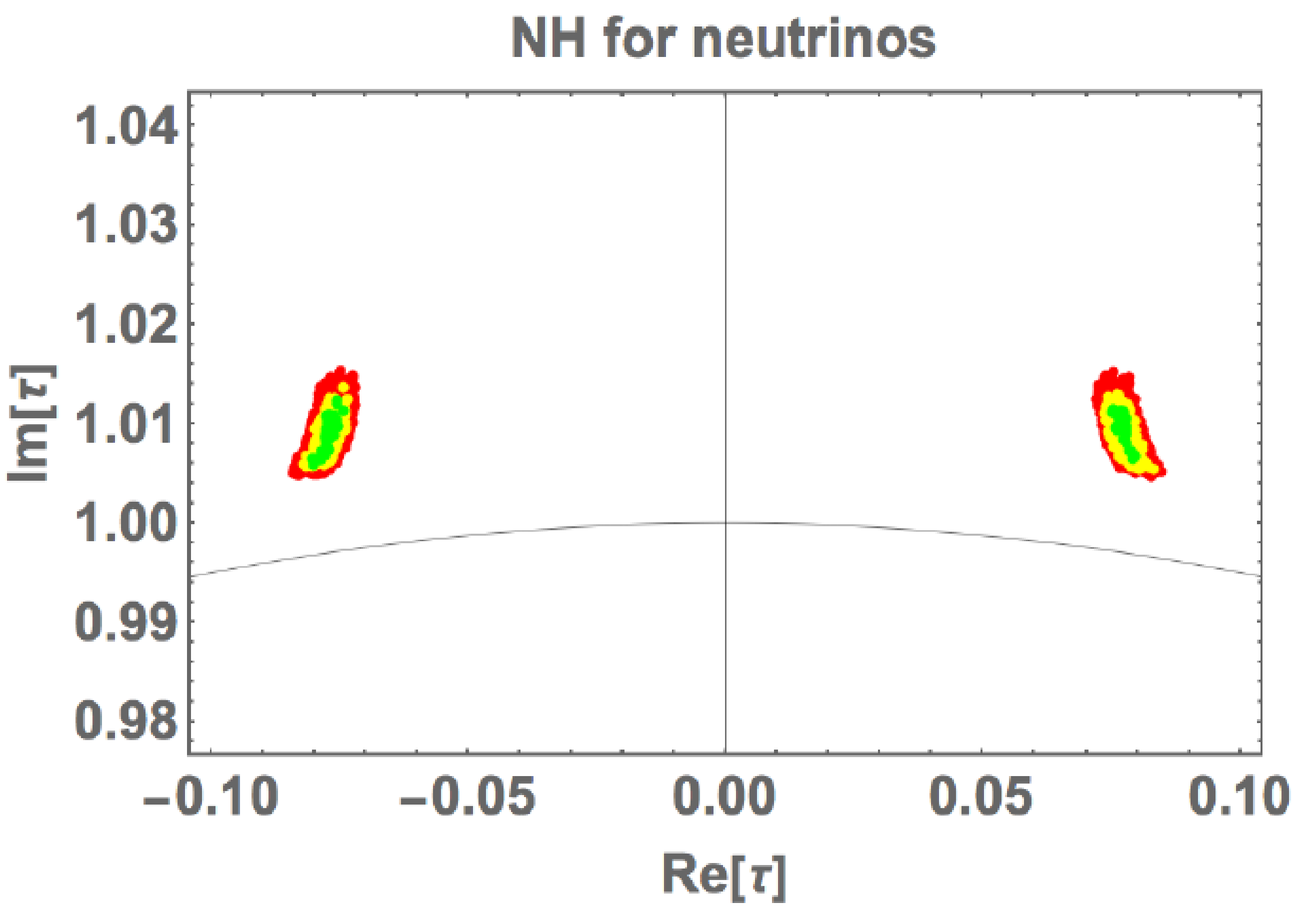
