Bootstrapping dS Exchanges

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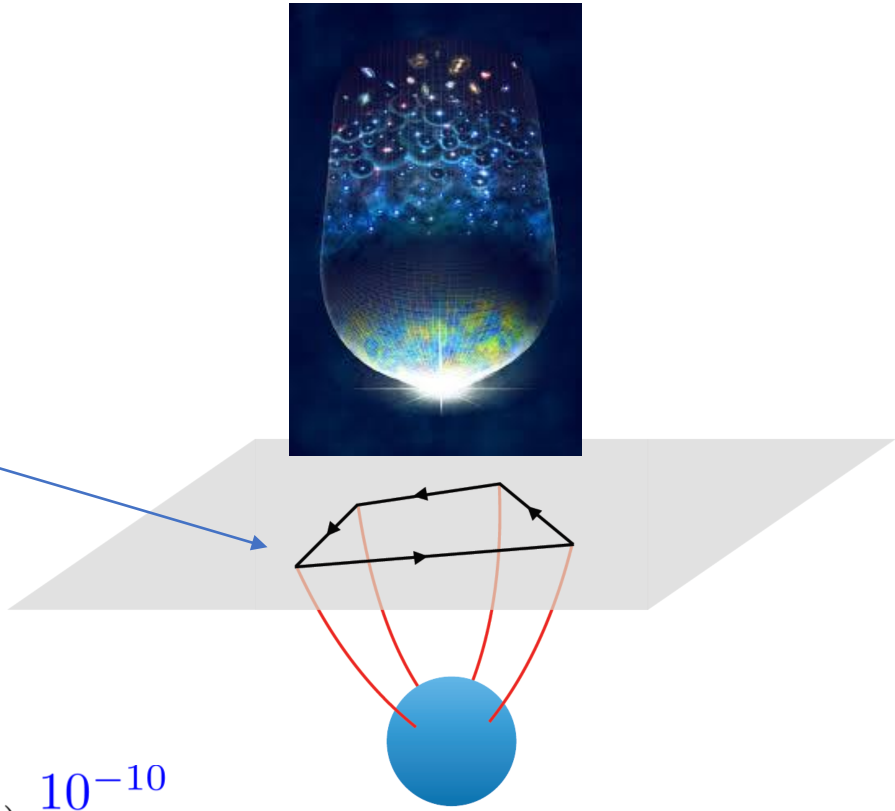
Based on arXiv:1907.01143 w. Charlotte Sleight

Outline

- Introduction
- Mellin space & AdS exchanges
- dS exchanges
- Soft limit & Inflationary Correlators

Cosmological Colliders



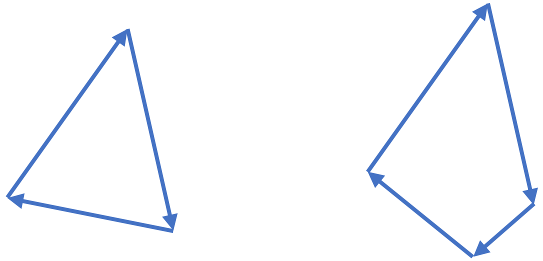
Fluctuations during inflation gave rise to the universe we see today!

Today we can measure correlations of density fluctuations:

$$\left\langle \frac{\delta\rho}{\rho}(x)\frac{\delta\rho}{\rho}(y)\right\rangle \to \delta(\vec{k}_1+\vec{k}_2)\,\frac{10^{-10}}{k^{3-\epsilon}}$$

In the sky there is infinite amount of information!

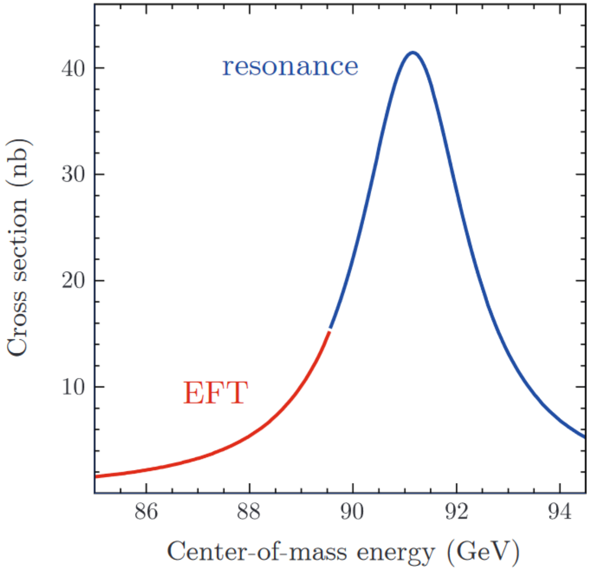
$$\left\langle \frac{\delta\rho}{\rho}(x_1)\cdots\frac{\delta\rho}{\rho}(x_n)\right\rangle$$