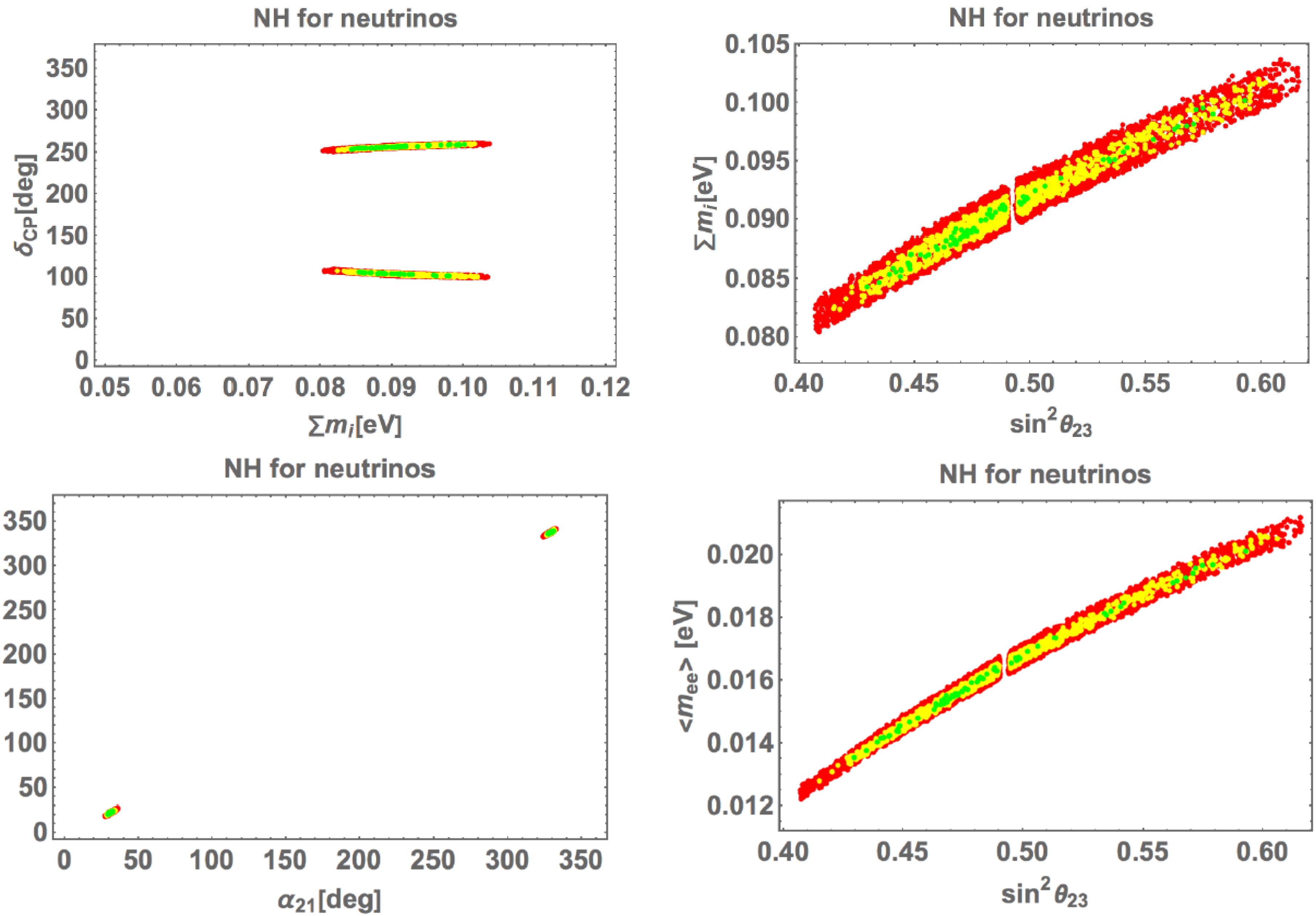


Figure 1: Allowed regions of τ for NH. Green, yellow and red correspond to 2σ, 3σ, 5σ confidence levels, respectively. The solid curve is the boundary of the fundamental domain, $|\tau| = 1$.

δ_{CP} is predicted clearly in $[98°, 110°]$ and $[250°, 262°]$ at 3σ confidence level.



Alternative: Seesaw Model of Neutrinos

H.Okada, Y.Shimizu, M.Tanimoto,T.YoshidaJHEP 07(2021) 184 [arXiv:2105.14292 [hep-ph]].

	L	(e^c, μ^c, τ^c)	N^c	H_u	H_d	$Y_3^{(k)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1}'',\ \mathbf{1}')$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$
weight	-1	$(k_e,\ k_\mu,\ k_\tau)$	-1	0	0	k

$$k_e = -1,\ k_\mu = -3,\ k_\tau = -5$$

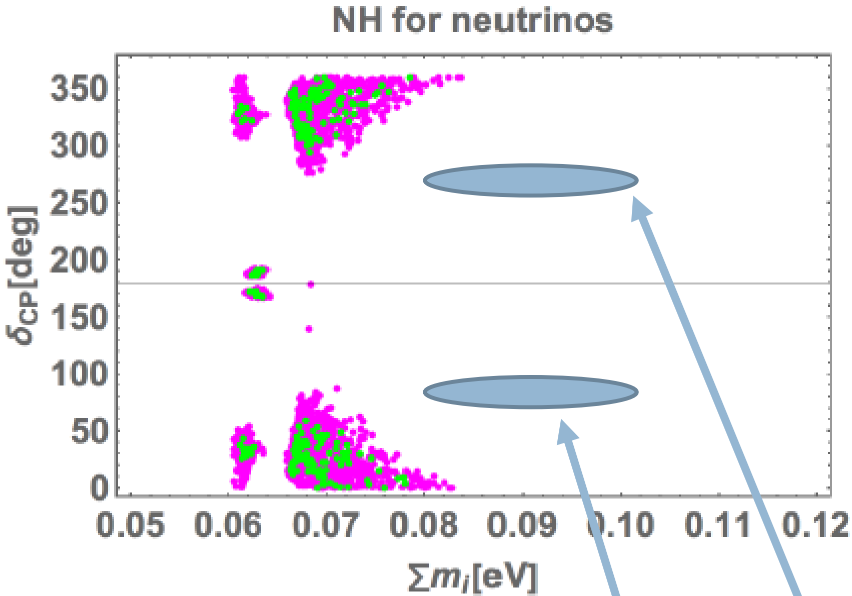
$$M_E(\tau) = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ Y_2^{(4)}(\tau) & Y_1^{(4)}(\tau) & Y_3^{(4)}(\tau) \\ Y_3^{(6)}(\tau) + g_e Y_3'^{(6)}(\tau) & Y_2^{(6)}(\tau) + g_e Y_2'^{(6)}(\tau) & Y_1^{(6)}(\tau) + g_e Y_1'^{(6)}(\tau) \end{pmatrix}$$

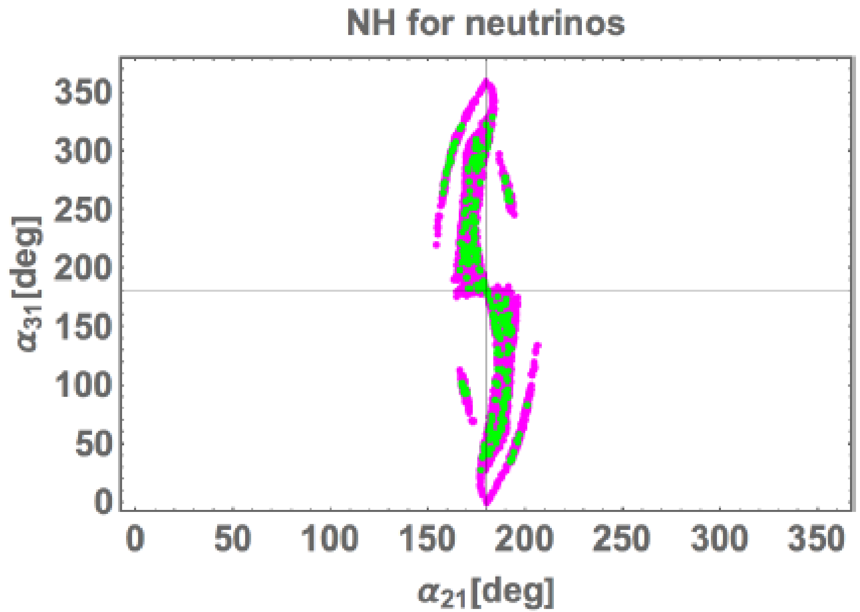
$$M_D = \gamma_\nu v_u \begin{pmatrix} 2Y_1 & (-1+g_D)Y_3 & (-1-g_D)Y_2 \\ (-1-g_D)Y_3 & 2Y_2 & (-1+g_D)Y_1 \\ (-1+g_D)Y_2 & (-1-g_D)Y_1 & 2Y_3 \end{pmatrix}$$

Also 8 parameters
g_e and g_D are real.

$$M_N = \Lambda \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$$

$$M_\nu = M_D^{\mathrm{T}} M_N^{-1} M_D$$





Prediction by Weinberg operator

5　Prospect

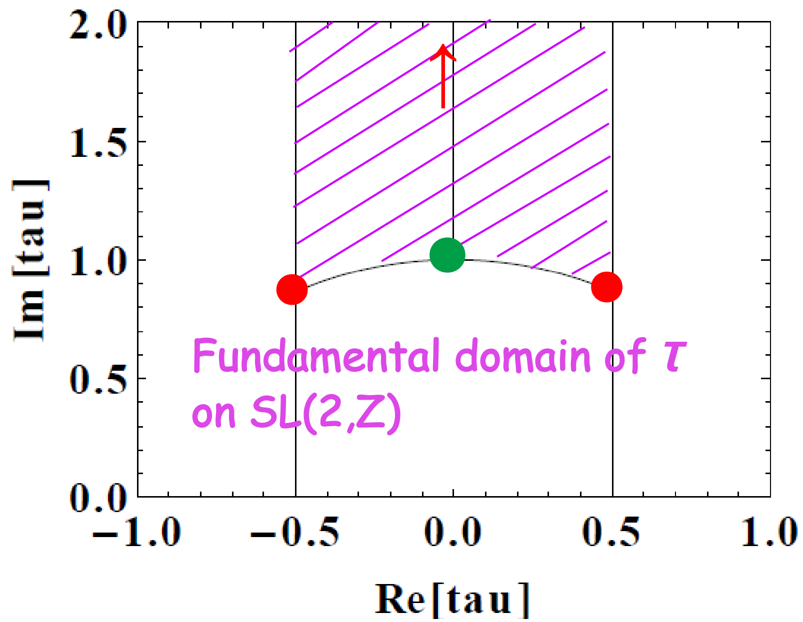
● **What is a Principle of fixing modulus τ ?**

☆ **Moduli stabilization** (non-perturbative effect, model dependent)

Some models indicate the potential minimum at nearby boundary of fundamental domain.