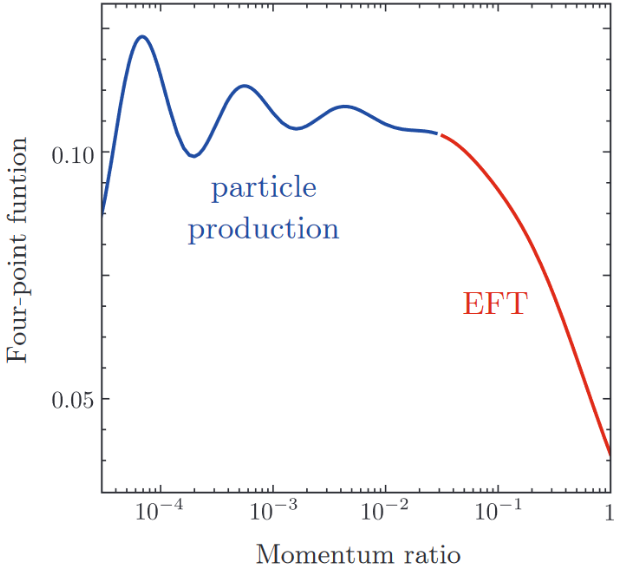


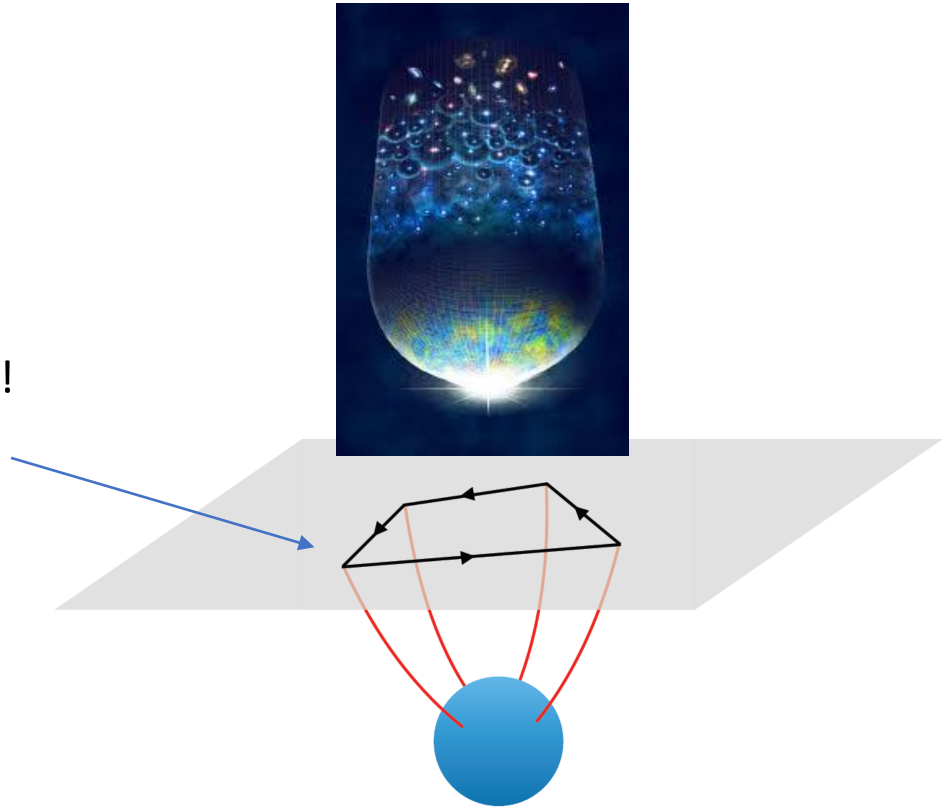
Cosmological Colliders

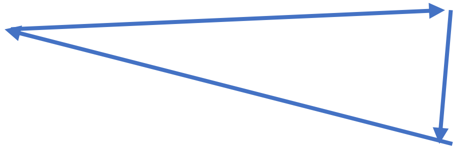
Fluctuations during inflation gave rise to the universe we see today!

$$M^2 \sim H \sim 10^{14}\ \mathrm{GeV}$$









$k_3 \to 0$

$$\left\langle \frac{\delta\rho}{\rho}(x_1)\frac{\delta\rho}{\rho}(x_2)\frac{\delta\rho}{\rho}(x_3)\right\rangle \sim \# e^{i\pi(\dots)}\left(\frac{k_3}{k_1}\right)^{\frac{d}{2}+i\nu} + c.c.$$

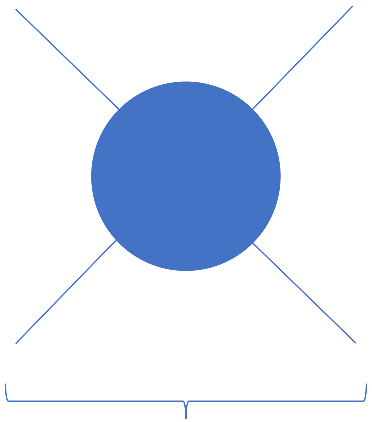
Like in LHC new particles produced during inflation leave a clear imprint on the density fluctuations TODAY!

Inflationary Spectroscopy

Can we bootstrap such imprints? Time without time!

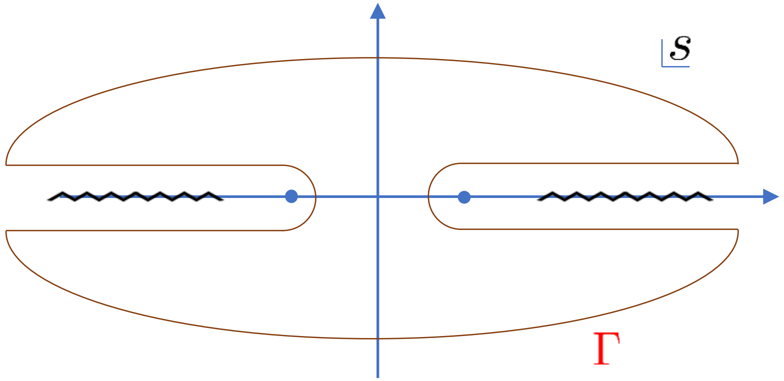
Flat Space Amplitudes

Singularities fix observables (important but hard problem)



$A(s,t)$

$$\sim G_N\Big[\sum \frac{\#}{-s+M^2} + p(s,t)\Big] + G_N^2\Big[\log(-s) + \ldots\Big] + \ldots$$



Dispersion relations:

$$A(s,t) = \frac{1}{2\pi i}\oint_{\Gamma} d\bar{s}\,\frac{A(\bar{s},t)}{\bar{s}-s} \sim \text{arc at } \infty + \int_{sing.} d\bar{s}\,\frac{\text{disc}\,A(\bar{s},t)}{\bar{s}-s}$$

Under mild analyticity assumptions Amplitudes are entirely fixed by the structure of their singularities

(…up to some possible ambiguities)

Tree-level discontinuities

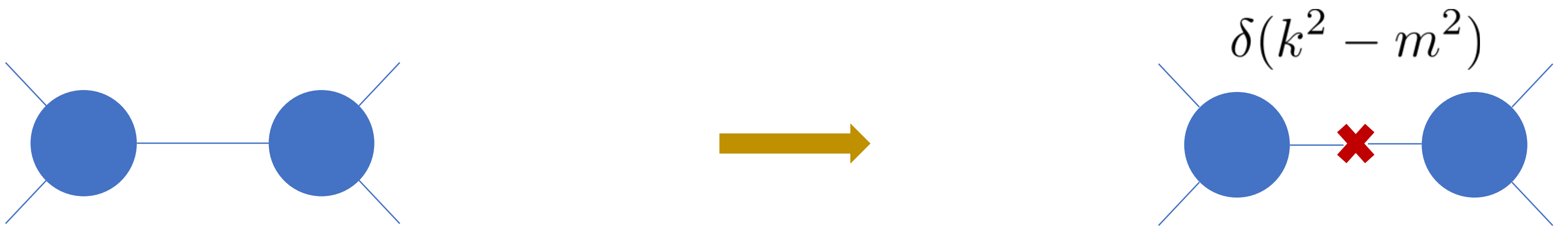
At tree level discontinuities are tied to the concept of Harmonic functions:

$$\Pi_k \sim \frac{1}{k^2 - m^2} \qquad\qquad \mathrm{Disc}_{k^2}\left[\Pi_k\right] \sim \delta(k^2 - m^2)$$