P.P.Novichkov, J.T.Penedo, S.T.Petcov ,JHEP03 (2022) 149 , arXiv:2201.02020

☆ **Fixed point of τ　Residual symmetry**



● Z_2: τ = i　S symmetry

● Z_3: τ = -1/2+√3/2i　ST symmetery

P.P.Novichkov, J.T.Penedo, S.T.Petcov arXiv:2201.02020

Okada and Tanimoto、PRD103(2021)015005 arXiv:2009.14242

Interesting physics of flavors

$\Gamma_2 \simeq S_3 \quad \Gamma_3 \simeq A_4 \quad \Gamma_4 \simeq S_4 \quad \Gamma_5 \simeq A_5 \quad \ldots\ldots$

$\Gamma_3' \simeq A_4' \quad \Gamma_4' \simeq S_4' \quad \Gamma_5' \simeq A_5'$ (double covering groups)

- Realization of quarks/leptons mass hierarchy at nearby fixed points
- EDM, g-2, FCNC in SMEFT
- Multi-Higgs
- Leptogenesis from modulus τ (CP violating source)

Modular symmerty provides new approachs for flavor problems !

Back up slides

Generalized CP Symmetry

Neufeld, Grimus, Ecker,1987

CP symmetry is non-trivial when flavor symmetry is set in the Yukawa sector.

Transformations of Irreducible representation in flavor group G

$$\psi(x) \xrightarrow{g} \rho_{\mathrm{r}}(g)\,\psi(x)\,, \quad g \in G_f$$ under a Flavor Symmetry

$$\psi(x) \xrightarrow{CP} X_{\mathrm{r}}\,\overline{\psi}(x_P)$$ $x=(t,\mathbf{x}),\ x_P=(t,-\mathbf{x})$

If $X_{\mathrm{r}} = \mathbb{1}_{\mathrm{r}}$, we have canonical CP transformation.
However, under the CP invariance of Lagrangian,
X_r is not necessary element of flavor group G

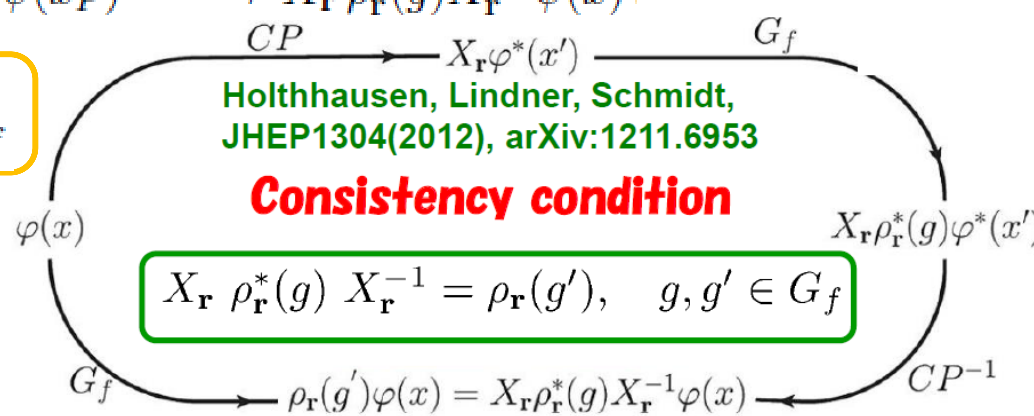
X_r should satisfy a condition

$$\psi(x) \xrightarrow{CP} X_{\mathrm{r}}\,\overline{\psi}(x_P) \xrightarrow{g} X_{\mathrm{r}}\,\rho^*_{\mathrm{r}}(g)\overline{\psi}(x_P) \xrightarrow{CP^{-1}} X_{\mathrm{r}}\,\rho^*_{\mathrm{r}}(g)X_{\mathrm{r}}^{-1}\psi(x)$$

$$X_{\mathrm{r}}\,\rho^*_{\mathrm{r}}(g)X_{\mathrm{r}}^{-1} = \rho_{\mathrm{r}}(g') \quad g,\, g' \in G_f$$

自己同型 automorphism of G_f

$X_{\mathrm{r}} = \rho_{\mathrm{r}}(g),\ g \in G_f$ **for S_3, S_4, A_4, A_5**



Mu-Chun Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys. B883 (2014) 267-305

Consider 3 fixed points in A_4 modular symmetry

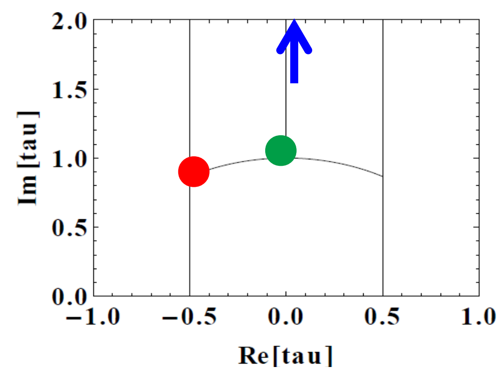
↑ **T ($\tau \rightarrow \tau+1$) preserved : $\langle\tau\rangle = \infty\, i$** ($q=0$) $(Y_1, Y_2, Y_3) = (1, 0, 0)$ Z_3: {1,T,T²}

● **S ($\tau \rightarrow -1/\tau$) preserved : $\langle\tau\rangle = i$** ($q=e^{-2\pi}$) $(Y_1, Y_2, Y_3) = Y_1(i)\,(1, 1-\sqrt{3}, -2+\sqrt{3})$ Z_2: {1,S}

● **ST preserved : $\langle\tau\rangle = \omega$** $(Y_1, Y_2, Y_3) = Y_1(i)\,(1, \omega, -1/2\,\omega^2)$ Z_3: {1,ST,(ST)²}

$$q = e^{2\pi i\tau}$$

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$



↑ Z_3 {1,T,T²}: $T^\dagger M_q^\dagger M_q T = M_q^\dagger M_q \Rightarrow [T, M_q^\dagger M_q] = 0$

● Z_2 {1,S}: $S^\dagger M_q^\dagger M_q S = M_q^\dagger M_q \Rightarrow [S, M_q^\dagger M_q] = 0$

● Z_3 {1,ST,(ST)²}: $(ST)^\dagger M_q^\dagger M_q ST = M_q^\dagger M_q \Rightarrow [ST, M_q^\dagger M_q] = 0$

Mixing matrices which diagonalise $M_q^\dagger M_q$ also diagonalize T, S and ST, respectively !

Modular transformation is the transformation of modulus τ

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}, \quad T : \tau \longrightarrow \tau + 1.$$

weight 2; k=2
3 modular forms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

S transformation

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix},$$

$(c\tau + d)^k$ cτ+d = -τ

T transformation

$$\begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$(c\tau + d)^k$ cτ+d = 1

$$\rho(\mathrm{S}) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega = \exp(i\frac{2}{3}\pi)$$

Flavor symmetry acts non-linealy (Modular forms).

Flavor mixing

Neutrino large mixing angles are reproduced easily thanks to Non-Abelian discrete symmetery, A_4, S_4 ....

Mass hierarchy

Mass hierarchy of quark and charged leptons could be reproduced thanks to fixed points

F. Feruglio, V.Gherardi, A.Romanino, A.Titov, 05 (2021)242 arXiv:2101.08718

P.P.Novichkov, J.T.Penedo, S.T.Petcov, JHEP04 (2021) 206 arXiv:2102.07488

CP violation

Modulus τ is a source of CP violation.

Spontaneous CP violation could be realized via modulus τ.

P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov, JHEP 07(2019)165, arXiv:1905.11970

H.Okada, M.Tanimoto, JHEP 03(2021),010, arXiv:2012.01688

Let us consider Modular forms with higher weights k=4, 6 …

# of modular forms is k+1

Weight 2
3 Modular forms

$$\mathbf{Y}_3{}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1}'} \oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1}''} \oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}}$$

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}\,, \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}''\,, \quad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'\,, \quad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}\,.$$

Modular forms with higher weights are constructed by the tensor product of modular forms of weight 2

J.T.Penedo, S.T.Petcov, Nucl.Phys.B939(2019)292

Weight 4
5 Modular forms

$$\mathbf{Y}_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3\,, \quad \mathbf{Y}_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1Y_2\,, \quad \mathbf{Y}_{\mathbf{1}''}^{(4)} = Y_2^2 + 2Y_1Y_3 = 0\,,$$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}\,,$$

Weight 6
7 Modular forms

$$\mathbf{Y}_{\mathbf{1}}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3\,,$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix}\,, \quad \mathbf{Y}_{\mathbf{3}'}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix}$$