propagator

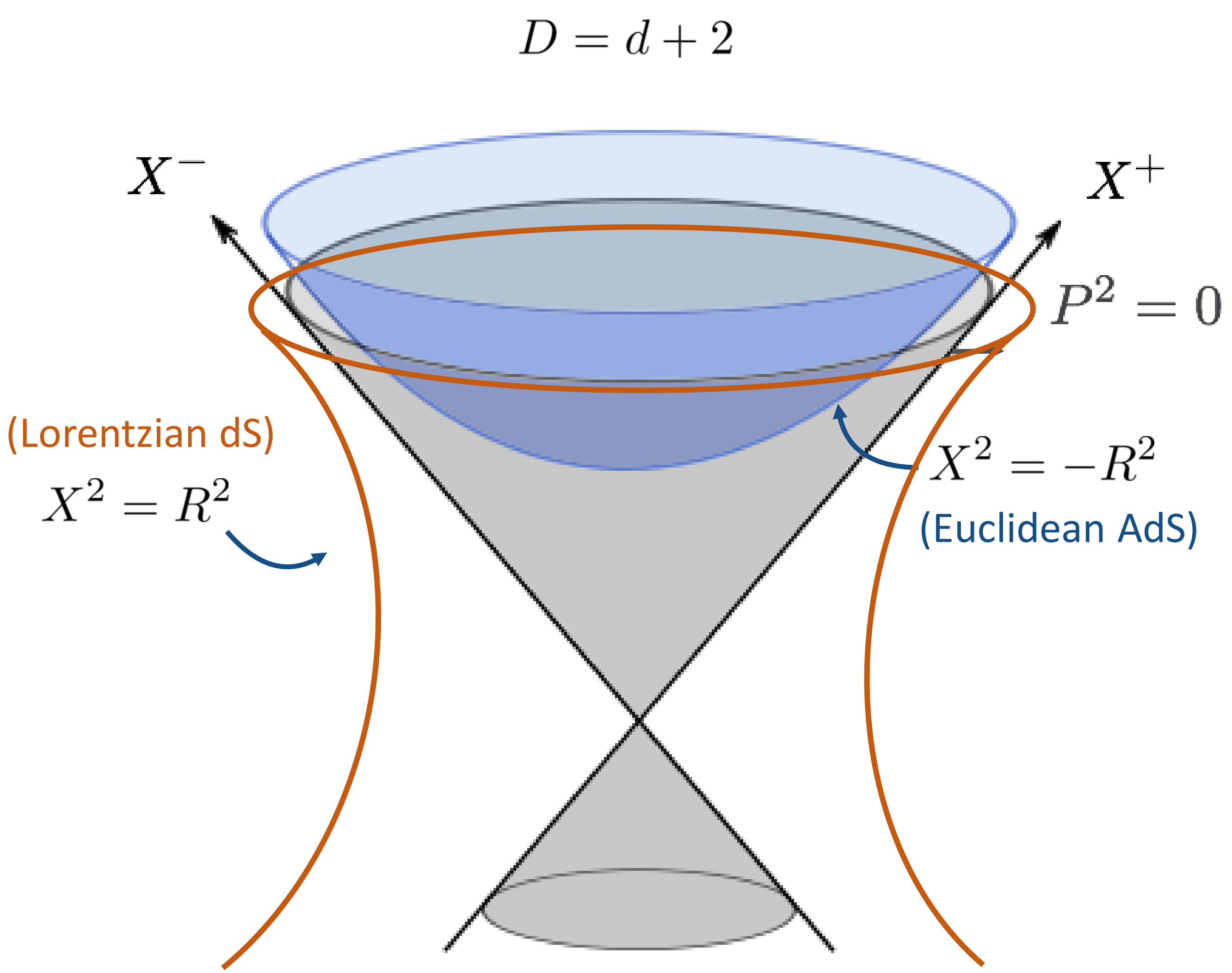
Harmonic function: $$\Omega_m(x) \sim \int d^d k \, e^{ik\cdot x} \, \delta(k^2 - m^2)$$

Put internal leg on-shell



It turns out that this picture can be generalized to constant curvature backgrounds (both AdS and dS)

Ambient Space



Flat space can be foliated by AdS and dS hypersurfaces (Ambient space methods)

$$X \leftrightarrow \pm i\, X$$

$$\{Z = X + iY \in \mathbb{C}^{d+2} \quad | \quad Z^2 = L^2\}$$

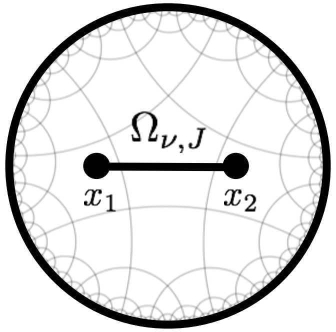
Moschella et al.

Ambient space has the advantage of putting on the same footing (A)dS and flat space physics

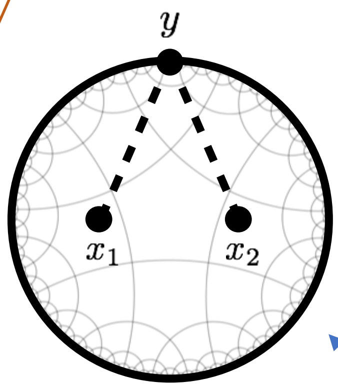
EAdS & CPW

The concept of partial wave admits a straightforward generalization from flat to (A)dS:

$$\Omega_m(x_1, x_2) \sim \int d^d k \, e^{ik\cdot x_1} e^{-ik\cdot x_2} \, \delta(k^2 - m^2)$$



$$= \int_{\partial \mathrm{AdS}} d^d y$$



$$= \frac{\nu^2}{\pi} \int dP \, K_{\frac{d}{2}+i\nu}(X_1; P) K_{\frac{d}{2}-i\nu}(X_2; P)$$

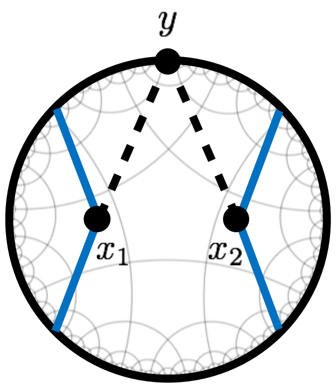
$P^2 = 0$

Light-cone

Bulk to boundary propagator

Split representation

EAdS & CPW

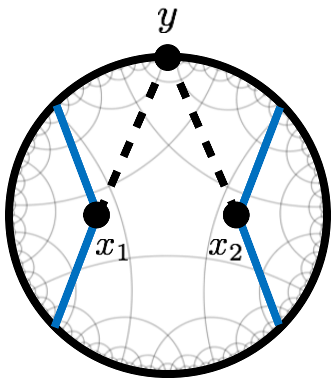


Its useful to think purely in terms of boundary observables:

$$\sim \int d^d x_0 \, \langle \Phi(x_1)\Phi(x_2)\mathcal{O}_\Delta(x_0)\rangle \langle \widetilde{\mathcal{O}}_{d-\Delta}(x_0)\Phi(x_3)\Phi(x_4)\rangle$$

This is a formidable integral! How do we read off it some physics? (Mellin!)

$\mathcal{F}(u,v) \sim$



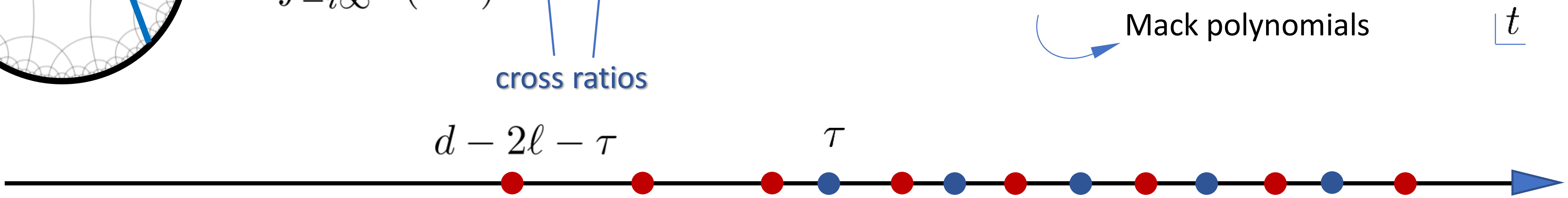
$$\sim \int_{-i\infty}^{+i\infty} \frac{ds\,dt}{(4\pi i)^2}\, u^{t/2} v^{-(s+t)/2}\, \rho(s,t)\, \Omega_\ell(t) \left[s^\ell + \dots \right]$$

$$\rho(s,t): \quad \Gamma\left(-\tfrac{s}{2}\right)^2 \Gamma\left(\tfrac{s+t}{2}\right)^2 \Gamma\left(\tfrac{2\Delta - t}{2}\right)^2$$

$$\Omega_\ell(t) \sim \frac{\Gamma(\frac{\tau - t}{2})\Gamma(\frac{d-\tau-2\ell-t}{2})}{\Gamma(\frac{-t+2\Delta}{2})^2}$$

u, v: cross ratios

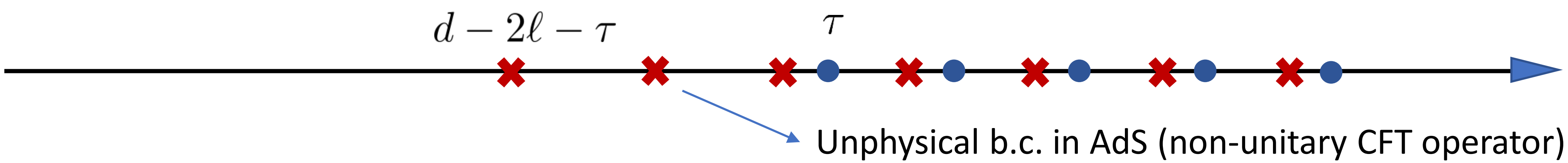
$\left[s^\ell + \dots \right]$: Mack polynomials



Bootstrapping AdS exchanges

In AdS imposing a b.c. is achieved projecting out the unwanted residues:

t



CPW are the basic objects fixed by conformal symmetry! However they fail to satisfy standard boundary conditions

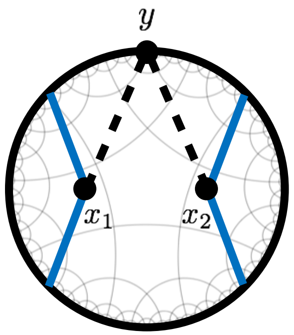
$$\frac{\Gamma(\frac{\tau-t}{2})\Gamma(\frac{d-\tau-2\ell-t}{2})}{\Gamma(\frac{-t+2\Delta}{2})^2} \quad \Longrightarrow \quad \sum_n \frac{\#}{-t+\tau+2n}$$

bootstrap

Generalisation of Legendre Polynomials

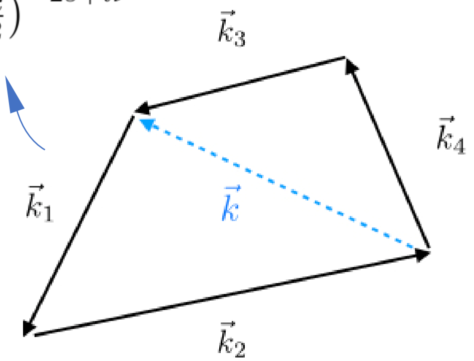
In the flat limit it reproduces the pole singularities

From EAdS to dS (momentum space)



Mellin representation:

$$\sim \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\,\Gamma(s+\tfrac{i\nu}{2})\Gamma(s-\tfrac{i\nu}{2})\left(\tfrac{k}{2}\right)^{-2s+i\nu}$$



$$\sim \mathcal{F}(s_i|u,\bar{u}|k_i;k) \ni \delta(s_1+s_2+u+\ldots)\delta(s_3+s_4+\bar{u}+\ldots)$$

A function of 4 variables requires in general 4 Mellin variables!

The z representation follows directly as an integral representation of boundary Mellin delta-function!

$$2\pi\,\delta(i(s+\alpha)) = \int_0^\infty dz\, z^{s-1}\, z^{\alpha}$$

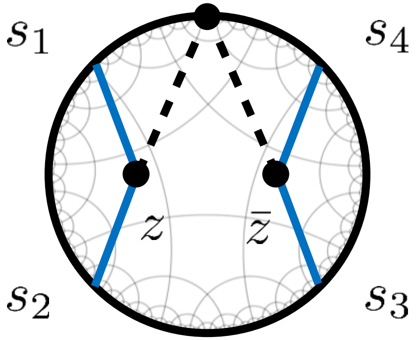
Bulk reconstruction

In the z-variable representation one can consider the following split CPW:

$$\mathcal{F}^{\gtrless}(z,\bar{z}|s_i|k_i;k) = \theta(|z| \gtrless |\bar{z}|) z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k)$$

From EAdS to dS (momentum space)

One therefore arrives to the following split form of the CPW:

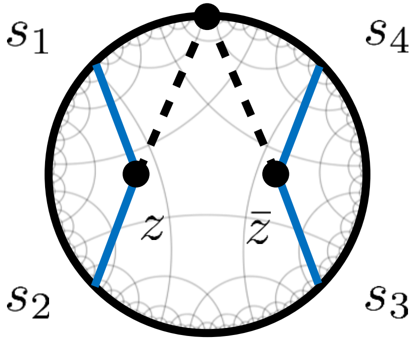


$$\mathcal{F}^{\gtrless}(z,\bar{z}|s_i|k_i;k) = \theta(|z| \gtrless |\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k)$$

$$\sim \theta(|z| > |\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k) + \theta(|z| < |\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k)$$

From EAdS to dS (momentum space)