Consider the case of Normal neutrino mass hierarchy

$m_1 < m_2 < m_3$

A_4 triplet $3\ (L_e, L_\mu, L_\tau)$ $3\ (\nu_{eR}, \nu_{\mu R}, \nu_{\tau R})$

A_4 singlets $e_R\ 1\ ;\ \mu_R\ 1''\ ;\ \tau_R\ 1'$

$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix}$$

$$\mathcal{Y}_\nu = \begin{pmatrix} 2g_1 Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1 Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1 Y_3 \end{pmatrix}$$

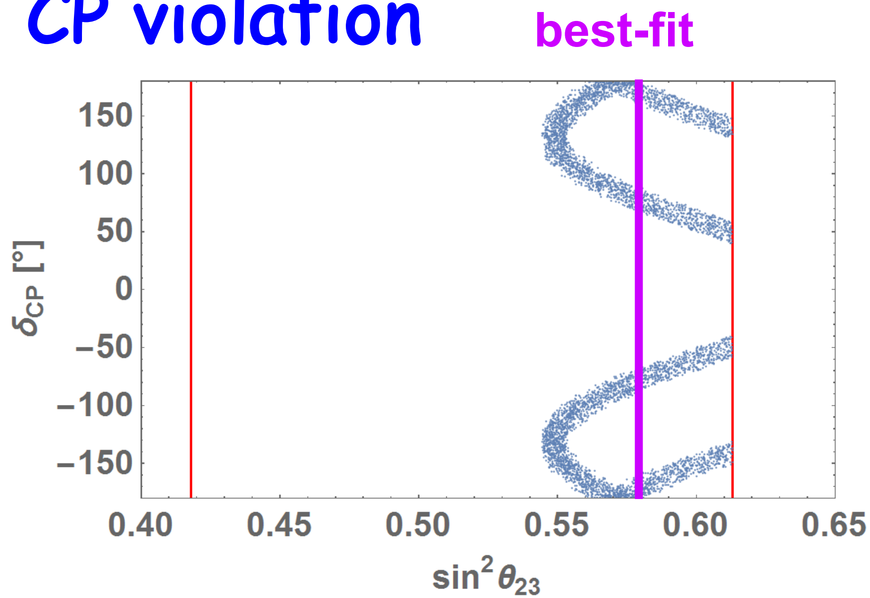
$$\mathcal{M}_R = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \Lambda$$

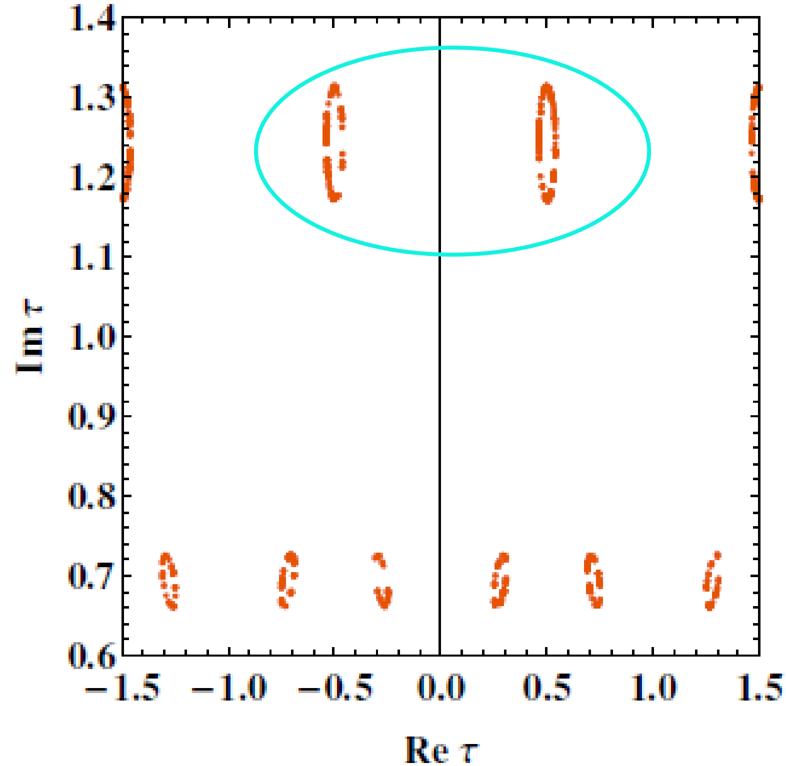
Parameters:
α, β, γ, $g_2/g_1=g$, τ

m_e, m_μ, m_τ fix α, β, γ .