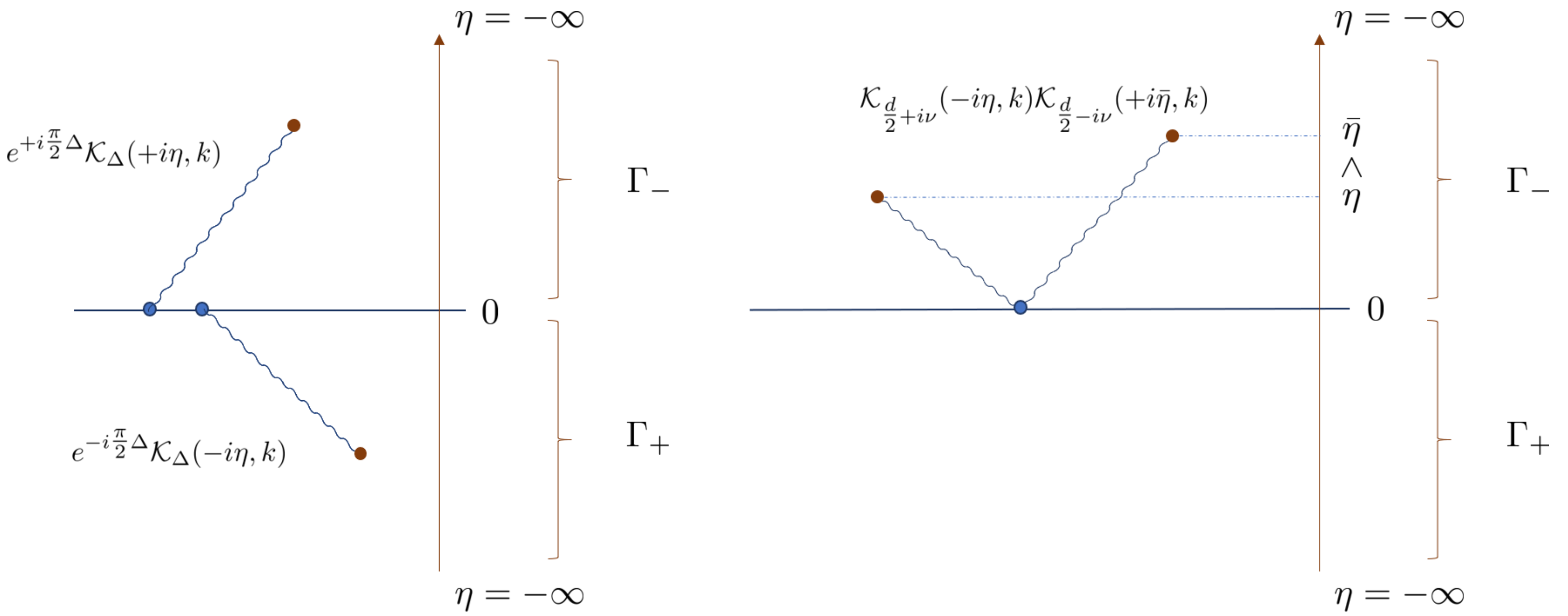
An advantage of momentum space is that bulk (time-)ordering is associated to a simple splitting of the CPW



$$\mathcal{F}^{\gtrless}(z,\bar{z}|s_i|k_i;k) = \theta(|z| \gtrless |\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k)$$

$$\sim \theta(|z|>|\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k) + \theta(|z|<|\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k)$$

$$z = -e^{\pm i\frac{\pi}{2}}\eta$$

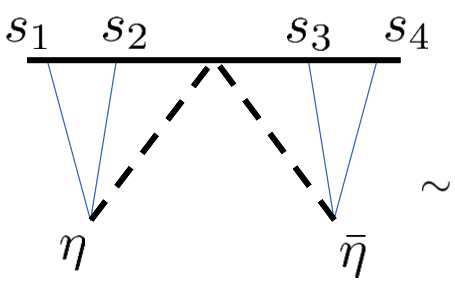


Causality & boundary conditions can be implemented by appropriate analytic continuation in both AdS & dS

From EAdS to dS (momentum space)

An advantage of momentum space is that bulk (time-)ordering is associated to a simple splitting of the CPW

$$\mathcal{F}^{\gtrless}(z,\bar{z}|s_i|k_i;k) = \theta(|z| \gtrless |\bar{z}|)z^{\#}\bar{z}^{\#}\,\mathcal{F}(s_i|k_i;k)$$



$$\sim \lim_{\eta_0\to 0}\int_{-\infty}^{\eta_0} d\eta\, d\bar{\eta}\left[e^{i\pi(\dots)}\theta(|\eta|>|\bar{\eta}|)(-\eta)^{\#}(-\bar{\eta})^{\#}\,\mathcal{F}(s_i|k_i;k) + e^{-i\pi(\dots)}\theta(|\eta|<|\bar{\eta}|)(-\eta)^{\#}(-\bar{\eta})^{\#}\,\mathcal{F}(s_i|k_i;k)\right]$$

Analytic continuation to dS along the various branches in-in contour

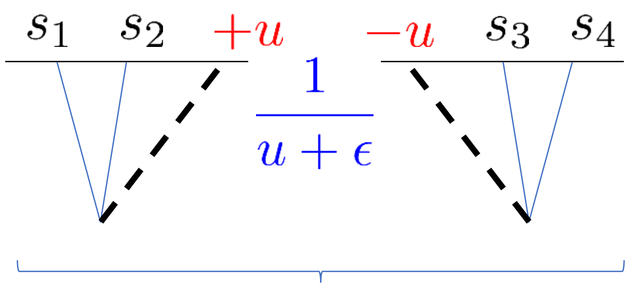
$$z = -e^{\pm i\frac{\pi}{2}}\eta$$

N.B. The phases which arise cannot be fixed by conformal symmetry alone! They are entirely fixed by causality & b.c. while the momentum dependent part factorises.

Contact terms are generated because of the relative phase between the two terms! *Contact term ambiguity*

dS Exchange:

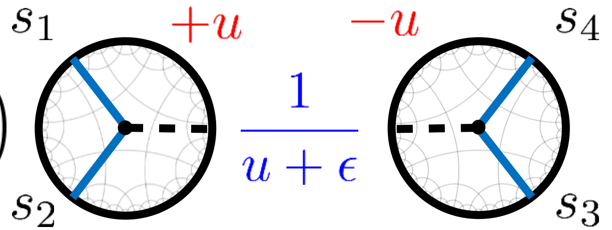
$$\int_{-i\infty}^{+i\infty} \frac{du}{2\pi i}$$



Weighted convolution of 3pt CFT 3pt functions in Mellin space

dS Exchange:

$$\int_{-i\infty}^{+i\infty} \frac{du}{2\pi i} \left(e^{i\pi(u+(s_i,\nu_i,\dots)} + \dots \right)$$

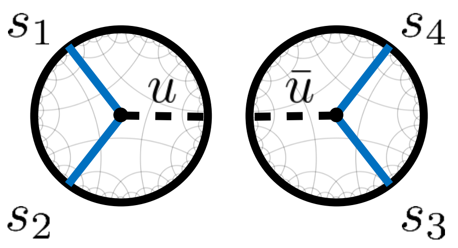


$\delta(u)$

Encodes the interference effect associated with a choice of boundary conditions (zeros are not fixed by conformal symmetry)

Weighted convolution of CFT 3pt functions in Mellin space (poles are fixed by conformal symmetry)

$$\csc(\pi(u+\bar{u}))\delta(u,\bar{u})$$



$$u = -s_1 - s_2 - \dots$$
$$\bar{u} = -s_3 - s_4 - \dots$$

EFT

Disc. = Factorised part $\left(e^{i\pi(\dots)}\right) g_{i\nu,\ell} + \left(e^{-i\pi(\dots)}\right) g_{-i\nu,\ell}$

Spinning 3pt correlators

Mellin independent polarization factor

Spinning 3pt correlators in Mellin space can be written as polynomials in the Mellin variables:

$$\left\langle \Phi_{k_1}^{(\nu_1)} \Phi_{k_2}^{(\nu_2)} \mathcal{O}_{k_3}^{(\nu_3,\ell)}(\xi) \right\rangle \sim \frac{1}{\left(u + \frac{i\nu_3}{2} - \ell\right)_\ell} \sum_{\alpha,\beta} \left(s_1 + \frac{i\nu_1}{2}\right)_{\alpha-\beta} \left(s_2 + \frac{i\nu_2}{2}\right)_\beta \left(u + \frac{i\nu 3}{2} - \ell\right)_{\ell-\alpha} f_{\alpha,\beta}(\xi \cdot k_i)$$

Contribution analogue to Ω

"Polynomials in s_i"

$$\delta(s_1 + s_2 + u + \ldots) \longrightarrow u \sim -s_1 - s_2 + \ldots$$

"Mack-like polynomials" in momentum space:

To be decomposed into helicities

$$P_{\ell=1} = (i\nu + 2u - 2)(-i\nu + 2\bar{u} - 2)(\vec{k}_{12} \cdot \vec{k}_{34})$$
$$- \frac{(s_1 - s_2)(s_3 - s_4)}{2} \left[\frac{1}{2} + \frac{(s_1+s_2)(i\nu+2u-4)}{\nu^2+4(u-1)^2} + \frac{(s_3+s_4)(-i\nu+2\bar{u}-4)}{\nu^2+4(\bar{u}-1)^2}\right] k^2$$

Example

$$p_{ij} = \frac{k_i + k_j}{k}$$

$$\langle \phi^{(i/2)}_{\vec{k}_1} \phi^{(i/2)}_{\vec{k}_2} \phi^{(i/2)}_{\vec{k}_3} \phi^{(i/2)}_{\vec{k}_4} \rangle' = \frac{1}{\pi} \frac{1}{k_1 k_2 k_3 k_4} \left(\frac{k}{2}\right)^{2-d} \int_{-i\infty}^{+i\infty} \frac{du d\bar{u}}{(2\pi i)^2} (2p_{12})^{1-\frac{d}{4}+u} (2p_{34})^{1-\frac{d}{4}+\bar{u}}$$

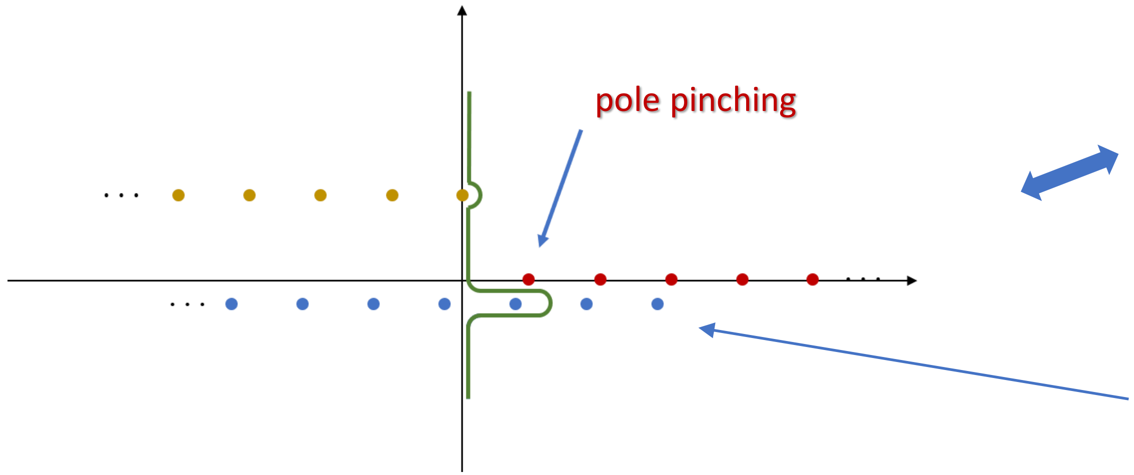
$$\times \csc\left(\pi(u+\bar{u})\right) \delta(u,\bar{u})$$

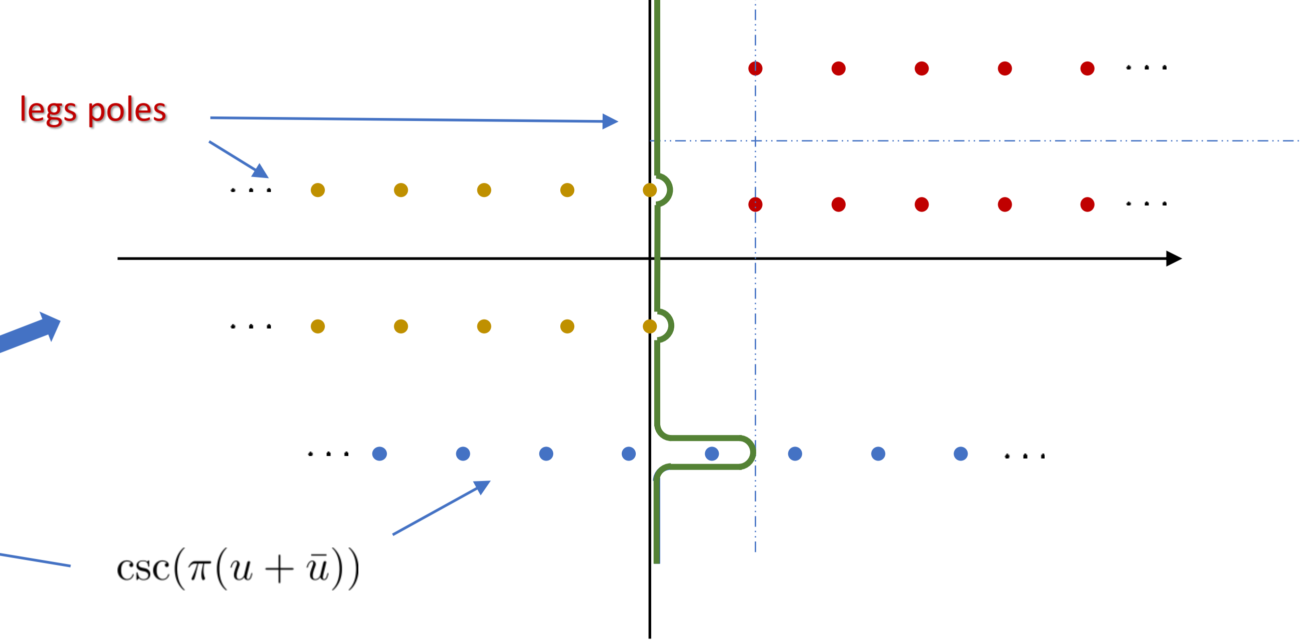
$$\times \underbrace{\Gamma\left(\tfrac{d-2}{2} - 2u\right) \Gamma\left(\tfrac{d-2}{2} - 2\bar{u}\right) \Gamma\left(u + \tfrac{i\nu}{2}\right) \Gamma\left(u - \tfrac{i\nu}{2}\right) \Gamma\left(\bar{u} + \tfrac{i\nu}{2}\right) \Gamma\left(\bar{u} - \tfrac{i\nu}{2}\right)}_{\text{CPW}}$$

$$\delta(u,\bar{u}) = \cos^2\left(\pi\left(\tfrac{d}{4} - \bar{u}\right)\right) \sin\left(\pi\left(u+\bar{u}\right)\right) - \sin\left(\pi\left(\tfrac{d}{2} - u - \bar{u}\right)\right) \sin\left(\pi\left(\bar{u} - \tfrac{i\nu}{2}\right)\right) \sin\left(\pi\left(\bar{u} + \tfrac{i\nu}{2}\right)\right)$$