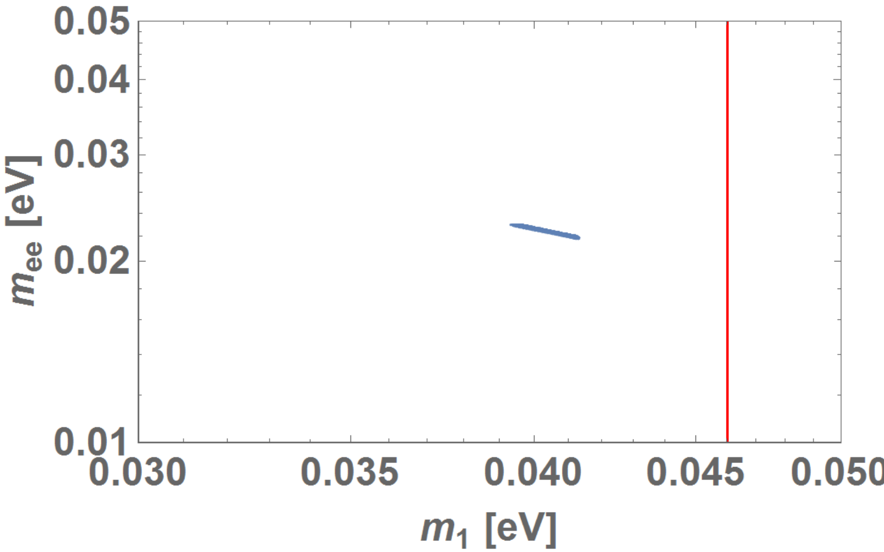
$\Delta m^2_{sol}/\Delta m^2_{atm}$ and θ_{23}, θ_{12}, θ_{13} fix $g_2/g_1=g$ and τ .

CP violation





Predicted <m_{ee}>



$m_1 \simeq m_2 \simeq 40\text{meV}$ and $m_3 \simeq 60\text{meV}$

$\sum m_i \sim 140$ meV

Planck 2018 results < 0.12 eV@ΛCDM model

Alternative successful charged lepton mass matrices

Original $$M_E(\tau) = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ Y_2^{(4)}(\tau) & Y_1^{(4)}(\tau) & Y_3^{(4)}(\tau) \\ Y_3^{(6)}(\tau) + g_e Y_3'^{(6)}(\tau) & Y_2^{(6)}(\tau) + g_e Y_2'^{(6)}(\tau) & Y_1^{(6)}(\tau) + g_e Y_1'^{(6)}(\tau) \end{pmatrix}$$

$$M_E(\tau) = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ Y_2(\tau) & Y_1(\tau) & Y_3(\tau) \\ Y_3^{(6)}(\tau) + g_e Y_3'^{(6)}(\tau) & Y_2^{(6)}(\tau) + g_e Y_2'^{(6)}(\tau) & Y_1^{(6)}(\tau) + g_e Y_1'^{(6)}(\tau) \end{pmatrix}$$

$$M_E(\tau) = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ Y_2(\tau) & Y_1(\tau) & Y_3(\tau) \\ Y_3^{(8)}(\tau) + g_e Y_3'^{(8)}(\tau) & Y_2^{(8)}(\tau) + g_e Y_2'^{(8)}(\tau) & Y_1^{(8)}(\tau) + g_e Y_1'^{(8)}(\tau) \end{pmatrix}$$

$$M_E(\tau) = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ Y_2^{(4)}(\tau) & Y_1^{(4)}(\tau) & Y_3^{(4)}(\tau) \\ Y_3^{(8)}(\tau) + g_e Y_3'^{(8)}(\tau) & Y_2^{(8)}(\tau) + g_e Y_2'^{(8)}(\tau) & Y_1^{(8)}(\tau) + g_e Y_1'^{(8)}(\tau) \end{pmatrix}$$

BAU via Leptogenesis

$$Y_B = \frac{n_B}{s} = (0.852 - 0.888) \times 10^{-10}$$

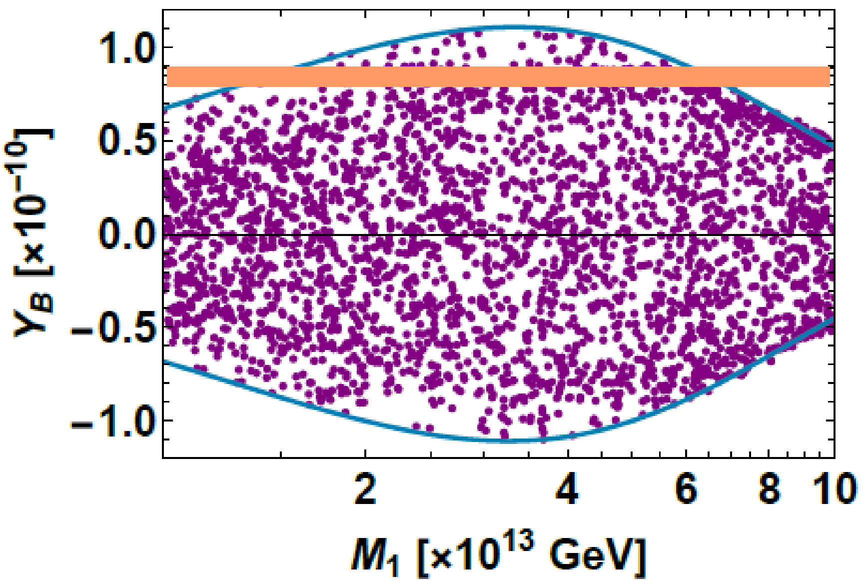


Figure 12: Predictive Y_B versus M_1. Points correspond to the output of section 4 at $\sqrt{\chi^2} \leq 3$. Horizontal lines denote the upper and lower bounds of observed Y_B in Eq. (39). The blue solid curves denote the boundary of Y_B.

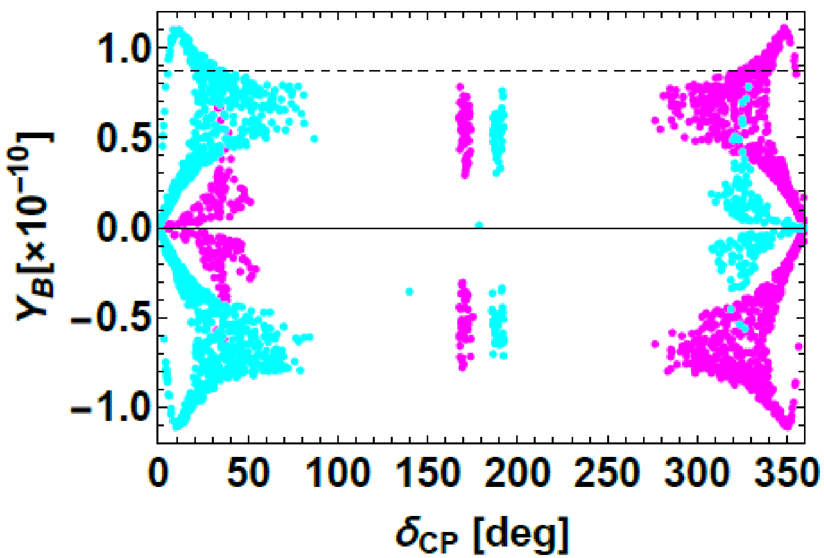


Figure 14: Predictive Y_B versus δ_{CP} at $M_1 = 3.36 \times 10^{13}$ GeV. Points correspond to the low energy output of section 4 at $\sqrt{\chi^2} \leq 3$, where cyan and magenta correspond to positive and negative Re $[\tau]$, respectively. Horizontal dashed line denotes the central value of observed Y_B.

	NH	IH
τ	$-0.0796 + 1.0065\,i$	$0.0103 + 1.0812\,i$
g_1^ν	0.124	-1.17
g_2^ν	-0.802	6.79
α_e/γ_e	6.82×10^{-2}	6.76×10^{-2}
β_e/γ_e	1.02×10^{-3}	1.02×10^{-3}
$\sin^2\theta_{12}$	0.290	0.291
$\sin^2\theta_{23}$	0.564	0.579
$\sin^2\theta_{13}$	0.0225	0.0219
δ_{CP}^{ℓ}	$258°$	$262°$
$[\alpha_{21}, \alpha_{31}]$	$[330°, 338°]$	$[3.24°, 182°]$
$\sum m_i$	97.9 meV	153 meV
$\langle m_{ee}\rangle$	19.2 meV	59.1 meV
χ^2	1.98	4.12

$S : \tau \longrightarrow -\frac{1}{\tau}$, Duality

±1 is identified

$S^2 = 1, \qquad (ST)^3 = 1.$

$T : \tau \longrightarrow \tau + 1.$ Dicrete shift symmetry

generate infinite discrete group

Modular group

4D effective theory

Coupling is not constant !

- **depends on a modulus τ**
- **is independent under modular transformation**

An example

$\mathcal{L}_1 = f(\tau)\phi_1\phi_2 \cdots \phi_n$

- $f(\tau)$: coupling constant
- ϕ_i: scalar fields

Automorphy factor

$f(\tau) \xrightarrow{\gamma} (c\tau + d)^k f(\tau)$ ⬅ Modular form with weight k **weight k**

$\phi_i \rightarrow (c\tau + d)^{-k_i}\phi_i$

$f(\tau) = (-1)^k f(\tau)$

When $k = \sum_i k_i$, $\mathcal{L}_1$ is modular invariant.

$\mathrm{f}(\gamma\tau) = \mathrm{f}(\tau)$ Too restrictive condition ! $\Rightarrow \frac{\mathrm{f}(\gamma\tau)}{\mathrm{f}(\tau)} = (\mathrm{c}\tau + \mathrm{d})^{\mathrm{k}}$; $\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = (\mathrm{c}\tau + \mathrm{d})^{-2}$