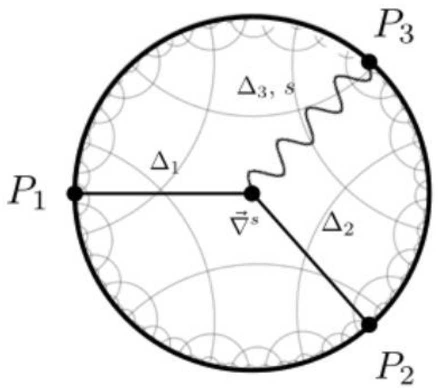
$\frac{d}{4}$

For some values of nu (outside princ. series) some (spurious) singularities might arise

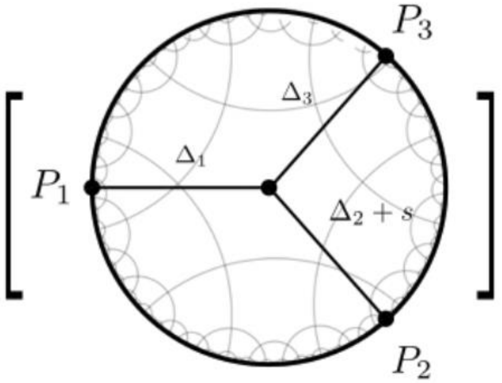




Differential Relations:



$$= \left(\mathcal{D}_{P_3}\left(Z|P_2\right)\right)^s \left[\quad \right]$$



Sleight 2016

Mellin space makes manifest a plethora of recursion/differential relations between 3pt/4pt correlators:

$$z_\mu \langle \mathcal{J}^{\mu}_{\nu_3}(\vec{k}_3)\mathcal{O}_{\frac{i}{2}}(\vec{k}_1)\mathcal{O}_{\frac{i}{2}}(\vec{k}_2)\rangle^{(d)} = -\frac{i}{2}\left[z\cdot\partial_{\vec{k}_1}(\vec{k}_1\cdot\partial_{\vec{k}_1}+1) - z\cdot\partial_{\vec{k}_2}(\vec{k}_2\cdot\partial_{\vec{k}_2}+1) - (\vec{k}_1\cdot\partial_{\vec{k}_1}-\vec{k}_2\cdot\partial_{\vec{k}_2})z\cdot\partial_{\vec{k}_3}\right] \langle \mathcal{O}_{\nu_3}(\vec{k}_3)\mathcal{O}_{\frac{i}{2}}(\vec{k}_1)\mathcal{O}_{\frac{i}{2}}(\vec{k}_2)\rangle^{(d-4)}$$

Basic relation valid for each Mellin polynomial:

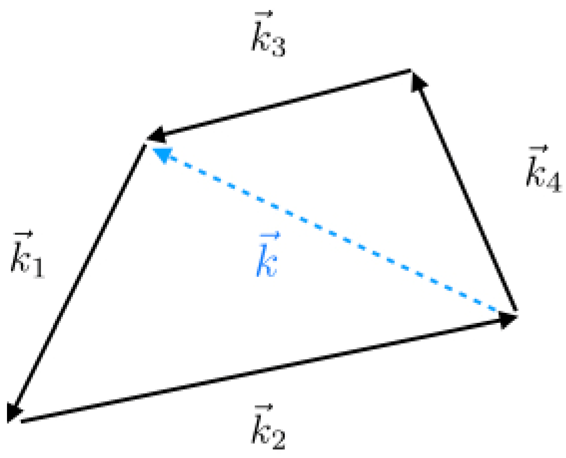
$$s\, k_1^{-2s+i\nu} = -\tfrac{1}{2}(\vec{k}_1\cdot\partial_{\vec{k}_1} - i\nu)k_1^{-2s+i\nu}$$

New weight shifting relations (which extend also to 4pt):

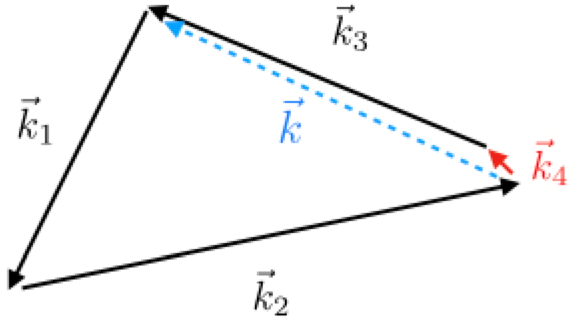
$$\langle \mathcal{O}_{\nu_1+i}(\vec{k}_1)\mathcal{O}_{\nu_2}(\vec{k}_2)\mathcal{O}_{\nu_3}(\vec{k}_3)\rangle^{(d)} = -\frac{2}{k_1}\partial_{k_1}\langle \mathcal{O}_{\nu_1}(\vec{k}_1)\mathcal{O}_{\nu_2}(\vec{k}_2)\mathcal{O}_{\nu_3}(\vec{k}_3)\rangle^{(d-2)}$$

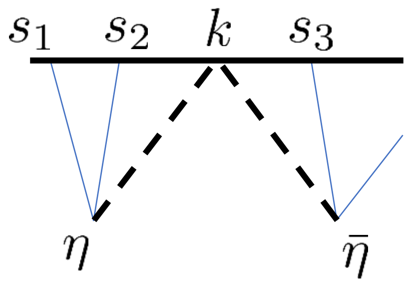
Soft Limit

Generic configuration of momenta



Soft limit $\vec{k}_4 \to 0$





$$k_4 \to 0 \qquad k \sim k_3 \qquad \sim \mathrm{Res}_{s_4} \int \frac{ds_i}{2\pi} (\dots) \Big(\underbrace{\frac{k_3}{k}}_{\sim 1} \Big)^{-s_3+\dots} \dots \sim O(\epsilon)$$

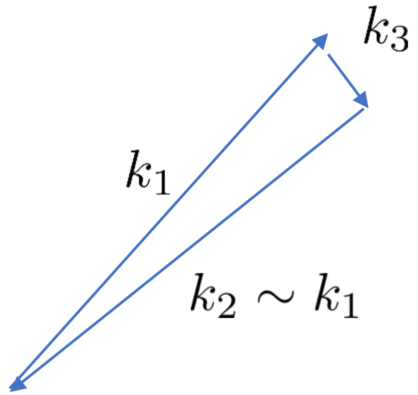
We get the Mellin representation of the 3pt Inflationary correlator!

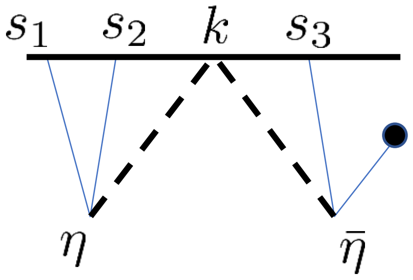
$\Delta_4 = d - \epsilon$

We take the soft limit of a massless leg to break conformal symmetry

Shift symmetric EFT (Adler zero) up to fine tuned contact terms

Soft Limit & squeezed limit





s_4

$k_4 \to 0$

$k \sim k_3$

$$\sim \mathrm{Res}_{s_4} \int \frac{ds_i}{2\pi} \left(\dots \right) \left(\frac{k_3}{k_1} \right)^{-s_2+\dots} \sim \epsilon \, \# \left(\frac{k_3}{k_1} \right)^{\frac{d}{2}-i\nu} + c.c. + O\left(\frac{k_3}{k_1} \right)$$

Reduces to simple instances of I and II Barnes Lemma

Signature of new particles: important to compute the phase and the modulus of the coefficient

The Mellin space representation automatically generates an asympotitc expansion in k3/k1. The result includes also the EFT contribution!

Squeezed limit

The general result is a rather complicated functions of the spectral parameters:

$$\langle \phi^{(\nu_1)}_{\vec{k}_1} \phi^{(\nu_2)}_{\vec{k}_2} \phi^{(\nu_3)}_{\vec{k}_3} \rangle'_{(\text{infl.})} \sim -\frac{\epsilon \mathcal{N}_3}{32\sqrt{\pi}} \frac{(-2)^\ell \ell!}{\left(\frac{d}{2}-1\right)_\ell} \left(\frac{k_1}{2}\right)^{i(\nu_1+\nu_2)} \left(\frac{k_3}{2}\right)^{-\frac{d}{2}+i\nu_3}$$
$$\times\, C_\ell^{\left(\frac{d-2}{2}\right)}(\cos\theta) \Bigg[\left(\frac{k_3}{k_1}\right)^{\frac{d}{2}-i\nu} \frac{\Gamma(i\nu) \left(-\frac{\ell}{2} - \frac{i\nu}{2} \pm \frac{i\nu_3}{2} + 1\right)_{\ell-1}}{\left(\frac{d}{2} - i\nu - 1\right)_\ell \Gamma\left(\frac{d}{2} + \ell - i\nu\right)}$$
$$\times\, \sin\left(\frac{\pi}{4}(d + 2\ell - 2i(\nu - \nu_1 - \nu_2))\right) \operatorname{csch}\left(\frac{\pi}{2}(i\ell + \nu + \nu_3)\right) \prod_{\pm\hat{\pm}} \Gamma\left(\frac{d+2\ell-2i(\nu\pm\nu_1\hat{\pm}\nu_2)}{4}\right)$$
$$+\, \nu \to -\nu \Bigg]$$

Which nicely reduces to known expressions (Arkani-hamed, Maldacena):

$$\langle \phi^{(3i/2)}_{\vec{k}_1} \phi^{(3i/2)}_{\vec{k}_2} \phi^{(3i/2)}_{\vec{k}_3} \rangle'_{(\text{infl.})} \sim -\epsilon \frac{(-8)^{-\ell}\, \ell!}{4\, k_1^3 k_3^3\, \Gamma\left(\ell + \frac{1}{2}\right)} P_\ell(\cos\theta) \left(\frac{k_3}{4k_1}\right)^{\frac{3}{2}-i\nu} \frac{\Gamma\left(\ell + i\nu + \frac{1}{2}\right) \Gamma\left(\ell - i\nu + \frac{1}{2}\right)}{\left(\ell - \frac{3}{2}\right)^2 + \nu^2}$$
$$\times \frac{\pi}{\cosh(\pi\nu)} \frac{\left(\frac{5}{2} + \ell - i\nu\right)\left(1 - i(-1)^\ell \sinh(\pi\nu)\right)\Gamma(i\nu)}{\left(\frac{3}{2} - \ell + i\nu\right)\Gamma\left(i\nu + \frac{1}{2}\right)} + \nu \to -\nu$$

Generalisations to spinning external legs also available!

Conclusions & Outlook

- The discontinuity of dS exchanges has been bootstrapped and the full exchange amplitude has been shown to be fully specified from it
- Our formalism clearly generalize beyond tree level! Can we understand the singularity structure of CFT correlators dual to Inflation at one loop?
- Can we generalize analytic bootstrap methods to dS? Role of crossing decomposition?
- Program: classify singularities of (Euclidean) CFT correlators dual to dS beyond tree-level (CPW=discontinuity fix the full answer up to b.c.)

Some properties of CPW

CPW is the simplest object with well defined crossing decomposition!
(conformal blocks have incompatible branch-cut singularities under crossing…)

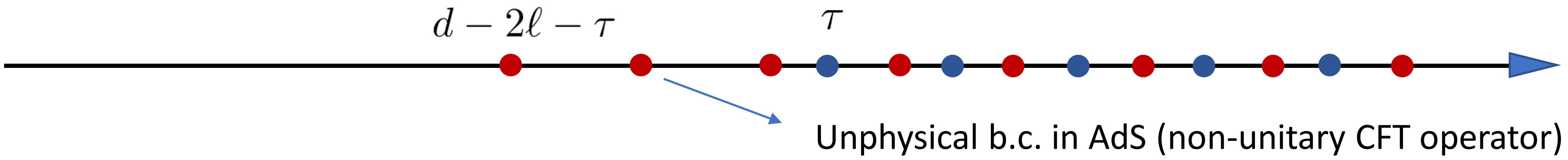
$$^{(\mathrm{t})}\mathcal{F}_{\tau,\ell}(u,v) = \sum_{n,\ell=0}^{\infty} A_{n,\ell}\, g_{2\Delta+2n,\ell}(u,v) + \sum_{n,\ell=0}^{\infty} A^{\partial}_{n,\ell}\, \partial g_{2\Delta+2n,\ell}(u,v)$$

CPW is the simplest combination of conformal blocks which is single valued in Euclidean region (compatible with standard CFT singularities on light-cones)

$$\mathcal{F}_{\tau,\ell}(u,v) \sim \#\, g_{\tau,\ell}(u,v) + \#\, g_{d-\tau-2\ell,\ell}(u,v)$$

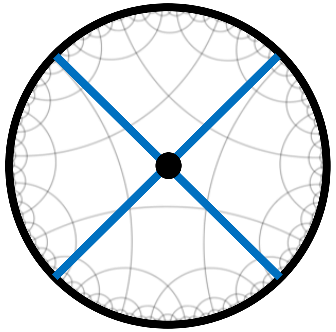
CPW are the basic objects fixed by conformal symmetry! However they fail to satisfy standard boundary conditions

t



Mellin

Mellin space allows to efficiently deal with quite cumbersome functions of cross ratios

$\int_{AdS}$ 

$$= D_{\Delta,\Delta,\Delta,\Delta}(x_i) \sim \int \frac{dsdt}{(4\pi i)^2} u^{t/2} v^{-(s+t)/2} \Gamma\left(-\tfrac{s}{2}\right)^2 \Gamma\left(\tfrac{s+t}{2}\right)^2 \Gamma\left(\tfrac{2\Delta-t}{2}\right)^2 \; 1$$

Poles have physical interpretation: some fixed by conformal symmetry, others fixed by b.c.

$$D_{1111}(u,v) \sim \frac{1}{z-\bar{z}} \left[2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z}) \log \frac{1-z}{1-\bar{z}} \right]$$

$$u = z\bar{z}$$

$$u = (1-z)(1-\bar{z})$$

Only for a few values of Delta there exist explicit representations! (plus recursion relations)